RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Distribution of solutions of diophantine equations $f_1(x_1)f_2(x_2) = f_3(x_3)$, where f_i are polynomials. - II

Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 29-46

<http://www.numdam.org/item?id=RSMUP_1994__92__29_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

Distribution of Solutions of Diophantine Equations $f_1(x_1)f_2(x_2) = f_3(x_3)$, where f_i are Polynomials. - II.

A. SCHINZEL - U. ZANNIER (*)

Introduction and statement of results.

The present paper is a sequel to [1] and the notation of that paper is retained, in particular a_i is the leading coefficient and Δ_i the discriminant of the quadratic polynomial f_i ($1 \leq i \leq 3$). The purpose of the paper is to perform the programme outlined in Remark 7 of [1], at least in the simplest case, when $a_i = 1$, $\Delta_i = 16$ ($i = 1, 2$), $a_3 = 1$, $\sqrt{\Delta_3} \in 4\mathbb{Z}$.

As a by-product one obtains the following purely algebraic

THEOREM 1. *Let k be a field, $\delta_1, \delta_2, \delta_3 \in k$, $\delta_1\delta_2 \neq 0$, $\delta_0 = \delta_1\delta_2 + \delta_3$. The equation*

$$(1) \qquad (p_1^2 - \delta_1)(p_2^2 - \delta_2) = p_3^2 - \delta_3$$

has infinitely many solutions in polynomials $p_i \in k[t]$ not all constant only if one of the following three conditions is satisfied

$$(2a) \qquad \sqrt{\delta_3} \in k \text{ and } \sqrt{\delta_i} \in k \text{ for an } i \in \{0, 1, 2\},$$

$$(2b) \qquad \sqrt{\frac{\delta_0}{\delta_i}} \in k \text{ and } \sqrt{\frac{\delta_3}{\delta_i}} \in k \text{ for an } i \in \{1, 2\},$$

$$(2c) \qquad \sqrt{\frac{\delta_3}{\delta_1}}, \quad \sqrt{\frac{\delta_3}{\delta_2}} \in k.$$

Then the set S of all solutions of (1) in polynomials $p_i \in k[t]$ not all constant is the minimal set S_0 with the following properties. For every choice of quadratic roots, every choice of $\varepsilon \in \{1, -1\}$ and every

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$p \in k[t]\backslash k$

$$\text{(3a)} \quad \left\langle p,\, p\,\frac{\sqrt{\delta_0}+\sqrt{\delta_3}}{\delta_1},\, \varepsilon\left(p^2\,\frac{\sqrt{\delta_0}+\sqrt{\delta_3}}{\delta_1} - \sqrt{\delta_0}\right)\right\rangle \in S_0$$

if we have (2*a*) *with* $i = 0$,

(3*b*) $\langle\sqrt{\delta_1}, p, \sqrt{\delta_3}\rangle \in S_0$ *if we have* (2*a*) *with* $i = 1$,

(3*c*) $\langle p, \sqrt{\delta_2}, \sqrt{\delta_3}\rangle \in S_0$ *if we have* (2*a*) *with* $i = 2$,

(3*d*) $\left\langle p, \sqrt{\frac{\delta_0}{\delta_1}}, \sqrt{\frac{\delta_3}{\delta_1}}\,p\right\rangle \in S_0$ *if we have* (2*b*) *with* $i = 1$,

(3*e*) $\left\langle \sqrt{\frac{\delta_0}{\delta_2}}, p, \sqrt{\frac{\delta_3}{\delta_2}}\,p\right\rangle \in S_0$ *if we have* (2*b*) *with* $i = 2$,

$$\text{(3f)} \quad \left\langle p,\, p\sqrt{\frac{\delta_2}{\delta_1}} + \sqrt{\frac{\delta_3}{\delta_1}},\, \varepsilon\left((p^2-\delta_1)\sqrt{\frac{\delta_2}{\delta_1}} + p\sqrt{\frac{\delta_3}{\delta_1}}\right)\right\rangle \in S_0$$

if (2*c*) *holds*.

Moreover if $\langle p_1, p_2, p_3\rangle \in S_0$, $\varepsilon \in \{1, -1\}$, *then*

$$\text{(4a)} \quad \left\langle p_1,\, \frac{2p_1^2 p_2 + 2\varepsilon p_1 p_3 - \delta_1 p_2}{\delta_1},\, \frac{2p_1^2 p_3 + 2\varepsilon p_1 p_2 (p_1^2 - \delta_1) - \delta_1 p_3}{\delta_1}\right\rangle \in S_0,$$

$$\text{(4b)} \quad \left\langle \frac{2p_1 p_2^2 + 2\varepsilon p_1 p_3 - \delta_2 p_1}{\delta_2},\, p_2,\, \frac{2p_2^2 p_3 + 2\delta p_1 p_2 (p_2^2 - \delta_2) - \delta_2 p_3}{\delta_2}\right\rangle \in S_0.$$

If $\sqrt{\delta_i} \in k$ ($i = 1, 2, 3$) *then the set* S *can be obtained simpler as the minimal set* S_1 *with the following properties. For every choice of quadratic roots and every* $p \in k[t]\backslash k$ *we have* (3*b*), (3*c*), (3*d*) *and* (3*e*) *with* S_0 *replaced by* S_1. *Moreover if* $\langle p_1, p_2, p_3\rangle \in S_1$, $\eta \in \{1, -1\}$, *then*

$$\left\langle p_1,\, \frac{p_1 p_2 + \eta p_3}{\sqrt{\delta_1}},\, \frac{p_1 p_3 + \eta p_2 (p_1^2 - \delta_1)}{\sqrt{\delta_1}}\right\rangle \in S_1,$$

$$\left\langle \frac{p_1 p_2 + \eta p_3}{\sqrt{\delta_2}},\, p_2,\, \frac{p_1 p_3 + \eta p_1 (p_2^2 - \delta_2)}{\sqrt{\delta_2}}\right\rangle \in S_1,$$

The principal result runs as follows.

THEOREM 2. *If* $a_1 = a_2 = a_3 = 1$, $\Delta_1 = \Delta_2 = 16$, $\sqrt{\Delta_3} \in 4\mathbb{Z}$, *then the number* $N(x)$ *of integers* x_3 *such that* $|x_3| \leq x$ *and there exist integers* x_1, x_2 *such that*

$$f_1(x_1)\, f_2(x_2) = f_3(x_3)$$

satisfies the asymptotic formula

$$N(x) = 2\sqrt{x} + O(x^{1/3}).$$

1. Proof of Theorem 1.

LEMMA 1. *Under the assumptions of the theorem all solutions of the equation* (1) *in which one but not each polynomial* $p_i \in k[t]$ *is constant are given by*

(5*a*) $$\langle p, \sqrt{\delta_2}, \sqrt{\delta_3}\rangle, \quad \text{if } \sqrt{\delta_2}, \sqrt{\delta_3} \in k,$$

(5*b*) $$\langle \sqrt{\delta_1}, p, \sqrt{\delta_3}\rangle, \quad \text{if } \sqrt{\delta_1}, \sqrt{\delta_3} \in k,$$

(5*c*) $$\left\langle p, \sqrt{\frac{\delta_0}{\delta_1}}, \sqrt{\frac{\delta_3}{\delta_1}}\, p\right\rangle, \quad \text{if } \sqrt{\frac{\delta_0}{\delta_1}}, \sqrt{\frac{\delta_3}{\delta_1}} \in k,$$

(5*d*) $$\left\langle \sqrt{\frac{\delta_0}{\delta_2}}, p, \sqrt{\frac{\delta_3}{\delta_2}}\, p\right\rangle, \quad \text{if } \sqrt{\frac{\delta_0}{\delta_2}}, \sqrt{\frac{\delta_3}{\delta_2}} \in k,$$

where $p \in k[t]\backslash k$ *and the choice of quadratic roots is arbitrary.*

PROOF. Suppose that $\langle p_1, p_2, p_3\rangle$ is a required solution. By symmetry we may assume that $p_1 \in k$. If $p_1^2 = \delta_1$, we obtain the case (5*b*). If $p_1^2 \neq \delta_1$, we have

$$(p_3 - \sqrt{p_1^2 - \delta_1}\, p_2)(p_3 + \sqrt{p_1^2 - \delta_1}\, p_2) = \delta_3 - (p_1^2 - \delta_1)\delta_2$$

and since the two factors cannot simultaneously be constant we have $\delta_3 - (p_1^2 - \delta_1)\delta_2 = 0$, which gives (5*d*). On the other hand, (5*a*), (5*b*), (5*c*), (5*d*) are solutions of (1) with the required properties. ■

LEMMA 2. *Under the assumptions of the theorem all solutions of the equation* (1) *in polynomials* $p_i \in k[t]$ *such that* $\deg p_1 = \deg p_2 > 0$

are given by

$$\text{(6a)}\quad \left\langle p,\, p\,\frac{\sqrt{\delta_0}+\sqrt{\delta_3}}{\delta_1},\, \varepsilon\left(p^2\,\frac{\sqrt{\delta_0}+\sqrt{\delta_3}}{\delta_1}-\sqrt{\delta_0}\right)\right\rangle \quad \textit{if } \sqrt{\delta_0}, \sqrt{\delta_3}\in k,$$

$$\text{(6b)}\quad \left\langle p,\, p\sqrt{\frac{\delta_2}{\delta_1}}+\sqrt{\frac{\delta_3}{\delta_1}},\, \varepsilon\left((p^2-\delta_1)\sqrt{\frac{\delta_2}{\delta_1}}+p\sqrt{\frac{\delta_3}{\delta_1}}\right)\right\rangle$$

$$\textit{if } \sqrt{\frac{\delta_2}{\delta_1}}, \sqrt{\frac{\delta_3}{\delta_1}}\in k,$$

where $p\in k[t]\backslash k$, *the choice of quadratic roots is arbitrary and* $\varepsilon\in\{1,-1\}$.

PROOF. Suppose that $\langle p_1, p_2, p_3\rangle$ is a required solution. Let $l(p_i)$ denote the leading coefficient of p_i and choose $\varepsilon = l(p_1p_2)/l(p_3)$ and $\sqrt{p_1^2-\delta_1}$ so that at $t=\infty$, $\sqrt{p_1^2-\delta_1}=p_1(t)+o(1)$, which implies

$$\sqrt{p_1^2-\delta_1}=p_1(t)+O(|t|^{-\deg p_1}).$$

(If char $k>0$ the notation should be suitably interpreted). Then define

$$p_3'+\varepsilon\sqrt{p_1^2-\delta_1}p_2'=(p_3+\varepsilon\sqrt{p_1^2-\delta_1}p_2)\,\frac{p_1-\sqrt{p_1^2-\delta_1}}{\sqrt{\delta_1}}\,.$$

We have at $t=\infty$

$$p_3'+\varepsilon\sqrt{p_1^2-\delta_1}p_2'=O(|t|^{\deg p_3-\deg p_1}),$$

but also

$$p_3-\varepsilon\sqrt{p_1^2-\delta_1}p_2=\frac{\delta_3-\delta_2(p_1^2-\delta_2)}{p_3+\varepsilon\sqrt{p_1^2-\delta_1}p_2}=O(|t|^{2\deg p_1-\deg p_3})=O(1),$$

hence

$$p_3'-\varepsilon\sqrt{p_1^2-\delta_1}p_2'=(p_3-\varepsilon\sqrt{p_1^2-\delta_1}p_2)\,\frac{p_1+\sqrt{p_1^2-\delta_1}}{\sqrt{\delta_1}}=$$

$$=O(|t|^{3\deg p_1-\deg p_3})$$

and so

$$p_2' = O(1).$$

However p_1', p_2', p_3' satisfy (1) and $p_i' \in k(\sqrt{\delta_1})[t]$, hence by virtue of Lemma 1 applied with $k(\sqrt{\delta_1})$ instead of k we have for a suitable choice of quadratic roots either

$$\langle p_2', p_3' \rangle = \langle \sqrt{\delta_2}, \varepsilon\sqrt{\delta_3} \rangle$$

or

$$\langle p_2', p_3' \rangle = \left\langle \sqrt{\frac{\delta_0}{\delta_1}}, \varepsilon\sqrt{\frac{\delta_3}{\delta_1}}\, p_1 \right\rangle.$$

In the first case we obtain

$$p_3 + \varepsilon\sqrt{p_1^2 - \delta_1}\, p_2 = \varepsilon(\sqrt{\delta_3} + \sqrt{p_1^2 - \delta_1} \cdot \sqrt{\delta_2})\, \frac{p_1 + \sqrt{p_1^2 - \delta_1}}{\sqrt{\delta_1}}$$

which gives (6*b*).

In the second case we obtain

$$p_3 + \varepsilon\sqrt{p_1^2 - \delta_1}\, p_2 = \varepsilon\left(\sqrt{\frac{\delta_3}{\delta_1}}\, p_2 + \sqrt{p_1^2 - \delta_1} \cdot \sqrt{\frac{\delta_0}{\delta_1}}\right) \frac{p_1 + \sqrt{p_1^2 - \delta_1}}{\sqrt{\delta_1}}$$

which gives (*a*). On the other hand, (6*a*) and (6*b*) are solutions of (1) with the required properties. ∎

PROOF OF THEOREM 1. We have under the specified conditions (2*a*), (2*b*), or (2*c*) for $p \in k[t]\backslash k$

$$\left\langle p, p\, \frac{\sqrt{\delta_0} + \sqrt{\delta_3}}{\delta_1}, \varepsilon\left(p^2\, \frac{\sqrt{\delta_0} + \sqrt{\delta_3}}{\delta_1} - \sqrt{\delta_0}\right)\right\rangle, \quad \langle p_1\sqrt{\delta_2}, \sqrt{\delta_3} \rangle,$$

$$\langle \sqrt{\delta_1}, p, \sqrt{\delta_3} \rangle, \quad \left\langle p, \sqrt{\frac{\delta_0}{\delta_1}}, \sqrt{\frac{\delta_3}{\delta_1}}\, p \right\rangle, \quad \left\langle \sqrt{\frac{\delta_0}{\delta_2}}, p, \sqrt{\frac{\delta_3}{\delta_2}}\, p \right\rangle,$$

$$\left\langle p, p\sqrt{\frac{\delta_2}{\delta_1}} + \sqrt{\frac{\delta_3}{\delta_1}}, \varepsilon\left((p^2 - \delta_1)\sqrt{\frac{\delta_2}{\delta_1}} + p\sqrt{\frac{\delta_3}{\delta_1}}\right)\right\rangle \in S$$

and if $\langle p_1, p_2, p_3\rangle \in S$ then for $i = 1, 2$

$$(p_i^2 - \delta_i)\left(\left(\frac{2p_i^2 p_{3-i} + 2\varepsilon p_i p_3 - \delta_i p_{3-i}}{\delta_i}\right)^2 - \delta_{3-i}\right) -$$

$$-\left(\frac{2p_i^2 p_3 + 2\varepsilon p_i p_{3-i}(p_i^2 - \delta_i) - \delta_i p_3}{\delta_i}\right)^2 = (p_1^2 - \delta_1)(p_2^2 - \delta_2) - p_3^2 = -\delta_3 .$$

Moreover $\begin{vmatrix} 2p_i^2 - \delta_i & 2\varepsilon p_i \\ 2\varepsilon p_i(p_i^2 - \delta_i) & 2p_i^2 - \delta_i \end{vmatrix} = \delta_i^2 \neq 0 \quad (i = 1, 2)$, thus if $\langle p_1, p_2, p_3\rangle \notin k^3$ then also for $i = 1, 2$

$$\left\langle p_i, \frac{2p_i^2 p_{3-i} + 2\varepsilon p_i p_3 - \delta_i p_{3-i}}{\delta_i}, \frac{2p_i^2 p_3 + 2\varepsilon p_i p_{3-i}(p_i^2 - \delta_i) - \delta_i p_3}{\delta_i}\right\rangle \notin k^3 ,$$

hence $S_0 \subset S$. In order to prove that $S \subset S_0$ we proceed by induction with respect to $m = \max\{\deg p_1, \deg p_2, \deg p_3\}$. If $m = 1$(1) implies that $p_1^2 - \delta_1 \in k$ or $p_2^2 - \delta_2 \in k$, hence by Lemma 1 either (2*a*) holds with $i \in \{1, 2\}$ and $\langle p_1, p_2, p_3\rangle \in S_0$ by (3*b*) or (3*c*) or (2*b*) holds and $\langle p_1, p_2, p_3\rangle \in S_0$ by (3*d*) or (3*e*).

Assume now that (1) implies $\langle p_1, p_2, p_3\rangle \in S_0$ provided $m < n$ and let

$$\max\{\deg p_1, \deg p_2, \deg p_3\} = n > 1 .$$

If $p_1^2 - \delta_1 = 0$ or $p_2^2 - \delta_2 = 0$ we have (2*a*) with $i = 1$ or 2 and $\langle p_1, p_2, p_3\rangle \in S_0$ by (3*b*) or (3*c*). If $p_1^2 - \delta_1 \neq 0 \neq p_2^2 - \delta_2$ we have $\deg p_3 = n$. If $\deg p_2 = 0$ we have (2*a*) with $i = 1$ and $\langle p_1, p_2, p_3\rangle \in S_0$ by (3*d*). If $\deg p_1 = 0$ we have similarly (2*a*) with $i = 2$ and $\langle p_1, p_2, p_3\rangle \in S_0$ by (3*e*). We may assume therefore that $\deg p_1 > 0$, $\deg p_2 > 0$.

Consider first the case, where $\deg p_1 = \deg p_2$. By Lemma 2 this can happen only if we have either

$$(7) \quad p_1 = p , \quad p_2 = p\,\frac{\sqrt{\delta_0} + \sqrt{\delta_3}}{\delta_1} , \quad p_3 = \varepsilon\left(p^2\,\frac{\sqrt{\delta_0} + \sqrt{\delta_3}}{\delta_1} - \sqrt{\delta_0}\right),$$

or

$$(8) \quad p_1 = p , \quad p_2 = p\sqrt{\frac{\delta_2}{\delta_1}} + \sqrt{\frac{\delta_3}{\delta_1}} , \quad p_3 = \varepsilon\left((p^2 - \delta_1)\sqrt{\frac{\delta_2}{\delta_1}} + p\sqrt{\frac{\delta_3}{\delta_1}}\right).$$

Now, (7) implies that (2a) holds with $i = 0$ and we have (3a); (8) implies that (2c) holds and we have (3f), hence $\langle p_1, p_2, p_3 \rangle \in S_0$.

Consider now the case, where $\deg p_1 < \deg p_2$. Choose $\varepsilon = l(p_1 p_2)/l(p_3)$ and $\sqrt{p_1^2 - \delta_1}$ so that at $t = \infty$ $\sqrt{p_1^2 - \delta_1} = p_1(t) + o(1)$, which implies

$$\sqrt{p_1^2 - \delta_1} = p_1(t) + O(|t|^{-\deg p_1}).$$

Then define

$$p_3' + \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2' = (p_3 + \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2)\left(\frac{p_1 - \sqrt{p_1^2 - \delta_1}}{\sqrt{\delta_1}}\right)^2. \tag{9}$$

We have at $t = \infty$

$$p_3' + \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2' = O(|t|^{\deg p_3 - 2\deg p_1}),$$

but also

$$p_3 - \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2 = \frac{\delta_3 - \delta_2(p_1^2 - \delta_1)}{p_3 + \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2} = O(|t|^{2\deg p_1 - \deg p_3}),$$

hence

$$p_3' - \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2' = (p_3 - \varepsilon\sqrt{p_1^2 - \delta_1}\,p_2)\left(\frac{p_1 + \sqrt{p_1^2 - \delta_1}}{\sqrt{\delta_1}}\right)^2 = \tag{10}$$

$$= O(|t|^{4\deg p_1 - \deg p_3})$$

and so

$$p_3' = O(|t|^{\max\{\deg p_3 - 2\deg p_1,\, 4\deg p_1 - \deg p_3\}}),$$

$$p_2' = O(|t|^{\max\{\deg p_3 - 3\deg p_1,\, 3\deg p_1 - \deg p_3\}}).$$

Since $1 \leqslant \deg p_1 < \deg p_2 = \deg p_3 - \deg p_1$ we obtain

$$\max\{\deg p_1, \deg p_2', \deg p_3'\} < n.$$

On the other hand, $p_1', p_2', p_3' \in k[t]$ and satisfy (1), hence by the inductive assumption $\langle p_1, p_2', p_3' \rangle \in S_0$. Now however

$$(11)\quad p_3 - \varepsilon\sqrt{p_1^2 - \delta_1}\, p_2 = (p_3' + \varepsilon\sqrt{p_1^2 - \delta_1}\, p_2')\left(\frac{p_1 + \sqrt{p_1^2 - \delta_1}}{\sqrt{\delta_1}}\right)^2$$

and so

$$p_2 = \frac{2p_1^2 p_2' + 2\varepsilon p_1 p_3' - \delta_1 p_2'}{\delta_1}, \quad p_3 = \frac{2p_1^2 p_3' + 2\varepsilon p_1 p_3'(p_1^2 - \delta_1) - \delta_1 p_3'}{\delta_1},$$

hence by (4*a*) $\langle p_1, p_2, p_3 \rangle \in S_0$.

By symmetry between (4*a*) and (4*b*) the same holds if

$$\deg p_2 < \deg p_1 .$$

In order to prove the last assertion of the theorem we observe that for $i = 1, 2$,

$$(12)\quad (p_i^2 - \delta_i)\left(\left(\frac{p_i p_{3-i} + \eta p_3}{\sqrt{\delta_i}}\right)^2 - \delta_{3-i}\right) - \left(\frac{p_i p_3 + \eta p_{3-i}(p_i^2 - \delta_i)}{\sqrt{\delta_i}}\right)^2 = (p_1^2 - \delta_1)(p_2^2 - \delta_2) - p_3^2$$

and

$$\begin{vmatrix} p_i & \eta \\ \eta(p_i^2 - \delta_i) & p_i \end{vmatrix} = \delta_i \neq 0,$$

which implies $S_1 \subset S$. The proof of the inclusion $S \subset S_1$ is similar to that of the inclusion $S \subset S_0$ with this difference that the case $\deg p_1 = \deg p_2$ is treated in the same way as $\deg p_1 < \deg p_2$ and in the formulae (9), (10) and (11) the exponent 2 is replaced by 1.

2. Proof of Theorem 2.

We shall deal with the equation

$$(13)\quad y_3^2 - \delta_3 = (y_1^2 - 4)(y_2^2 - 4)$$

where $\delta_3 = 4b^2$, $b \in \mathbb{Z}$.

We define τ as the minimal set of triples $\langle p_1, p_2, p_3 \rangle$ of polynomials in $\mathbb{Q}[x]$ with the following properties

(i) For every choice of $\varepsilon, \eta \in \{1, -1\}$ we have

(i*a*) $$\langle x, 2\varepsilon, 2b\eta\rangle \in \tau,$$

(i*b*) $$\langle 2\varepsilon, x, 2b\eta\rangle \in \tau.$$

(ii) For every choice of $\eta \in \{1, -1\}$, putting $\boldsymbol{p} = \langle p_1, p_2, p_3\rangle \in \tau$ both

(ii*a*) $$\varphi_{1,\eta}(\boldsymbol{p}) = \left\langle p_1, \frac{p_1p_2 + \eta p_3}{2}, \frac{p_1p_3 + \eta p_2(p_1^2 - 4)}{2}\right\rangle \in \tau$$

and

(ii*b*) $$\varphi_{2,\eta}(\boldsymbol{p}) = \left\langle \frac{p_1p_2 + \eta p_3}{2}, p_2, \frac{p_2p_3 + \eta p_1(p_2^2 - 4)}{2}\right\rangle \in \tau.$$

LEMMA 3. *If* $\langle p_1, p_2, p_3\rangle \in \tau$, *then*

(iii) $p_i \in \mathbb{Z}[x]$ *for* $i = 1, 2, 3$;

(iv) $\langle p_1, p_2, p_3\rangle$ *satisfies* (13);

(v) *for all* $\eta_1, \eta_2, \eta_3 \in \{1, -1\}$ *there exists* $\langle q_1, q_2, q_3\rangle \in \tau$ *such that* $\eta_i p_i(x) = q_i(\varepsilon x)$ *for* $i = 1, 2, 3$;

(vi) *if* $x_0 \in \mathbb{C}$ *is such that* $p_i(x_0) \in \mathbb{Z}$, $i = 1, 2, 3$, *then* $x_0 \in \mathbb{Z}$.

PROOF. Since $\langle p_1, p_2, p_3\rangle$ is obtained by a finite iteration of operations of type $\varphi_{j,\eta}$ starting from either (i*a*) or (i*b*), and since (i*a*) and (i*b*) satisfy the four assertions of the Lemma, it suffices to prove that if some triple $\boldsymbol{p}$ satisfies the assertions, then $\varphi_{j,\eta}(\boldsymbol{p})$ does.

Now, since $\boldsymbol{p}$ satisfies both (iii) and (iv) by assumption, we have $p_3 \equiv p_1p_2 \pmod{2\mathbb{Z}[x]}$ whence (iii) holds for $\varphi_{j,\eta}(\boldsymbol{p})$. Also, (iv) follows from (12). (v) follows from the identity

$$\left\langle \eta_1 p_1, \eta_2 \frac{p_1p_2 + \eta p_3}{2}, \eta_3 \frac{p_1p_3 + \eta p_2(p_1^2 - 4)}{2}\right\rangle =$$

$$= \varphi_{1,\eta_1\eta_2\eta_3\eta}(\eta_1p_1, \eta_1\eta_2p_2, \eta_1\eta_3p_3).$$

As to (vi) we observe that if the components of $\varphi_{j,\eta}(\boldsymbol{p})(x_0)$ belong to $\mathbb{Z}$, the same is true for the components of $\varphi_{j,-\eta}\circ\varphi_{j,\eta}(\boldsymbol{p})(x_0)$, by the same argument which proved $p_i \in \mathbb{Z}[x]$. But $\varphi_{j,-\eta}\circ\varphi_{j,\eta}$ is the identity, and induction applies. ■

LEMMA 4. *Let* $\langle s_1, s_2, s_3\rangle$ *be a solution of* (13) *in polynomials*

$s_i \in \mathbb{Q}[t]$ not all constant. Then there exists a polynomial $p \in \mathbb{Q}[t]$ and a triple $\langle p_1, p_2, p_3\rangle \in \tau$ such that $s_i(t) = p_i(p(t))$ $i = 1, 2, 3$.

This follows from the last assertion of Th. 1 on noticing that $b^2 + 4$ is not a square, for $b \neq 0$. ■

The proof of the last assertion of Th. 1 also shows.

LEMMA 5. *Each triple $\boldsymbol{p} = \langle p_1, p_2, p_3\rangle \in \tau$ may be obtained starting from some triple $\boldsymbol{p}_0 \in \tau$ such that the maximum degree of the components of $\boldsymbol{p}_0$ is 1, and applying successively operations of type $\varphi_{j,\eta}$ in such a way as to increase strictly the maximum degree at each step.*

Define $|\boldsymbol{p}| = \max\deg p_i$.

LEMMA 6. *If $\boldsymbol{p} \in \tau$ and $|\boldsymbol{p}| = 1$, then, either*

$$\boldsymbol{p} = \langle \eta_1(x+a), 2\eta_2, 2b\eta_3\rangle$$

or

$$\boldsymbol{p} = (2\eta_1, \eta_2(x+a), 2b\eta_3\rangle$$

for some $a \in \mathbb{Z}$ and some $\eta_1, \eta_2, \eta_3 \in \{1, -1\}$.

PROOF. Let $\boldsymbol{p} = (p_1, p_2, p_3)$. By (13) either $p_2 \in \mathbb{Q}$ or $p_1 \in \mathbb{Q}$ and by Lemma 1 either $\boldsymbol{p} = \langle p(x), 2\eta_2, 2b\eta_3\rangle$ or $\boldsymbol{p} = \langle 2\eta, p(x), 2b\eta_3\rangle$ for some $p \in \mathbb{Z}[x]$, $\deg p = 1$. Now, for instance by (vi) of Lemma 3 the leading coefficient of p must be ± 1. (Alternatively one may show by induction that leading coefficients of nonconstant polynomials appearing in some triple in τ are ± 1). ■

LEMMA 7. *Let $\boldsymbol{p} = \langle p_1, p_2, p_3\rangle \in \tau$, $|\boldsymbol{p}| \geqslant 2$, $d_1 = \deg p_1 \leqslant d_2 = \deg p_2$, and assume $\boldsymbol{p}' = \varphi_{j,\eta}(\boldsymbol{p}) = \langle p_1', p_2', p_3'\rangle$ is such that $|\boldsymbol{p}'| > |\boldsymbol{p}|$, $d_1' = \deg p_1' \leqslant d_2' = \deg p_2'$. Then*

$$\langle d_1', d_2'\rangle = \begin{cases} \langle d_1, d_1 + d_2\rangle & \text{or} \\ \langle d_2 - d_1, d_2\rangle & \text{or} \\ \langle d_2, d_1 + d_2\rangle \end{cases}$$

where the second possibility may happen only if $d_2 > 2d_1$. (Observe that $|\boldsymbol{p}| \geqslant 2$ implies that min $\deg p_i \geqslant 1$.)

PROOF. From (13) one gets $p_3^2 - p_1^2 p_2^2 = \delta_3 - 4(p_1^2 + p_2^2) + 16$, whence

$$\deg(p_1 p_2 + p_3) + \deg(p_1 p_2 - p_3) = 2d_2 ,$$

say that $\deg(p_1 p_2 + p_3) = \deg p_3 = d_1 + d_2$ (the argument being symmetrical if $\deg(p_1 p_2 - p_3) = \deg p_3$). So $\deg(p_1 p_2 - p_3) = d_2 - d_1$.

If $j = 1$, since $|\boldsymbol{p}'| > |\boldsymbol{p}|$ we must have $\eta = 1$, and we fall in the first case. If $j = 2$ we may have either $\eta = 1$ falling in the third case, or $\eta = -1$, provided $d_2 - d_1 > d_1$, and we fall in the second case. ■

Let now $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \tau$. We define $\boldsymbol{p}_1 \sim \boldsymbol{p}_2$ if $\boldsymbol{p}_2(x) = \boldsymbol{p}_1(\eta(x + a))$ for some $a \in \mathbb{Z}$, $\eta \in \{1, -1\}$. Clearly this is an equivalence relation which preserves the max deg function, so we may define $\mathcal{N}(D)$ to be the number of equivalence classes of triples $\boldsymbol{p}$ in τ such that $|\boldsymbol{p}| \leqslant D$.

LEMMA 8. *$\mathcal{N}(D) \leqslant D^4$, for all $D \geqslant 2$.*

PROOF. By Lemma 5 we obtain each triple in τ starting from the equivalence class of either (i*a*) or (i*b*) and applying operators $\varphi_{j,\eta}$ to increase the degree at each step (observe $\varphi_{j,\eta}$ preserves the equivalence). After the first step we obtain an equivalence class of one of the eight triples given by (6*b*) with p replaced by x, i.e. $\langle x, \varepsilon_1(x + \eta b), \varepsilon_2(x^2 + \eta bx - 4)\rangle$, where $\varepsilon_1, \varepsilon_2, \eta \in \{1, -1\}$. Define for pairs $\langle d_1, d_2\rangle$ of integers with $1 \leqslant d_1 \leqslant d_2$ three operations

$$\alpha(d_1, d_2) = \langle d_1, d_1 + d_2\rangle,$$

$$\beta(d_1, d_2) = \langle d_2, d_1 + d_2\rangle,$$

$$\gamma(d_1, d_2) = \langle d_2 - d_1, d_2\rangle,$$

where γ is defined only if $d_2 > 2d_1$. In view of Lemma 7 we have $\mathcal{N}(D) \leqslant 8C(D) + 8$ where $C(D)$ is the number of sequences $\delta_1, \ldots, \delta_k$ such that each δ_i is either α, or β, or γ and the sum of the components of $\delta_k \circ \ldots \circ \delta_1\langle 1, 1\rangle$ is bounded by D. If every δ_i is of type α we have at most $D - 1$ such sequences. Otherwise let r be the greatest index such that δ_r is either β or γ, so δ_μ is α for $r < \mu \leqslant k$.

Put $\delta_r \circ \ldots \circ \delta_1\langle 1, 1\rangle = \langle m, n\rangle$. Then, setting $k = r + s$ we have $n + (s + 1)m \leqslant D$. Also $\langle m, n\rangle$ is in the image of either β of γ so $m \geqslant n/2$, whence

$$n + m \leqslant \frac{3}{s+3}(n + (s + 1)m) \leqslant \frac{3}{s+3} D . \tag{14}$$

Assume $\delta_r = \beta$. Then, if $\langle m', n'\rangle = \delta_{r-1} \circ \ldots \circ \delta_1\langle 1, 1\rangle$ we have

$\beta(m', n') = \langle m, n\rangle$, whence

$$m' + n' \leqslant \frac{2}{3}(m+n). \tag{15}$$

If on the other hand $\delta_r = \gamma$, necessarily $\delta_{r-1} = \alpha$. Observing that $\gamma \circ \alpha = \beta$, and setting $\langle m', n'\rangle = \delta_{r-2} \circ \ldots \circ \delta_1 \langle 1, 1\rangle$ again we have $m' + n' \leqslant (2/3)(m+n)$. In conclusion we may write

$$C(D) + 1 \leqslant D + 2 \sum_{s=0}^{\infty} C\left(\frac{2D}{s+3}\right). \tag{16}$$

Clearly, $C(2) = 1$. Assume that $C(y) + 1 \leqslant (1/8)y^4$ for $2 \leqslant y \leqslant D-1$, $D \geqslant 3$. Then (16) shows

$$C(D) + 1 \leqslant D + \frac{1}{4}\left(D^4 \sum_{s=0}^{\infty}\left(\frac{2}{s+3}\right)^4\right) < D + \frac{1}{12} D^4 < \frac{1}{8} D^4 .$$

The inequality $C(D) + 1 \leqslant D^4$ holds, by induction, for all integers $D \geqslant 2$. ■

REMARK 1. The inequality $\mathcal{N}(D) \ll D^B$ (B constant) may be proved also by the (essentially equivalent) method of proof of Th. 4 in [1], cf. Remark 7. Perhaps the present method is slightly simpler.

DEFINITION 1. We say that a solution of (13) $\langle y_1, y_2, y_3\rangle$ in integers y_1, y_2, y_3 is polynomial if there is a polynomial solution $\langle s_1, s_2, s_3\rangle$ of (13) with $s_i \in \mathbb{Q}[t]$, not all constant, and a complex number t_0 such that

$$y_i = s_i(t_0) \quad i = 1, 2, 3 .$$

By Lemma 4 we have $y_i = (p(t_0))$ for some triple $\langle p_1, p_2, p_3\rangle \in \tau$, $p \in \mathbb{Q}[t]$. By (vi) of Lemma 3 we have

$$y_i = p_i(x_0) \quad i = 1, 2, 3 \tag{17}$$

for some $x_0 \in \mathbb{Z}$.

DEFINITION 2. We define the degree $\partial(\boldsymbol{y})$ of a polynomial solution $\boldsymbol{y} = \langle y_1, y_2, y_3\rangle$ to be $\min_{\boldsymbol{p}} |\boldsymbol{p}|$ where $\boldsymbol{p} = \langle p_1, p_2, p_3\rangle$ is a triple in τ such that (17) holds for some $x_0 \in \mathbb{Z}$.

We choose now $C_1 = \max\{1/2\sqrt{\delta_3 + 16}, \exp 720\}$.

Let $\langle y_1, y_2, y_3\rangle$ be an integer solution of (13), where $0 \leqslant y_1 \leqslant$

$\leq y_2$, $0 \leq y_3$. If $y_1^2 < 4$ we have clearly a finite number of possibilities for y_2, y_3. If $y_1^2 > 4$ we set $w = y_1^2 - 4$, $\zeta = (y_1 + \sqrt{w})/2$, $\xi = y_3 + y_2\sqrt{w}$, $B = \delta_3 - 4w$, $C = |B|\,\zeta^{-1}$. There exists a unique $a \in \mathbb{Z}$ such that

$$C^{1/2} \leq \xi\zeta^a < C^{1/2}\zeta .$$

In view of the choice of C_1 such a satisfies

$$(18)\quad |a| \leq \frac{\log(y_3 + y_2\sqrt{w})}{\log\zeta} + \frac{1}{2} \leq \begin{cases} C_1 & \text{if } y_1 < C_1 , \\ 2\log y_3/\log y_1 & \text{if } y_1 \geq C_1 . \end{cases}$$

We set $y_3^* + y_2^*\sqrt{w} = \xi\zeta^a$ and observe that y_2^*, y_3^* are integers and that $\langle y_1, y_2^*, y_3^* \rangle$ is a solution of (13).

Moreover we have easily (cf. [1], formula (33))

$$(19)\qquad |y_2^*| \leq \frac{|B|^{1/2}\zeta^{1/2}}{\sqrt{w}} \leq \begin{cases} C_1 & \text{if } 3 \leq y_1 \leq C_1 , \\ 2y_1^{1/2} & \text{if } y_1 \geq C_1 ; \end{cases}$$

$$(20)\qquad |y_3^*| \leq \begin{cases} C_1 & \text{if } 3 \leq y_1 \leq C_1 , \\ 2y_1^{3/2} & \text{if } y_1 \geq C_1 . \end{cases}$$

We set

$$(21)\qquad \begin{cases} \varphi(y_1, y_2, y_3) = \begin{cases} \langle y_1, |y_2^*|, |y_3^*| \rangle, & \text{if } y_1 \leq |y_2^*| , \\ \langle |y_2^*|, y_1, |y_3^*| \rangle, & \text{if } y_1 > |y_2^*| , \end{cases} \\ a(y_1, y_2, y_3) = |a| . \end{cases}$$

LEMMA 9. *If the solution $\boldsymbol{y} = \langle y_1, y_2, y_3 \rangle$ is polynomial, the same is true of $\varphi(\boldsymbol{y})$, and conversely. Moreover if $\varphi(\boldsymbol{y})$ is of degree d, then $\partial(\boldsymbol{y}) \leq 2^{|a|}d$.*

The proof of the first statement is immediate. We prove the second statement by induction on $|a|$. If $a = 0$ we have $\varphi(\boldsymbol{y}) = \boldsymbol{y}$, hence the inequality $\partial(\boldsymbol{y}) \leq 2^{|a|}\partial(\varphi(\boldsymbol{y}))$ holds. Assume that it is true whenever $|a(\boldsymbol{y})| < n$ and that

$$\varphi(\boldsymbol{y}) = \begin{cases} \langle y_1, |y_2^0|, |y_3^0| \rangle & \text{or} \\ \langle |y_2^0|, y_1, |y_3^0| \rangle, \end{cases}$$

where

$$y_3^* + y_2^* \sqrt{w} = \xi\zeta^a , \qquad |a| = n .$$

Put

$$\boldsymbol{y}' = \langle y_1, |y_2'|, |y_3'|\rangle, \qquad \text{where } y_3' + y_2' \sqrt{w} = \xi\zeta^{\operatorname{sgn} a} .$$

By the inductive assumption

$$\partial(\boldsymbol{y}') \leqslant 2^{n-1} \partial(\varphi(\boldsymbol{y})) . \tag{22}$$

By the definition of $\partial(\boldsymbol{y}')$ and by Lemma 3 (vi) there exists a triple $\boldsymbol{p}' = \langle p_1', p_2', p_3'\rangle \in \tau$ and an integer $x_0 \in \mathbb{Z}$ such that

$$y_1 = p_1'(x_0), \qquad y_i' = p_i'(x_0) \quad (i = 2, 3),$$

and $|\boldsymbol{p}'| = \partial \boldsymbol{y}'$. However

$$y_3 + y_2 \sqrt{w} = \xi = (y_3' + y_2' \sqrt{w})\, \xi^{-\operatorname{sgn} a}$$

hence

$$\langle y_1, y_2, y_3\rangle = \varphi_{1, -\operatorname{sgn} a}(\boldsymbol{p}')(x_0)$$

and

$$\partial(\boldsymbol{y}) \leqslant |\varphi_{1, -\operatorname{sgn} a}(\boldsymbol{p}')| \leqslant \deg p_1' + \deg p_3' \leqslant 2\, |\boldsymbol{p}'| . \tag{23}$$

The inequalities (14) and (15) imply the desired inequality

$$\partial(\boldsymbol{p}) \leqslant 2^n \partial(\varphi(\boldsymbol{p})) . \qquad \blacksquare$$

We now define the following procedure, applied to any solution of (13) in natural numbers y_1, y_2, y_3; namely we apply several times the function φ until we reach a solution z_1, z_2, z_3 such that $z_i \geqslant 0$ $i = 1, 2, 3$, $z_1 \leqslant z_2$, $z_1 \leqslant C_1$. By (19) this will happen sooner or later. We have $\boldsymbol{z} = \langle z_1, z_2, z_3\rangle = \varphi^m(y_1, y_2, y_3)$ (possibly $m = 0$). Three cases may occur.

A) $z_1^2 = 4$: now the solution $\boldsymbol{z}$ is clearly polynomial and $\partial(\boldsymbol{z}) = 1$.

B) $z_1^2 < 4$: as observed before we have $z_2 \leqslant C_1$, $z_3 \leqslant C_1$.

C) $3 \leqslant z_1 \leqslant C_1$. Now a new application of φ produces a solution with components bounded by C_1, by (19), (20).

We set

$$\boldsymbol{\rho}(y_1, y_2, y_3) = \begin{cases} \boldsymbol{z} & \text{if we are in case } B), \\ \varphi(\boldsymbol{z}) & \text{if we are in case } C). \end{cases}$$

We first deal with case A):

By Lemma 9 the solution $\langle y_1, y_2, y_3 \rangle$ is polynomial of degree $\leqslant 2^{|a_1| + \dots + |a_m|}\,\partial(\boldsymbol{z}) = 2^{|a_1| + \dots + |a_m|}$. To estimate such degree we define $\Pi(Y) = \max\{|a_1| + \dots + |a_m|\}$ where the maximum is taken over all solutions with $y_3 \leqslant Y$.

If $y_1 < C_1$ we have $m = 0$ and no a_i appears. Thus if $Y < C_1$ we have $\Pi(Y) = 0$. If $y_1 \geqslant C_1$, then, by (18), $|a_1| \leqslant 2(\log Y/\log y_1)$, whence by (22)

$$\Pi(Y) \leqslant 2\,\frac{\log Y}{\log y_1} + \Pi(y_1^{3/2}).$$

Since $y_1 \leqslant y_2$ we have $y_3^2 \geqslant (y_1^2 - 4)^2$, whence $y_1 \leqslant \sqrt{Y + 4}$, so

$$\Pi(Y) \leqslant 2\,\frac{\log Y}{\log C_1} + \Pi((Y+4)^{3/4}) \leqslant 2\,\frac{\log Y}{\log C_1} + \Pi(Y^{4/5}).$$

By an easy induction we get

$$\Pi(Y) \leqslant 10\,\frac{\log Y}{\log C_1} \leqslant \frac{\log Y}{72} \tag{24}$$

for all integers $y \geqslant 1$.

LEMMA 10. *Let $\boldsymbol{s} = \langle s_1, s_2, s_3 \rangle$ be a solution of* (13) *in natural numbers s_i. Then the number $H(Y)$ of solutions $\langle y_1, y_2, y_3 \rangle$, $y_i \geqslant 0$, $y_3 \leqslant Y$ such that the solution $\boldsymbol{\rho}(y_1, y_2, y_3)$ introduced above coincides with $\boldsymbol{s}$ satisfies*

$$H(Y) \ll_{\delta_3} Y^{1/3}.$$

(Clearly we assume that $\langle y_1, y_2, y_3 \rangle$ falls in case B) or C))

PROOF. Let $\boldsymbol{y} = \langle y_1, y_2, y_3 \rangle$ be counted in $H(Y)$. If $y_1 \leqslant C_1$ then either $\boldsymbol{y} = \boldsymbol{s}$ or $\varphi(\boldsymbol{y}) = \boldsymbol{s}$. If $y_1 > C_1$ then, by (20), $\varphi(\boldsymbol{y})$ is counted in $H((Y+4)^{3/4})$.

On the other hand the number of solutions $\langle y_1, y_2, y_3 \rangle$, $y_3 \leqslant Y$, such that $\varphi(y_1, y_2, y_3)$, is a given solution is easily seen to be bounded by $C_2 \log Y$, $C_2 = C_2(\delta_3)$ (cf. [1], formula (30)) whence

$$H(Y) \leqslant C_2 \log Y \cdot \{H((Y+4)^{3/4}) + 1\}.$$

Iterating this inequality we get the Lemma, and even

$$H(Y) \ll_{\delta_3} \exp\left(3(\log\log Y)^2\right). \quad \blacksquare$$

REMARK 2. Imitating the proof of Theorem 4 in [1] one can prove $H(Y) \ll \log^c Y$.

LEMMA 11. *Let $p \in \mathbb{R}[x]$ be a polynomial of degree $n \geqslant 1$ with the leading coefficient at least 1 in absolute value, and let I be an interval of length T. Then*

$$\#(p(\mathbb{Z}) \cap I) \leqslant \#\{x \in \mathbb{Z} \mid p(x) \in I\} \leqslant nT^{1/n} + n.$$

PROOF. The first inequality is trivial. Let us prove the second by induction on n, the case $n = 1$ being immediate. Set $x_1 = \min\{x \in \mathbb{Z} \mid p(x) \in I\}$ and put

$$p_1(x) = \frac{p(x + x_1) - p(x_1)}{x}.$$

Observe that

$$-x_0 \in \mathbb{Z},\ p(x_0) \in I \Leftrightarrow x_0 \geqslant x_1,\ p(x_0) - p(x_1) \in I - p(x_1)$$

$$\Leftrightarrow (x_0 - x_1)\, p_1(x_0 - x_1) \in I - p(x_1) = I_1,$$

say. I_1 is an interval of length T containing 0. So

$$\#\{x \in \mathbb{Z} \mid p(x) \in I\} \leqslant \#\{x \in \mathbb{N} \mid x p_1(x) \in I_1\} \leqslant$$

$$\leqslant \#\{z \in \mathbb{N} \mid z \leqslant T^{1/n}\} + \#\{x \in \mathbb{N} \mid x > T^{1/n},\ x p_i(x) \in I_1\} \leqslant$$

$$\leqslant T^{1/n} + 1 + \#\left\{z \in \mathbb{N} \mid p_1(z) \in \frac{1}{T^{1/n}} I_1\right\}$$

since I_1 contains 0. By induction the last set contains $\leqslant (n-1)\cdot(T^{1-1/n})^{1/n-1} + n - 1 = (n-1)\,T^{1/n} + n - 1$ elements, whence the Lemma follows. $\blacksquare$

REMARK 3. Dr. F. Amoroso has observed that the lemma can be improved by using the properties of the transfinite diameter.

PROOF OF THEOREM 2. Define now $\mathscr{P}_\mu(Y)$ as the number of natural numbers $y_3 \leqslant Y$ such that there exist y_1, y_2 such that y_1, y_2, y_3 is a polynomial solution of degree μ.

Then $\mathscr{P}_\mu(Y)$ is bounded by the cardinality of the union of sets of type $y_3(\mathbb{Z}) \cap \{0, 1, \ldots, Y\}$, where y_3 runs over the third components of triples in τ of degree $= \mu$. But we may clearly take one triple only from each equivalence class.

By Lemma 11 each such set has $\leqslant \mu Y^{1/\mu} + \mu \leqslant 2\mu Y^{1/\mu}$ elements, and, combining this with Lemma 8 we get

$$\# \mathscr{P}_\mu(Y) \leqslant 2\mu^2 Y^{1/\nu}. \tag{25}$$

When $\mu = 2$ we directly see that $y_3(x)$ may be taken as $\pm(x^2 + \eta bx - 4)$ where $\eta \in \{1, -1\}$. The set $y_3(\mathbb{Z}) \cap \{0, \ldots, Y\}$ contains either $O(1)$ elements or $\sqrt{Y} + O(1)$ elements, whence

$$\# \mathscr{P}_2(Y) = \sqrt{Y} + O(1) \tag{26}$$

(In fact $(x-b)^2 + b(x-b) - 4 = x^2 - bx - 4$.)

By the observations following Lemma 11 and by (24) we see that each solution $\langle y_1, y_2, y_3 \rangle$ is either polynomial of degree $\leqslant 2^{\log Y/72}$ or we fall in case B) or C), where Lemma 10 applies. Thus, since $2^{\log Y/72} \leqslant$ $\leqslant Y^{1/72}$, we get, putting everything together

$$\# \{ y_3 \mid 0 \leqslant y_3 \leqslant Y,\ y_3^2 - \delta_3 = (y_1^2 - 4)(y_2^2 - 4) \text{ for some } y_1, y_2 \in \mathbb{Z}\} =$$

$$= \sqrt{Y} + O\Bigg(\sum_{2 \leqslant \mu \leqslant Y^{1/72}} \mu^5 Y^{1/\mu}\Bigg) + O(H(Y)) =$$

$$= O(Y^{1/3}) + O\Bigg(Y^{1/4} \sum_{\mu \leqslant Y^{1/72}} \mu^5\Bigg) = \sqrt{Y} + O(Y^{1/3}).$$

The equation $f_1(x_1) f_2(x_2) = f_2(x_3)$ under the assumption of the theorem reduces to (13) by a linear substitution. The condition $|x_3| \leqslant x$ becomes $|y_3| \leqslant x + O(1)$ and the theorem follows from (27) on setting $Y = x + O(1)$ and allowing y_3 both positive and negative. ■

Note added in proof.

K. Kashihara in his paper *Explicit complete solution in integers of a class of equations* $(ax^2 - b)(ay^2 - b) = z^2 - c$, Manuscripta Math., **80** (1993), pp. 373-392, gives a method to find all integer solutions of the equation in the title for $a \neq 0$, c and $b = \pm 1, \pm 2, \pm 4$. However Kashihara does not give any asymptotic formulae.

Paper [1] requires a small correction at page 62, lines 3, 5; namely in the expressions under square root on the left, a minus sign must be inserted before $\Delta_3/4$. Also, we point out the following paper related to [1]: S. D. COHEN, P. ERDÖS, M. B. NATHANSON, *Prime Polynomial sequences*, J. London Math. Soc. (2), **14** (1976), pp. 559-562.

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Manoscritto pervenuto in redazione il 23 novembre 1992.