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Wiener Tauberian Theorems for Ultradistributions.

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SUMMARY - The purpose of this paper is the extension of Wiener Tauberian theorems for distributions ([6]) on ultradistribution spaces. Because of that, we give the versions of Beurling's and Wiener's theorems for bounded ultradistributions. The corollary of our main theorem is the following one. Let f be an ultradistribution such that f/c is a bounded ultradistribution, where c is a smooth function which behaves as $L(e^x)e^{\alpha x}$, $x \to \infty$, L is a slowly varying function at ∞ and $\alpha \in \boldsymbol{R}$. If for an ultradifferentiable function ϕ with the property $\mathcal{F}[\phi](\xi - i\alpha) \neq 0$, $\xi \in \boldsymbol{R}$,

$$\lim_{x \to \infty} \frac{(f * \phi)(x)}{L(e^x)e^{\alpha x}} = a \int \phi(t) e^{-\alpha t}\, dt, \qquad a \in \boldsymbol{R},$$

then for every ultradifferentiable function ψ

$$\frac{(f * \psi)(x)}{L(e^x)e^{\alpha x}} \to a \int \psi(t) e^{-\alpha t}\, dt, \qquad x \to \infty .$$

1. Notation and notions.

With $\boldsymbol{N}$ and $\boldsymbol{R}$ are denoted the sets of natural and real numbers; $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$. If f is a function on $\boldsymbol{R}$, then $\check{f}$ denotes the function defined by $\check{f}(x) = f(-x)$, $x \in \boldsymbol{R}$. $\mathcal{C}^\infty$ denotes the space of smooth functions on $\boldsymbol{R}$ and $\mathcal{L}^1$ is the space of Lebesgue integrable funstions (classes) on $\boldsymbol{R}$ with the usual norm $\| \|_{\mathcal{L}^1}$. For an f from $\mathcal{L}^1$ the Fourier transform is denoted by $\mathcal{F}f$ or $\widehat{f}$. $\mathcal{L}^1_{\text{loc}}$ and $\mathcal{L}^\infty$ are defined in a usual way.

We shall always denote by L a slowly varying function ([1]). Recall,

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L is measurable, positive and

$$\frac{L(xh)}{L(h)} \to 1, \qquad h \to \infty, \quad x > 0.$$

For every $\delta > 0$ there is $C_\delta > 0$ such that

$$(1) \qquad \frac{1}{C_\delta} \min\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\} \leqslant \frac{L(x)}{L(y)} \leqslant C_\delta \max\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\},$$
$$x > 0, \quad y > 0.$$

For the notation and properties of the spaces $\mathcal{D}^{(M_\alpha)}$, $\mathcal{D}^{\{M_\alpha\}}$ and their strong duals $\mathcal{D}'^{(M_\alpha)}$ (the space of Beurling ultradistributions) and $\mathcal{D}'^{\{M_\alpha\}}$ (the space of Roumieau ultradistributions) we refer to [3]. We shall assume that the sequence M_α satisfies conditions (M.1), (M.2) and (M.3)′ ([3]).

As in [5] we put

$$\mathcal{D}_{\mathcal{L}^1}^{(M_\alpha)} = \operatorname{proj}_{h\to\infty} \mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha}, \qquad \mathcal{D}_{\mathcal{L}^1}^{\{M_\alpha\}} = \operatorname{ind}_{h\to 0} \mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha},$$

where $\mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha}$, $h > 0$, is a Banach space of smooth functions φ on $\boldsymbol{R}$ with the finite norm

$$\|\varphi\|_{\mathcal{L}^1, h} = \sup\left\{\frac{h^\alpha}{M_\alpha} \|\phi^{(\alpha)}\|_{\mathcal{L}^1};\ \alpha \in \boldsymbol{N}_0\right\}.$$

The common notation for (M_α) and $\{M_\alpha\}$ will be $*$. The space $\mathcal{D}^*$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$ and the inclusion mapping is continuous. The strong dual of $\mathcal{D}^*_{\mathcal{L}^1}$ is denoted by $\mathcal{B}'^*$.

The spaces of tempered ultradistributions are defined as the strong duals of the following testing function spaces ([4])

$$\mathcal{S}^{(M_\alpha)} = \operatorname{proj}_{h\to\infty} \mathcal{S}_h^{M_\alpha}, \qquad \mathcal{S}^{\{M_\alpha\}} = \operatorname{ind}_{h\to 0} \mathcal{S}_h^{M_\alpha},$$

where $\mathcal{S}_h^{M_\alpha}$, $h > 0$, is the Banach space of smooth functions φ on $\boldsymbol{R}$ with the finite norm:

$$\gamma_h(\varphi) = \sup\left\{\frac{h^{\alpha+\beta}}{M_\alpha M_\beta} \|(1+x^2)^{\alpha/2} \varphi^{(\beta)}\|_{\mathcal{L}^\infty},\ \alpha, \beta \in \boldsymbol{N}_0\right\}.$$

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto $\mathcal{S}^*$ and $\mathcal{D}^*$ is dense in S^*. Clearly, $\mathcal{S}^*$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$ and the inclusion mapping is continuous.

Recall from [2] that a sequence of continuous and bounded functions

f_n on $\boldsymbol{R}$ converges narrowly to a continuous bounded function f_0 if and only if f_n converges to f_0 uniformly on bounded sets in $\boldsymbol{R}$ and

$$\|f_n\|_{\mathcal{L}^\infty} \to \|f_0\|_{\mathcal{L}^\infty}, \qquad n \to \infty .$$

We shall always assume that $\alpha, \beta \in \boldsymbol{R}$ and $\alpha > \beta$. Put

(2) $$c_0(x) = \begin{cases} L(e^x)e^{\alpha x}, & x \geqslant 0, \\ e^{\beta x}, & x < 0. \end{cases}$$

The following regularization of this function will be used.

(3) $$c(x) = (c_0 * \omega)(x), \qquad x \in \boldsymbol{R},$$

where $\omega \in \mathcal{D}^*$, $\operatorname{supp}\omega \subset [-1, 1]$, $\omega \geqslant 0$ and $\int_{\boldsymbol{R}} \omega(t)dt = 1$.

We shall denote by η a function from $\mathcal{E}^*$ with the properties

(4) $$\eta(x) = 1, \qquad x > x_0 > 0, \qquad \eta(x) = 0, \qquad x < -x_0 .$$

2. Assertions.

THEOREM 1. *Let $f \in \mathcal{B}'^*$ and $K \in \mathcal{D}^*_{\mathcal{L}^1}$ such that $\mathcal{F}[K](\xi) \neq 0$, $\xi \in \boldsymbol{R}$. If*

$$\lim_{x \to \infty} (f * K)(x) = a \int_{\boldsymbol{R}} K(t)\,dt, \qquad a \in \boldsymbol{R},$$

then for every $\psi \in \mathcal{D}^*_{\mathcal{L}^1}$,

$$\lim_{x \to \infty} (f * \psi)(x) = a \int_{\boldsymbol{R}} \psi(t)\,dt .$$

THEOREM 2. *Let $f \in \mathcal{D}'^*$ and $K \in C^\infty$. Assume:*

(i) $f/c \in \mathcal{B}'^*$.

(ii) *There exists $\delta > 0$ such that*

$$\eta\check{K}e^{(\alpha+\delta)\cdot}, \ (1-\eta)\check{K}e^{\beta\cdot} \in \mathcal{D}^*_{\mathcal{L}^1} .$$

(iii) $\mathcal{F}[K](\xi - i\alpha) \neq 0$, $\xi \in \boldsymbol{R}$.

(iv) $\lim_{x \to \infty} (f * K)(x)/(L(e^x)e^{\alpha x}) = a \int_{\boldsymbol{R}} K(t)e^{-\alpha t}\,dt, \qquad a \in \boldsymbol{R}.$

Then for every $\psi \in C^\infty$ *for which*

$$(*) \qquad \eta\check{\psi} e^{(\alpha+\delta)\cdot}, \ (1-\eta)\check{\psi} e^{\beta\cdot} \in \mathcal{D}^*_{L^1},$$

there holds

$$\lim_{x \to \infty} \frac{(f * \psi)(x)}{L(e^x)e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} dt .$$

REMARK. It is an open problem whether the assumption that the set $\{(f(\cdot + h))/c(h);\ h \in \boldsymbol{R}\}$ is bounded in $\mathcal{D}'^*$ implies that $f/c \in \mathcal{B}'^*$. Note that for distributions the corresponding assertion holds (see [7]).

COROLLARY 1. *Let* $f \in \mathcal{D}'^*$ *such that* $f/c \in \mathcal{B}'^*$ *and let* $\phi \in \mathcal{D}^*$ *such that* $\mathcal{F}[\phi](\xi - i\alpha) \neq 0$, $\xi \in \boldsymbol{R}$. *If*

$$\lim_{x \to \infty} \frac{(f * \phi)(x)}{L(e^x)e^{\alpha x}} = a \int \phi(t) e^{-\alpha t} dt , \qquad a \in \boldsymbol{R},$$

then for every $\psi \in \mathcal{D}^*$

$$\lim_{x \to \infty} \frac{(f * \psi)(x)}{L(e^x)e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} dt .$$

3. Proofs.

PROOF OF THEOREM 1. First we need the following version of Beurling's theorem ([2]) for bounded ultradistributions.

«Let $f \in \mathcal{B}'^*$. A point ξ_0 belongs to supp$\hat{f}$ if and only if there is a sequence of functions $\{\varphi_n\}$ from $\mathcal{S}^*$ such that

$$f_n(x) = (f * \varphi_n)(x), \qquad x \in \boldsymbol{R}, \qquad n \in \boldsymbol{N},$$

convergences narrowly to $f_0(x) = e^{ix\xi_0}$, $x \in \boldsymbol{R}$, $n \to \infty$».

The proof of this assertion is the same as for bounded distributions since all the properties of Schwartz's test functions which were used in [2, pp. 230-231], have been proved in [3] and [5] for ultradifferentiable functions. The same holds for the next assertion, based on the previous one, which is analogous to the Theorem on p. 232 in [2].

«Let $f \in \mathcal{B}'^*$ and $K \in \mathcal{D}^*_{\mathcal{L}^1}$. If $K^* f \equiv 0$ on $\boldsymbol{R}$, then $\widehat{K}(\xi) = 0$ for $\xi \in \operatorname{supp} \widehat{f}$».

First, we shal prove that the set $\mathcal{M}$ which consists of finite linear combinations of translations of $K \in \mathcal{D}^*_{\mathcal{L}^1}$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$. By the property of dual pairing, $\mathcal{M}$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$ if and only if for every $S \in \mathcal{B}'^*$, $S * \check{K} = 0 \Leftrightarrow S = 0$. For, if $\mathcal{M}$ is not dense, there exists an $S_0 \in \mathcal{B}^*$, $S_0 \neq 0$ such that $S_0 * \check{K} = 0$. Thus $\mathcal{F}[\check{K}](\xi) = 0$, $\xi \in \operatorname{supp} \mathcal{F}[S_0]$. Since we assume $\mathcal{F}[\check{K}](\xi) = \mathcal{F}[K](-\xi)$ is never zero, we conclude that $\mathcal{M}$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$.

From that and previous statement we obtain the proof of the quoted Wiener theorem.

Note ([5]), $f \in \mathcal{B}'^*$ if and only if it is of the form

$$f = \sum_{\alpha=0}^{\infty} D^\alpha F_\alpha , \qquad F_\alpha \in \mathcal{L}^\infty , \qquad \alpha \in N_0 ,$$

where D is derivative in $\mathcal{B}'^*$ and F_α, $\alpha \in N_0$, are such that for some $h > 0$ (in the (M_α)-case), respectively, for every $h > 0$ (in the $\{M_\alpha\}$-case)

$$\sum_{\alpha=0}^{\infty} \frac{M_\alpha}{h^\alpha} \|F_\alpha\|_{\mathcal{L}^\infty} = K_h < \infty . \tag{5}$$

Let $\psi \in \mathcal{D}^*_{\mathcal{L}^1}$. Since $\mathcal{M}$ is dense in $\mathcal{D}^*_{\mathcal{L}^1}$, then: In the (M_p) case, for every $\varepsilon > 0$ and every $h > 0$, there is $H_h \in \mathcal{M}$ such that

$$\|H_h - \psi\|_{\mathcal{L}^1, h} < \varepsilon . \tag{6}$$

In the $\{M_p\}$ case we have that for every $\varepsilon > 0$ there is $h > 0$ and H such that (6) holds.

In the $\{M_p\}$ case, the assumption of the theorem and Lebesgue's theorem give that for $x > x_0(\varepsilon)$, where $x_0(\varepsilon)$ is large enough,

$$\left| (f * \psi)(x) - a \int_{\boldsymbol{R}} \psi(\xi)\, dt \right| \leq \left| ((\psi - H) * f)(x) - a \int_{\boldsymbol{R}} (\psi(t) - H(t))\, dt \right| +$$

$$+ \left| (H * f)(x) - a \int_{\boldsymbol{R}} H(t)\, dt \right| \leq$$

$$\leq \left| \sum_{i=0}^{\infty} \int_{\boldsymbol{R}} (\psi - H)^{(\alpha)}(t) F_\alpha(x - t)\, dt \right| + a \int_{\boldsymbol{R}} |\psi(t) - H(t)|\, dt +$$

$$+ \left| (H * f)(x) - a \int_{\mathbf{R}} H(t)\, dt \right| \leqslant$$

$$\leqslant \sup_{\alpha} \frac{h^{\alpha}}{M_{\alpha}} \|(\psi - H)^{(\alpha)}\|_{\mathcal{L}^1} \sum_{\alpha=0}^{\infty} \frac{M_{\alpha}}{h^{\alpha}} \|F_{\alpha}\|_{\mathcal{L}^{\infty}} + a\varepsilon +$$

$$+ \left| (H * f)(x) - a \int H(t)\, dt \right| \leqslant \varepsilon K_h + a\varepsilon + \varepsilon .$$

The (M_p) - case can be proved similarly. The proof is completed.

Proof of Theorem 2. We shall only prove the (M_{α})-case since this proof can be simply transfered to the $\{M_{\alpha}\}$-case.

The proof is organized as follows. In Part I we shall prove estimations (7), (7′) and (8) which will be used in Part II for the proof that $\mathcal{F}(Ke^{-\alpha \cdot})(\xi)$, $\xi \in \mathbf{R}^n$, and the convolution $f * K$ exist. In Part III we will prove the assertion of Theorem 2.

Part I. Note, from the assumption that $(1-\eta)\check{K} e^{\beta \cdot}$ and $\eta \check{K} e^{(\alpha+\delta) \cdot}$ belong to $\mathcal{D}_{\mathcal{L}^1}^{(M_{\alpha})}$ and (M.2) it follows that for every $r > 0$

$$\text{(7)} \quad \sup \left\{ \frac{r^m}{M_m} [\, \|e^{\beta x}((1-\eta(x))\check{K}(x))^{(m)}\|_{\mathcal{L}^1} + \right.$$

$$\left. + \|e^{(\alpha+\delta)x}(\eta(x)\check{K}(x))^{(m)}\|_{\mathcal{L}^1}],\ m \in N_0 \right\} < \infty .$$

Since $e^{(\alpha+\delta)x} \leqslant e^{\beta x}$, for $x < 0$, we also have that for every $r > 0$

$$\text{(7′)} \quad \sup \left\{ \frac{r^m}{M_m} \|e^{(\alpha+\delta)x}((1-\eta(x))\check{K}(x))^{(m)}\|_{\mathcal{L}^1(-\infty,0)},\ m \in N \right\} < \infty .$$

We need the following estimate:

For every $r > 0$ there is $C > 0$ such that

$$\text{(8)} \quad \sup_{k \in N_0} \left\{ \frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h)\, e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \right\} \leqslant \begin{cases} Ce^{\alpha x + \delta |x|}, & x+h>0, \\ Ce^{\beta x}, & x+h<0, \end{cases}$$

where we choose δ such that $0 < \delta < \alpha - \beta$.

Let $r > 0$, $k \in N_0$ and $x + h > 1$. By using (3) and (1) we have (with suitable constants)

$$\frac{r^k}{M_k}\left|\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}}-e^{\alpha x}\right)^{(k)}\right| \leqslant$$

$$\leqslant \frac{r^k}{M_k}\int_{-1}^{1}\frac{L(e^{x+h-t})}{L(e^h)}e^{\alpha(x-t)}\,|\omega^{(k)}(t)|\,dt+\frac{|r\alpha|^k}{M_k}e^{\alpha x}\leqslant$$

$$\leqslant C_1\,e^{\alpha x+\delta|x|}\,\|\omega\|_{[-1,1],r}+C_2\,e^{\alpha x}\leqslant Ce^{\alpha x+\delta|x|}\,.$$

Similarly, for $x+h<-1$ we get that for given $r>0$ there is $C>0$ such that

$$\sup_{k\in N_0}\left\{\frac{r^k}{M_k}\left|\left(\frac{c_0(x+h)}{L(e^h)e^{\alpha h}}-e^{\alpha x}\right)^{(k)}\right|\right\}<Ce^{\beta x}\,.$$

Let $u=x+h\in[-1,1]$, $r>0$ and $k\in\boldsymbol{N}_0$. From (1) we have that

$$\frac{1}{L(e^{u-x})}\leqslant Ce^{\delta|x|}\,,\qquad x\in\boldsymbol{R}\,,$$

where $C>0$ and $\delta>0$. Thus,

$$\frac{r^k}{M_k}\left|\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}}-e^{\alpha x}\right)^{(k)}\right| \leqslant$$

$$\leqslant \|\omega\|_{[-1,1],r}\sup_{t\in[-1,1]}\{c_0(x+h-t)\}\frac{1}{L(e^{u-x})\,e^{\alpha(u-x)}}+$$

$$+C_2\,e^{\alpha x}\leqslant C_3\,e^{\alpha x+\delta|x|}+C_2e^{\alpha x}\,.$$

These inequalities and the assumption $\delta\in(0,\alpha-\beta)$ imply (8).

Part II. Let $\psi\in\mathcal{E}^{(M_\alpha)}$ be such that $\psi(x)=1$ on $(-\infty,-1)$ and $\psi(x)=0$ on $[0,\infty)$. We have

$$e^{\alpha x}\check{K}(x)=e^{\alpha x}\check{K}(x)(1-\eta(x))\psi(x)+ \tag{9}$$

$$+e^{\alpha x}\check{K}(x)(1-\eta(x))(1-\psi(x))+e^{\alpha x}\check{K}(x)\eta(x)\,,\qquad x\in\boldsymbol{R}\,.$$

Since the multiplication in $\mathcal{D}^{(M_\alpha)}_{\mathcal{L}^1}$ is the inner operation, one can easily prove that all the members on the right side of (9) are from $\mathcal{D}_{L^1}^{(M_\alpha)}$ and so the same holds for $e^{\alpha\cdot}\check{K}$. This implies that $\mathcal{F}(Ke^{-\alpha\cdot})(\xi) = \mathcal{F}(K)(\xi - i\alpha)$, $\xi \in \boldsymbol{R}^n$, exists.

Since

$$(f * K)(h) = \left\langle \frac{f(x+h)}{c(x+h)}, \; c(x+h)\check{K}(x) \right\rangle, \qquad h \in \boldsymbol{R},$$

the existence of the convolution $f * K$ will be proved if we prove that for every $h \in \boldsymbol{R}$,

$$c(.+h)\check{K} \in \mathcal{D}_{L^1}^{(M_\alpha)},$$

because by (i) $f(.+h)/c(.+h) \in \mathcal{B}'^{(M_\alpha)}$.

For a fixed $h \subset \boldsymbol{R}$ and a ψ as in (9) we have

$$\text{(10)} \quad c(x+h)\check{K}(x) = c(x+h)\check{K}(x)(1-\eta(x))\psi(x) + $$

$$+ c(x+h)\check{K}(x)(1-\eta(x))(1-\psi(x)) + c(x+h)\check{K}(x)\eta(x), \qquad x \in \boldsymbol{R}.$$

By using (8), (7), (7′) and that $e^{\alpha\cdot}\check{K} \in \mathcal{D}_{L^1}^{(M_\alpha)}$ we shall prove that $c(.+h)\check{K} \in \mathcal{D}_{L^1}^{(M_\alpha)}$. We shall only prove that $c(.+h)\check{K}(1-\eta)\psi$ is from $\mathcal{D}_{L^1}^{(M_\alpha)}$ for every $h \in \boldsymbol{R}$, because the proof that $c(.+h)\check{K}\eta$ belongs to $\mathcal{D}_{L^1}^{(M_\alpha)}$ is similar and then one can easily see that

$$c(.+h)\check{K}(1-\eta)(1-\psi) \in \mathcal{D}^{(M_\alpha)}.$$

Since

$$c(x+h)\check{K}(x)(1-\eta(x))\psi(x) = \left[e^{\alpha h} L(e^h) \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \check{K}(x) + \right.$$

$$\left. + e^{\alpha(x+h)} L(e^h) \check{K}(x) \right] (1-\eta(x))\psi(x), \qquad x \in \boldsymbol{R},$$

we have to prove that

$$(1-\eta)\psi \left(\frac{c(.+h)}{e^{\alpha h} L(e^h)} - e^{\alpha\cdot} \right) \check{K} \in \mathcal{D}_{L^1}^{(M_\alpha)}.$$

For every $r > 0$, $x \in \boldsymbol{R}$, $k \in \boldsymbol{N}_0$, by using (M.2) and (8) we have

$$\frac{r^k}{M_k}\left\|\sum_{j=0}^{k}\binom{k}{j}(\check{K}(x)(1-\eta(x))\psi(x))^{(k-j)}\left(\frac{c(x+h)}{e^{\alpha h}L(e^h)}-e^{\alpha x}\right)^{(j)}\right\|_{\mathcal{L}^1} \leqslant$$

$$\leqslant \frac{AC}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{(2rH)^{k-j}}{M_{k-j}}\{\|(\check{K}(x)(1-\eta(x))\psi(x))^{(k-j)}e^{\beta x}\|_{\mathcal{L}^1}\} \leqslant$$

$$\leqslant C_1 \sup_{\alpha\in\boldsymbol{N}_0}\left\{\frac{(2rH)^{\alpha}}{M_{\alpha}}\|(\check{K}(x)(1-\eta(x))\psi(x))^{(\alpha)}e^{\beta x}\|_{\mathcal{L}^1}\right\},$$

where C and C_1 are suitable constants.

For the proof that the last supremum is bounded we have to use the following estimates

$$\frac{(4rH)^{\alpha}}{M_{\alpha}}\frac{1}{2^{\alpha}}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\|\check{K}(x)(1-\eta(x))^{(j)}e^{\beta x}\psi^{(\alpha-j)}\|_{\mathcal{L}^1} \leqslant$$

$$\leqslant C_2 \sup_{j\in\boldsymbol{N}_0}\left\{\frac{(4rH^2)^j}{M_j}\|\check{K}(x)(1-\eta(x))^{(j)}e^{\beta x}\|_{\mathcal{L}^1}\right\}\cdot$$

$$\cdot \sup_{\substack{\alpha\in\boldsymbol{N}_0\\ j\leqslant\alpha}}\left\{\frac{(4rH^2)^{k-j}}{M_{\alpha-j}}\|\psi^{(\alpha-j)}\|_{\mathcal{L}^{\infty}(-1,0)}\right\} \leqslant C_3,$$

where C_2 and C_3 are suitable constants. Thus, we have proved that the convolution $f * K$ exists.

Part III. We are going to prove that the assumptions of the theorem imply

$$\left(\frac{f}{c} * Ke^{-\alpha\cdot}\right)(h) = \left\langle\frac{f(x+h)}{c(x+h)}, \check{K}(x)e^{\alpha x}\right\rangle \to a\int_{\boldsymbol{R}} K(t)e^{-\alpha t}\,dt, \qquad h\to\infty. \tag{11}$$

It is enough to prove that

$$\left\langle\frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{\alpha h}L(e^h)}-e^{\alpha x}\right)\check{K}(x)\right\rangle \to 0, \qquad \text{as } h\to 0.$$

Since $\mathscr{B}'^* \ni f/c = \sum_{i=0}^{\infty} D^i F_i$ such that (5) holds, we have to prove that

$$S_h = \sum_{i=0}^{\infty} (-1)^i \int_R F_i(x+h)\left(\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x}\right)\check{K}(x)\right)^{(i)} dx \to 0\,, \qquad h \to \infty\,.$$

We have

$$S_h = \sum_{i=0}^{N} (-1)^i \int_R F_i(x+h)\left(\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x}\right)\check{K}(x)\right)^{(i)} dx +$$

$$+ \sum_{i=N+1}^{\infty} (-1)^i \int_R F_i(x+h)\left(\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x}\right)\check{K}(x)\right)^{(i)} dx = S_{h,N} + S_{h,\infty}\,.$$

Because the sum in $S_{h,N}$ is finite, the proof that $S_{h,N} \to 0$, $h \to \infty$, is the same as in the main assertion of [6]. By using (5) we obtain

$$S_{h,\infty} \leqslant \sum_{i=N+1} \frac{1}{2^i} \frac{M_i \|F_i\|_{\mathscr{L}^\infty}}{r^i} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x}\right)\check{K}(x)\right]^{(i)} \right\|_{\mathscr{L}^1} \leqslant$$

$$\leqslant \frac{C}{2^{N+1}} \sum_{i=0}^{\infty} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x}\right)\check{K}(x)\right]^{(i)} \right\|_{\mathscr{L}^1}.$$

So, if we prove that the last series is bounded with respect to h for $h \geqslant h_0$, the proof that $S_{h,\infty} \to 0$, $h \to \infty$, simply follows. Put

$$I_{m,h} = \left\| \left(\left(\frac{c(.+h)}{e^{\alpha h}L(e^h)} - e^{\alpha .}\right)\check{K}\right)^{(m)} \right\|_{\mathscr{L}^1}, \qquad m \in N_0\,, \qquad h \geqslant h_0\,.$$

We are going to prove that for every $r > 0$, there is $C > 0$ such that

$$\text{(12)} \qquad \sup_{m \in N_0} \left\{ \frac{r^m}{M_m} I_{m,h} \right\} < C\,, \qquad h > h_0\,.$$

This implies that the quoted series is bounded.

Let $x_0 > 0$ be as in (4). We have

$$I_{m,h} = \int_{-\infty}^{-x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) (1 - \eta(x)) \check{K}(x) \right]^{(m)} \right| dx +$$

$$+ \int_{-x_0}^{x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \check{K}(x) \right]^{(m)} \right| dx +$$

$$+ \int_{x_0}^{\infty} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \eta(x) \check{K}(x) \right]^{(m)} \right| dx = I_1 + I_2 + I_3 .$$

By the Leibniz formula, (8), (7), (7′) and (1) there are constants C_1 and C which do not depend on m and p (but depend on r) such that for $\delta \in (0, \alpha - \beta)$

$$I_1 \leqslant \sum_{p=0}^{m} \begin{bmatrix} m \\ p \end{bmatrix} \left[\int_{-\infty}^{-h} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-\beta x} \right| \cdot \right.$$

$$\cdot \left| e^{\beta x} ((1 - \eta(x)) \check{K}(x))^{(m-p)} \right| dx +$$

$$\left. + \int_{-h}^{-x_0} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-(\alpha x - \delta x)} \right| \left| e^{\alpha x - \delta x} ((1 - \eta(x)) \check{K}(x))^{(m-p)} \right| dx \right] \leqslant$$

$$\leqslant \sum_{p=0}^{m} \binom{m}{p} C \left[\frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leqslant m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \| e^{\beta x} (1 - \eta(x)) (\check{K}(x))^{(m-p)} \|_{\mathcal{L}^1} \right\} + \right.$$

$$\left. + \frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leqslant m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \| e^{\alpha x - \delta x} (1 - \eta(x)) \check{K}(x) \|_{\mathcal{L}^1_{(-\infty, 0)}} \right\} \right] \leqslant$$

$$\leqslant C_1 \sum_{p=0}^{m} \binom{m}{p} \frac{M_m}{r^m} = C_1 \frac{M_m}{(r/2)^m} .$$

This gives that $\sup_{m \in N_0} \{ (r/2)^m / (M_m) I_1 \} < C_1$.

In a similar way one can prove the corresponding estimates for I_2 and I_3 and the proof of (12) is completed.

Thus, we have proved (11).

If $\psi \in \mathcal{E}^{(M_\alpha)}$ satisfies the assumption given in (*) then $\psi e^{\alpha \cdot} \in \mathcal{D}^{(M_\alpha)}_{\mathcal{L}^1}$ and we have

$$\left(\frac{f}{c} * \psi e^{-\alpha\cdot}\right)(h) \to a \int_{\boldsymbol{R}} \psi(x) e^{-\alpha x} dx, \qquad h \to \infty .$$

As above we can prove that $(f * \psi)(h)$, $h \in \boldsymbol{R}$, exists.

For the proof that

$$\left\langle \frac{f(x+h)}{L(e^h) e^{\alpha h}}, \check{\psi}(x) \right\rangle \to a \int \psi(x) e^{-\alpha x} dx, \qquad h \to \infty ,$$

we have to prove that

$$\left\langle \frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \psi(x) \right\rangle \to 0, \qquad h \to \infty$$

but this is already done (with K instead of ψ) and the proof of the Theorem is completed.

The proof of Corollary 1 simply follows from the given Theorem 2 since the ϕ in the Corollary satisfies conditions assumed for K and functions from $\mathcal{D}^*$ satisfy condition (c) of Theorem 2.

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