RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

L. FUCHS
G. VILJOEN

Completely decomposable pure subgroups of completely decomposable abelian groups

Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 63-69
<http://www.numdam.org/item?id=RSMUP_1994__92__63_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

Completely Decomposable Pure Subgroups of Completely Decomposable Abelian Groups.

L. FUCHS - G. VILJOEN (*)

When is a pure subgroup of a completely decomposable torsion-free abelian group again completely decomposable? Though this question has been the motivation of several articles, the question itself has not been given due attention in the literature.

Bican [B1] was the first to investigate pure subgroups in completely decomposable groups; he and Kravchenko [K] considered those completely decomposable groups in which all pure subgroups were completely decomposable. Bican [B2] and Kravchenko [K] dealt with the same question for regular subgroups. Hill and Megibben [HM] pointed out the interesting fact that a pure subgroup A of a completely decomposable group G can only be completely decomposable if it is a separative subgroup of G (i.e. it is separable in Hill's sense—this means that for each $g \in G$ there is a countable subset $\{a_n \mid n < \omega\}$ of A such that the characteristics $\chi(g + a_n)$ $(n < \omega)$ form a cofinal subset in the set $\{\chi(g + a) \mid a \in A\}$).

A general sufficient criterion for the complete decomposability of a pure subgroup in a completely decomposable group is due to Dugas and Rangaswamy [DR2]. They proved that if the completely decomposable group G admits an Axiom-3 family of separative subgroups over a pure subgroup A and if every countable subset in any subgroup H in this family is contained in a completely decomposable pure subgroup of H, then A is completely decomposable. (Note that in this case all the subgroups H in the family are completely decomposable.)

(*) Indirizzo degli AA.: L. FUCHS: Department of Mathematics, Tulane University, New Orleans, LA 70118, U.S.A.; G. VILJOEN: Department of Mathematics, University of Orange Free State, Bloemfontein 9300, O.F.S., South Africa.

Financial support to the second author by F.R.D. (C.S.I.R.) and the University of Orange Free State is gratefully acknowledged.

Our objective here is to establish a necessary and sufficient condition for a pure subgroup of a completely decomposable group to be again completely decomposable. The point of departure is the observation that the Dugas-Rangaswamy condition is parallel to the necessary and sufficient criterion established in Fuchs [F] for a pure subgroup of a Butler group (i.e. a B_2-group) to be again Butler. As a matter of fact, the cited sufficient condition in [DR] turns out to be necessary as well. Dugas-Rangaswamy used so-called h-maps in their proof; instead, we follow a more direct and simpler approach, utilizing ideas developed by Bican-Fuchs [BF2], in particular, relative balanced-projective resolutions.

1. A sufficient condition.

We start with a simple lemma; for a proof see e.g. [BF1].

LEMMA 1. *Suppose* $\{0 \to K \to H_\sigma \to A_\sigma \to 0\}$ *is a direct system of balanced-exact sequences* ($\sigma < \kappa$ *for some ordinal* κ) *where the* A_σ *are torsion-free groups and the connecting maps* $\phi_{\sigma\tau}\colon A_\sigma \to A_\tau$ *are monic* ($\sigma < \tau < \kappa$) *with* $\operatorname{Im}\phi_{\sigma\tau}$ *pure in* A_τ. *Then the direct limit of the system is again balanced-exact.* ■

Recall that by an $\aleph_0$-*prebalanced* subgroup of a torsion-free group G is meant a pure subgroup B such that for every pure subgroup A of G that contains B as a corank 1 subgroup the following holds: the lattice ideal generated by the types of rank one pure subgroups J in $A \setminus B$ in the lattice of all types is countably generated. For a discussion of $\aleph_0$-prebalanced subgroups we refer to Bican-Fuchs [BF2].

By an $\aleph_0$-*prebalanced chain* in a torsion-free group G we mean a continuous well-ordered ascending chain of $\aleph_0$-prebalanced subgroups with countable factors from 0 up to the group G.

The following lemma is crucial.

LEMMA 2. *Let* $0 \to K \to G \to A \to 0$ *be a balanced-exact sequence of torsion-free groups where* A *has the property that every countable subgroup of* A *can be embedded in a (countable) completely decomposable pure subgroup of* A. *If* A *is an* $\aleph_0$-*prebalanced subgroup of the torsion-free group* B *with* B/A *countable, then there is a commutative dia-*

gram with balanced-exact rows

$$
(1)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0.
\end{array}
$$

PROOF. Since A is $\aleph_0$-prebalanced in B and B/A is countable, by [BF2] we can select a relative balanced-projective resolution $0 \to D \to^{\alpha} A \oplus C \to B \to 0$ where C is a countable completely decomposable group and D is isomorphic to a pure subgroup of C. By hypothesis, there is a countable completely decomposable pure subgroup X of A that contains the projection of the image of D to A. Since $X \oplus C$ is completely decomposable and the sequence $0 \to K \to G \oplus C \to A \oplus C \to 0$ is balanced-exact, there is a homomorphism $\eta: X \oplus C \to G \oplus C$ which gives rise to the commutative upper right square in the diagram

$$
\begin{array}{ccccccccc}
 & & & & D & & D & & \\
 & & & & \downarrow{\scriptstyle \eta|D} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & K & \longrightarrow & G\oplus C & \longrightarrow & A\oplus C & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

where H is defined as Coker $\eta|D$. The 3×3 lemma guarantees that the bottom row is exact. It is straightforward to check that, moreover, it is balanced-exact. ∎

We can now prove a somewhat stronger version of Theorem 2 in [DR]:

THEOREM 3. *Suppose that A is a pure subgroup of a completely decomposable group G. A is completely decomposable if*

(i) *there is an $\aleph_0$-prebalanced chain $\{A_\sigma\}$ from A to G with countable factors;*

(ii) *for each member A_σ of the chain, every countable subgroup of A_σ can be embedded in a (countable) completely decomposable pure subgroup of A_σ.*

PROOF. Assume (i) and (ii) are satisfied, and let $A = A_0 <$ $< A_1 < \ldots < A_\sigma < \ldots < G$ be an $\aleph_0$-prebalanced chain from A to G with countable factor groups with property (ii). In order to verify that every balanced-exact sequence $0 \to K \to B \to A \to 0$ of torsion-free groups splits, we use Lemma 2 in a straightforward transfinite induction to obtain balanced-exact sequences $0 \to K \to H_\sigma \to A_\sigma \to 0$ along with commutative diagrams like (1) (with H, B replaced by H_σ, A_σ, respectively) and commutative diagrams between exact sequences for indices σ and $\sigma+1$. In view of Lemma 1, at limit ordinals σ we obtain balanced-exact sequences $0 \to K \to H_\sigma \to A_\sigma \to 0$, and the limit of the whole system will be a balanced-exact sequence $0 \to K \to H \to G \to 0$.

Since G is completely decomposable, this sequence splits. Because of the commutative diagram between the given balanced-exact sequence $0 \to K \to B \to A \to 0$ and the limit sequence, the former sequence ought to split. ■

It is easily seen that condition (ii) is satisfied if the groups A_σ are separable, i.e. finite subsets embed in completely decomposable direct summands.

2. Necessary and sufficient conditions.

Suppose A is a completely decomposable pure subgroup of the completely decomposable group B, and let $C = B/A$. Fix decompositions $A = \bigoplus_{i \in I} A_i$ and $B = \bigoplus_{j \in J} B_j$ with rank one summands A_i, B_j, and let $\mathcal{A}$, $\mathcal{B}$ be the families of summands of the form $A_\alpha = \bigoplus_{i \in \alpha} A_i$ and $B_\beta = \bigoplus_{j \in \beta} B_j$, for subsets α, β of I and J, respectively. Then $\mathcal{A}$ (and likewise $\mathcal{B}$) is a $G(\aleph_0)$*-family* in the sense that 1) $0, A \in \mathcal{A}$; 2) $\mathcal{A}$ is closed under unions of chains; 3) given $H \in \mathcal{A}$ and a countable subset X of A, there is an $H' \in \mathcal{A}$ that contains both H and X, and H'/H is countable. Let $\mathcal{C}$ be the $G(\aleph_0)$-family of all pure subgroups in C.

LEMMA 4. *Let $0 \to A \to B \to^{\phi} C \to 0$ be an exact sequence where A and B are completely decomposable and C is torsion-free. There are $G(\aleph_0)$-families $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{C}' \subset \mathcal{C}$ such that*

i) $\mathcal{A}' = \{A \cap B' \mid B' \in \mathcal{B}'\}$,

ii) $\mathcal{C}' = \{\phi B' \mid B' \in \mathcal{B}'\}$,

(iii) *for each $B' \in \mathcal{B}'$, the sequence $0 \to A' = A \cap B' \to B' \to^{\phi} C' = \phi B' \to 0$ is exact.*

PROOF. By a straightforward back-and-forth argument. ■.

LEMMA 5. *Again, let* $0 \to A \to B \to^{\phi} C \to 0$ *be an exact sequence with completely decomposable A, B, and torsion-free C. Suppose* B' *is a summand of B such that* $A' = A \cap B'$ *is a summand of A and* $C' = \phi B'$ *is pure in C. Then* $A + B'$ *is a pure completely decomposable subgroup of B.*

PROOF. As a complete inverse image of C', $A + B'$ is pure in B. We evidently have $(A + B')/B' \cong A/A'$ completely decomposable. B' is a summand of $A + B'$, so $A + B'$ is likewise completely decomposable. ■

We can now state and prove our main result.

THEOREM 6. *For a pure subgroup A of a completely decomposable group B, the following are equivalent:*

(*a*) *A is completely decomposable;*

(*b*) *there is an* $\aleph_0$*-prebalanced chain* $\{A_\sigma\}$ *from A to B with countable factor groups such that, for each member* A_σ *of the chain, every countable subgroup of* A_σ *can be embedded in a (countable) completely decomposable pure subgroup of* A_σ*;*

(*c*) *there is a continuous well-ordered ascending chain of pure subgroups* $\{A_\sigma\}$ *from A to B with countable factors such that each member* A_σ *is completely decomposable;*

(*d*) *there is a* $G(\aleph_0)$*-family of completely decomposable pure subgroups of B over A.*

PROOF. The implications $(d) \Rightarrow (c) \Rightarrow (b)$ are trivial, while $(b) \Rightarrow (a)$ has been proved in Theorem 3. To show that $(a) \Rightarrow (d)$, it is sufficient to observe that, by the preceding lemma, the subgroups $A + B'$ with $B' \in \mathcal{B}'$ ($\mathcal{B}'$ as in Lemma 4) form a $G(\aleph_0)$-family of the desired kind. ■

It is worth while pointing out that it is easy to verify the most relevant part of the last result, viz. the equivalence of conditions (*a*) and (*c*), directly, by utilizing relative balanced-projective resolutions. Evidently, it suffices to show that (*a*) implies (*c*). For a pure subgroup A of the torsion-free group B, there is a relative balanced-projective resolution $0 \to K \to A \oplus C \to B \to 0$ where C is completely decomposable. Now, if both A and B are completely decomposable, then this sequence splits and K is a summand of the completely decomposable group $A \times C$, so itself completely decomposable. Let $\mathcal{K}$ and $\mathcal{C}$ be $G(\aleph_0)$-families of summands in K and C, respectively. Using standard back-and-forth techniques, it is straightforward to construct a well-ordered direct system of splitting exact sequences $0 \to K_\sigma \to A \oplus C_\sigma \to A_\sigma \to 0$ $(\sigma < \tau)$ whose direct limit is $0 \to K \to A \oplus C \to B \to 0$, where the system is subject to the conditions: 1) $K_\sigma \in \mathcal{K}$, 2) $C_\sigma \in \mathcal{C}$, 3) $A \leqslant A_\sigma$ is pure in B, and 4) $A_{\sigma+1}/A_\sigma$ is countable for all $\sigma < \tau$. Then the chain $\{A_\sigma\}$ will be as desired, since A_σ is isomorphic to a summand of the completely decomposable group $A \oplus C_\sigma$ for each $\sigma < \tau$.

As an application, we give a proof for Kravchenko's theorem (the proof of Theorem 4 in [DR] is incorrect, since the subgroups S_n are not necessarily balanced).

COROLLARY 7. *Let G be a completely decomposable group whose typset T contains a countable ascending chain $\boldsymbol{t}_1 < \boldsymbol{t}_2 < \ldots < \boldsymbol{t}_n < \ldots$ such that*

1) *for every $\boldsymbol{t} \in T$ there is an n with $\boldsymbol{t} \leqslant \boldsymbol{t}_n$, and*
2) *the set $\{\boldsymbol{t} \in T \mid \boldsymbol{t} \leqslant \boldsymbol{t}_n\}$ is inversely well-ordered.*

Then every pure subgroup of G is completely decomposable.

PROOF. From the stated conditions it follows at once that every pure subgroup in G is $\aleph_0$-prebalanced in G; thus (i) in Theorem 3 holds. Furthermore, Lemma 1 in [K] shows that every pure subgroup of G is separable. Since countable separable groups are completely decomposable, (ii) of Theorem 3 is obviously satisfied. ∎

Another proof can be given by using Wang's Theorem 1 in [W].

REFERENCES

[B1] L. BICAN, *Completely decomposable abelian groups any pure subgroup of which is completely decomposable*, Czech. Math. J., 24 (1974), pp. 176-191.

[B2] L. BICAN, *Completely decomposable abelian groups any regular subgroup of which is completely decomposable*, Czech. Math. J., 25 (1975), pp. 71-76.

[BF1] L. BICAN - L. FUCHS, *On abelian groups by which balanced extensions of a rational group split*, J. Pure Appl. Alg., 78 (1992), pp. 221-238.

[BF2] L. BICAN - L. FUCHS, *Subgroups of Butler groups*, Comm. Algebra, 22 (1994), pp. 1037-1047.

[DR] M. DUGAS - K. M. RANGASWAMY, *Separable pure subgroups of completely decomposable torsion-free abelian groups*, Arch. Math., 58 (1992), pp. 332-337.

[F] L. FUCHS, *Butler groups of infinite rank*, J. Pure Appl. Algebra, to appear.

[HM] P. HILL - C. MEGIBBEN, *Torsion-free groups*, Trans. Amer. Math. Soc., **295** (1986), pp. 735-751.

[K] A. A. KRAVCHENKO, *Completely decomposable groups*, Mat. Zametki, **31** (1982), pp. 171-185.

[W] J. S. P. WANG, *On completely decomposable groups*, Proc. Amer. Math. Soc., **15** (1964), pp. 184-186.

Manoscritto pervenuto in redazione il 21 gennaio 1993.