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Introduction.

Let G be a Chevalley-Demazure group scheme. Chevalley and Demazure have showed in [5] and [6] that, as a representable covariant functor from the category of commutative rings with unity to the category of groups, G is uniquely determined by the semisimple complex Lie group $G(\mathbb{C})$. More precisely, they demonstrated that a Chevalley-Demazure group scheme G' is isomorphic to G if and only if, as semisimple Lie groups, $G'(\mathbb{C})$ and $G(\mathbb{C})$ are isomorphic to each other. The purpose of this paper is to generalize their result for simple Chevalley-Demazure group schemes, as well as for absolutely almost simple algebraic groups, by replacing the complex field $\mathbb{C}$ by an integral domain.

Suppose G and G' are simple Chevalley-Demazure group schemes. In general for a commutative integral domain R, the existence of an isomorphism between the groups $G(R)$ and $G'(R)$, which are Chevalley groups over R, does not necessarily imply an isomorphism between G and G'. A counter example is that there exists an isomorphism between the special linear group $SL_3(k)$ and the projective linear group $PGL_3(k)$, where k is a perfect field of characteristic 3, while SL_3 is not isomorphic to PGL_3 as Chevalley-Demazure group schemes. However, we find that when R contains an infinite field and when both G and G' are either simply connected or adjoint the existence of an isomorphism between $G(R)$ and $G'(R)$ does imply an isomorphism between the group schemes G and G' except in one special case (see Theorem 1.3). We also show in this paper that, if R and R' are integral domains containing infinite fields, then $G(R)$ being isomorphic to $G'(R')$ implies

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that G and G' belong to the same isogeny class except in one special case (see Theorem 1.2). For proving these results, we apply mainly the Borel-Tits theorem (see [3, §8.1]) on abstract homomorphisms of simple algebraic groups and our result is in fact a consequence of their work [3].

1. Main results.

1.1. For a field k, we fix an extension K of k, which is algebraically closed and plays a role of universal domain, i.e. all extensions of k dealt with in this paper are contained in K. The fields appearing in this paper are always infinite. Let G be a simple Chevalley-Demazure group scheme, then the group $G(K)$ is an absolutely almost simple algebraic group over K and splits over the prime field of K. In general we also denote by G an almost simple algebraic group over K for convenience without any confusion.

1.2. THEOREM. *Let G and G' be simple Chevalley-Demazure group schemes. If there exist integral domains R and R', which contain infinite fields, such that $G(R)$ is isomorphic to $G'(R')$, then G and G' have isomorphic root systems except when the characteristic of R is equal to 2 and G is of type B_n or C_n. In this exceptional case, the root system of G' is either isomorphic or dual (i.e. C_n or B_n) to that of G.*

1.3. THEOREM. *Let G and G' be absolutely almost simple algebraic groups defined over k. If both G and G' are either simply connected or adjoint, then G and G' are isomorphic to each other as algebraic groups if and only if there exists a k-subalgebra R of K such that $G(R)$ is isomorphic to $G'(R)$, except when the characteristic of K is equal to 2 and G is of type B_n or C_n. In this exceptional case, $G(R)$ being isomorphic to $G'(R)$ implies that either G is isomorphic to G' or G' is of the dual type of G.*

2. Proof of the theorems.

2.1. Let G be an absolutely almost simple algebraic group over K and let T be a maximal torus of G. Denote by $X^*(T)$ the character group $\mathrm{Hom}(T, K^*)$ of T, where K^* is the multiplication group of K. The root system of G related to T is denoted by Φ. Suppose G' is another absolutely almost simple algebraic group over K. If there is an isogeny β from G to G', then the image T' of T under β is a maximal torus of G'.

We denote by $X^*(T')$ and Φ' the character group of T' and the root system of G' related to T' respectively. It is easily seen that β induces a monomorphism $\beta^*: X^*(T) \to X^*(T')$ defined by

$$\beta^*(\chi)(t) = \chi(\beta(t)), \qquad \forall \chi \in X^*(T'), \qquad \forall t \in T.$$

It follows from [4] that there exist a bijection $\rho: \Phi \to \Phi'$ and a map

$$\lambda: \Phi \to \{p^n \mid n \in \mathbb{Z}, n \geqslant 0\},$$

where $p = 1$ if the characteristic of K is zero and otherwise p is equal to the characteristic of K, such that

$$\beta^*(\rho(a)) = \lambda(a) \cdot a, \qquad \forall a \in \Phi. \tag{2.1.1}$$

Recall that an isogeny β is called *special* if $1 \in \lambda(\Phi)$ (cf.[3, §3]), we have the following lemma.

2.2. LEMMA. *Suppose G and G' are absolutely almost simple algebraic groups over K. If there exists a special isogeny between G and G', then either G and G' have isomorphic root systems, or the characteristic of K is equal to 2 and G is of type B_n or C_n while G' is of the dual type of C_n or B_n.*

PROOF. See [3, §3] and [4].

2.3. Let G be an absolutely almost simple algebraic group defined over k. We denote by $E(k)$ the subgroup of G generated by all k-rational points in unipotent radicals of parabolic subgroups of G defined over k. Let k' be another field and K' be a universal domain of k'. Suppose G' is an almost simple algebraic group over K'. We have the following corollary of the Borel-Tis theorem [3, §8.1].

COROLLARY. *If there exists a homomorphism from $E(k)$ to G' with Zariski dense image, then G and G' have isomorphic root systems except when the characteristic of K is 2 and G is of type B_n or C_n. In this exceptional case, the root systems of G and G' are either isomorphic or dual to each other.*

PROOF. Let $\alpha: E(k) \to G'$ be a homomorphism with Zariski dense image. Consider the following k-universal covering of G

$$\varepsilon: G^{sc} \to G,$$

where G^{sc} is a simply connected algebraic group defined over k and is of the same type as G. We have by [3, §6.3]

$$\varepsilon(E^{sc}(k)) = E(k).$$

Hence the composition of homomorphisms $\alpha\varepsilon$: $E^{sc}(k) \to G'$ is also a homomorphism with Zariski dense image. Therefore, it follows from the Borel-Tits theorem [3, §8.1], there exist a homomorphism φ from k into K' and a special isogeny β from ${}^{\varphi}G^{sc}$, the group obtained by changing the base field of G^{sc} through φ [3, §1.7], onto G' such that the following diagram is commutative

$$\begin{array}{ccc} E^{sc}(k) & \xrightarrow{\varepsilon} & E(k) \\ \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \alpha} \\ {}^{\varphi}G^{sc} & \xrightarrow{\beta} & G' \end{array}$$

where φ^0 is the canonical homomorphism induced by φ. Note that ${}^{\varphi}G^{sc}$, G^{sc} and G have isomorphic root systems and that, since the isogeny β is special, ${}^{\varphi}G^{sc}$ and G' have isomorphic root systems except in one particular case as in Lemma 2.2. This implies immediately our lemma.

2.4. Corollary. *With notations as in Corollary* 2.3, *we have*

$$\dim G = \dim G'.$$

Proof. This comes directly from Corollary 2.3.

2.5. Proof of Theorem 1.2. We may assume that

$$k \subseteq R \subseteq K, \qquad k' \subseteq R' \subseteq K',$$

where k and k' are subfields of R and R' respectively. Consider $G(R)$ as R-rational points of the almost simple algebraic group G and $G'(R')$ as R'-rational points of G'. Suppose α: $G(R) \to G'(R')$ is an isomorphism. Let $\overline{\alpha(E(k))}$ be the Zariski closure of $\alpha(E(k))$ in G'. We show in the following that

$$\overline{\alpha(E(k))} = G', \tag{2.5.1}$$

which means that the restriction of α to $E(k)$ is a homomorphism with Zariski dense image. Hence our theorem comes directly from this and Corollary 2.3.

We claim first that $\overline{\alpha(E(k))}$ is a connected subgroup. In fact, denoting by $\overline{\alpha(E(k))}^0$ the connected component of $\overline{\alpha(E(k))}$ which contains the identity element, we have a composition of homomorphisms

$$E(k) \xrightarrow{\alpha} \overline{\alpha(E(k))} \xrightarrow{\delta} \overline{\alpha(E(k))} / \overline{\alpha(E(k))}^0$$

where δ is the natural homomorphism. Since $\overline{\alpha(E(k))} / \overline{\alpha(E(k))}^0$ is a finite group, it follows that

$$|E(k)/\ker \delta\alpha E(k)| < \infty .$$

This implies that, since $E(k)$ does not contain any proper normal subgroup of finite index [3, §6.7],

$$E(k) \subseteq \ker \delta\alpha .$$

Thus we have

$$\alpha(E(k)) \subseteq \overline{\alpha(E(k))}^0 \subseteq \overline{\alpha(E(k))} .$$

Taking the Zariski closures of above three groups simultaneously, we obtain immediately

$$\overline{\alpha(E(k))}^0 = \overline{\alpha(E(k))} .$$

We show secondly that

(2.5.2) $$\dim \overline{\alpha(E(k))} = \dim G' .$$

we denote by $[\overline{\alpha(E(k))}, \overline{\alpha(E(k))}]$ the commutative subgroup of $\overline{\alpha(E(k))}$ and by $\overline{[\alpha(E(k)), \alpha(E(k))]}$ the Zariski closure of the commutator subgroup of $\alpha(E(k))$. Since $E(k)$ is equal to its commutator subgroup [3, 6.4], by [2, Ch. I, §2.1] we have

$$[\overline{\alpha(E(k))}, \overline{\alpha(E(k))}] = \overline{[\alpha(E(k)), \alpha(E(k))]} = \overline{\alpha(E(k))} .$$

In particular, $\overline{\alpha(E(k))}$ is not a solvable group. Let $\mathfrak{R}$ be the solvable radical of $\overline{\alpha(E(k))}$. Then the quotient group $\overline{\alpha(E(k))} / \mathfrak{R}$ is a non-trivial semisimple algebraic group. Hence there exist non-trivial simple algebraic groups, say $G_1, G_2, \ldots, G_r$, of adjoint type and an isogeny (see [4])

$$\varepsilon : \overline{\alpha(E(k))} / \mathfrak{R} \to \prod_{i=1}^{r} G_i .$$

Let π be the natural homomorphism from $\overline{\alpha(E(k))}$ to its quotient group $\overline{\alpha(E(k))} / \mathfrak{R}$ and let $p_j : \prod_{i=1}^{r} G_i \to G_j$ be the canonical projection of $\prod_{i=1}^{r} G_i$

onto the j-th factor for $1 \leqslant j \leqslant r$. Note that $p_j \pi$ and δ send Zariski dense sets onto Zariski dense sets, so does the morphism $p_j \pi \delta$. In particular, we have for $1 \leqslant j \leqslant r$

$$\overline{p_j \pi \delta \alpha(E(k))} = p_j \pi \delta \, \overline{\alpha(E(k))} = G_j \,, \tag{2.5.3}$$

which means that $p_j \pi \delta \alpha$ is a homomorphism from $E(k)$ to G_j with Zariski dense image. It follows from Corollary 2.4 that

$$\dim G = \dim G_j \,.$$

Note that we also have from (2.5.3)

$$\dim \overline{\alpha(E(k))} \geqslant \dim G_j \,.$$

Thus we obtain, since $\overline{\alpha(E(k))}$ is a closed subgroup of G',

$$\dim \mathrm{G} \leqslant \dim \overline{\alpha(E(k))} \leqslant \dim G' \,. \tag{2.5.4}$$

Taking $G'(k')$, G' and α^{-1} in places of $E(k)$, G and α respectively, and vice-versa, and following a similar argument as above, we obtain on the other hand

$$\dim G' \leqslant \dim G \,.$$

This, together with (2.5.4), implies (2.5.2).

Finally, since $\overline{\alpha(E(k))}$ is a connected subgroup of G', the identity (2.5.2) gives rise to (2.5.1) immediately.

2.6. Proof of Theorem 1.3. By the classification theorem (cf. [4]) of almost simple algebraic groups, it is sufficient to show that G and G' have isomorphic root systems since both groups are either simply connected or adjoint, with an exceptional case as in our theorem. The existence of an isomorphism between the root systems of G and G', however, comes by following a similar proof as that of Theorem 1.2 and by using Lemma 2.3.

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