RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 1-8

<http://www.numdam.org/item?id=RSMUP_1994__92__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

Symmetry and Minimality Properties for Generalized Ruled Submanifolds.

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ABSTRACT - We prove the following theorem. Let $\overline{M}$ be a standard space form. If R is a submanifold of $\overline{M}$ foliated by totally geodesic submanifolds of $\overline{M}$, such that the geodesic reflection of $\overline{M}$ with respect to each leaf of R locally maps R into itself, leaf by leaf (R symmetric generalized ruled submanifold), then R is a minimal submanifold of $\overline{M}$.

1. Introduction.

Let $\overline{M} = \overline{M}(c)$ be a standard space of constant curvature c. We define *generalized ruled submanifold* of $\overline{M}$ a submanifold R of $\overline{M}$ foliated by totally geodesic submanifolds of $\overline{M}$. Each leaf of the foliation is called *ruling* of R. Moreover we say that R is a *symmetric generalized ruled submanifold* of $\overline{M}$ if R is locally mapped into itself, ruling by ruling, by the geodesic reflection of $\overline{M}$ with respect to each ruling.

Any multihelicoid in a standard space $\overline{M}$ of constant curvature is obviously a generalized ruled submanifold of $\overline{M}$ (see n. 2).

In Proposition 2.5 we even prove that any multihelicoid, associated to a nicely curved generalized 2-symmetric submanifold, is a symmetric generalized ruled submanifold in the sense above specified.

At n. 3, Theorem 3.16, we prove that any symmetric generalized ruled submanifold R of $\overline{M}$ is a minimal submanifold of $\overline{M}$.

In this work we use the same symbols as in [M-R]. In particular, if M is a submanifold of $\overline{M}$, we denote by $\overset{k}{N}_x M$ the k-th normal space of M

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Work supported by MURST 40% and 60%.

at x and by $\overset{k}{\sigma}: T_xM \times \overset{k}{N}_xM \to \overset{k+1}{N_x} M$ the k-th fundamental form of M at x.

(1.1) REMARKS. (a) If the codimension of the ruling of a generalized ruled submanifold R of $\overline{M}$ is equal to 1, then R is just a ruled submanifold of $\overline{M}$ in the sense of Barbosa, Dajczer and Jorge (see [B-D-J]).

(b) Let R be a generalized ruled submanifold of $\overline{M}$. Let, for each $x \in R$, the leaf F_x be the only totally geodesic submanifold of $\overline{M}$ passing through x, contained in R and of dimension $p = \dim F_x$. We recall that a geodesic reflection of $\overline{M}$ with respect to a totally geodesic submanifold of $\overline{M}$ is an isometry (see [C-V]). Then a sufficient condition for R to be a symmetric generalized ruled submanifold of $\overline{M}$ is that R is locally mapped into itself by the geodesic reflection with respect to each ruling of R.

2. Symmetry property of multihelicoids.

Let M be a nicely curved submanifold of a standard space $\overline{M} = \overline{M}(c)$, of constant curvature c, satisfying the following condition

(2.1) $$\nabla \overset{k}{\sigma} = 0, \qquad \text{for each } k.$$

We recall that any nicely curved 2-symmetric submanifold satisfies such a condition (see [CD-M-R]).

In [M-R] we have defined multihelicoid associated to M any submanifold R of $\overline{M}$ which is a tubular neighbourhood of M in the set

(2.2) $$\{\exp_x v\}_{x \in M,\, v \in V_x},$$

being

(2.3) $$V_x = \overset{[(l-1)/2]}{\underset{0}{\oplus_h}} \overset{2h+1}{N_x} M.$$

R is foliated by the totally geodesic submanifolds of $\overline{M}$

(2.4) $$F_x = \{\exp_x v\}_{v \in V_x} \cap R, \qquad x \in M.$$

So R is just a generalized ruled submanifold of $\overline{M}$ with ruling F_x passing through $x \in M$.

We also recall that any multihelicoid R is a minimal submanifold of $\overline{M}$. But the aim of this section is to prove that any multihelicoid R, associated to a nicely curved 2-symmetric submanifold M of $\overline{M}$, is even a symmetric generalized ruled submanifold of $\overline{M}$, in the sense specified at n. 1.

More precisely we want to prove the following proposition.

(2.5) PROPOSITION. *Let* M *be a nicely curved* 2*-symmetric submanifold of* $\overline{M}$. *If* R *is a multihelicoid associated to* M, *then for each* $x \in M$, *the geodesic reflection* S_x *of* $\overline{M}$ *with respect to the ruling* F_x *is such that*

$$S_x(F_y) = F_{S_x(y)}, \qquad y \in M. \tag{2.6}$$

PROOF. In [CD-M-R] it is proved that M is mapped into itself by S_x and $dS_x(\overset{k}{N}_y M) = \overset{k}{N}_{S_x(y)} M$, for each k.

So, for (2.3), we have

$$dS_x(V_y) = V_{S_x(y)}. \tag{2.7}$$

Moreover, as we recalled in Remark 1.1*(b)*, the geodesic reflection S_x is an isometry of $\overline{M}$. Then if $z_y = \exp_y v_y \in F_y$, $v_y \in V_y$, it will be, for (2.7),

$$S_x(z_y) = S_x(\exp_y v_y) = \exp_{S_x(y)} dS_x(v_y) \in F_{S_x(y)},$$

and hence formula (2.6).

3. Minimality of symmetric generalized ruled submanifolds.

Let R be a Riemannian manifold. If W is a subbundle of the tangent bundle TR, we denote by $\overset{W}{\nabla}$ the connection induced on the vector subbundle W by the Levi-Civita connection ∇ on R, i.e. we put

$$\overset{W}{\nabla}_X Y = P_W(\nabla_X Y), \quad X \in \Gamma(TR), \qquad Y \in \Gamma(W). \tag{3.1}$$

The following lemma holds.

(3.2) LEMMA. *For each* $x_0 \in R$ *and for each* $Y_{x_0} \in W_{x_0}$ *locally there is one and only one curve* γ *of* R, *passing through* x_0 *with tangent vector* Y_{x_0} *such that, if* Y *is the generic tangent vector of* γ, *it results* $Y \in \Gamma(W)$ *and*

$$\overset{W}{\nabla}_Y Y = 0. \tag{3.3}$$

We'll call such a curve γ *a geodesic of the distribution* W. *Obviously, if*

W is an integrable distribution of R, then γ is a geodesic of an integral submanifold of the distribution.

PROOF. If dim $R = r$ and dim $W = p$, it results dim $W^{\perp} = r - p$.

Now we choose a local coordinates system $x^1, \ldots, x^r$. We can describe the distribution W making equal to zero $r - p$ independent 1-forms. So we have the system

$$a_i^j \, dx^i = 0\,, \qquad j = 1, \ldots, r - p\,, \tag{3.4}$$

where the coefficients a_i^j are functions of the coordinates.

Analogously we can describe the orthogonal distribution $W^{\perp}$ by the system

$$b_i^h \, dx^i = 0\,, \qquad h = r - p + 1, \ldots, r\,, \tag{3.5}$$

where the coefficients b_i^h are functions of the coordinates.

Because of the orthogonality of W and $W^{\perp}$, if we put

$$A = ((a_i^j)) \quad \text{and} \quad B = ((b_i^h))\,,$$

we have

$$\det \begin{pmatrix} A \\ B \end{pmatrix} \neq 0\,. \tag{3.6}$$

Now let $x^i = x^i(t)$, $i = 1, \ldots, r$, be the equations of a generic curve γ of R. If we want that its tangent vector $X = (dx^i/dt)(\partial/\partial x^i)$ must be in W, it is necessary to put

$$a_i^j \, \frac{dx^i}{dt} = 0\,, \qquad j = 1, \ldots, r - p\,. \tag{3.7}$$

Moreover if we want that $\overset{W}{\nabla}_X X = 0$, that is

$$\nabla_X X = \left(\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \, \frac{dx^j}{dt} \, \frac{dx^k}{dt} \right) \frac{\partial}{\partial x^i} \in W^{\perp}\,,$$

where Γ^i_{jk} are the Christoffel symbols of the connection ∇, it is necessary to put

$$b_i^h \left(\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \, \frac{dx^j}{dt} \, \frac{dx^k}{dt} \right) = 0\,, \qquad h = r - p + 1, \ldots, r\,. \tag{3.8}$$

If we differentiate with respect to t in (3.7), we have the system

$$a_i^j \frac{d^2 x^i}{dt^2} + \dot{a}_i^j \frac{dx^i}{dt} = 0, \qquad j = 1, \dots, r - p, \tag{3.9}$$

where $\dot{a}_i^j = (\partial a_i^j / \partial x^k)(dx^k / dt)$.

System (3.7), (3.8) implies System (3.9), (3.8).

Reciprocally if $x^i = x^i(t)$, $i = 1, \dots, r$, is a solution of System (3.9), (3.8), which verifies the initial condition

$$a_i^j(x^1(0), \dots, x^r(0)) \left. \frac{dx^i}{dt} \right|_{t=0} = 0, \qquad j = 1, \dots, r - p. \tag{3.10}$$

Equality (3.9) insures us that System (3.7) is satisfied by such a solution. In fact Equality (3.9) can be written as

$$\frac{d}{dt}\left(a_i^j \frac{dx^i}{dt}\right) = 0, \qquad j = 1, \dots, r - p. \tag{3.11}$$

Therefore $a_i^j(dx^i/dt) = c^j$, But, for (3.10), it is $c^j = 0$, and hence System (3.7) is satisfied.

Then System (3.7), (3.8) is equivalent, for solutions which verify Condition (3.10), to System (3.9), (3.8).

But the matrix of the coefficients of the second derivatives in System (3.9), (3.8) is the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ and, for (3.6), it is $\det \begin{pmatrix} A \\ B \end{pmatrix} \neq 0$.

Then it is possible to esplicit System (3.9), (3.8) in the form

$$\frac{d^2 x^i}{dt^2} = f^i\left(t, x^1, \dots, x^r, \frac{dx^1}{dt}, \dots, \frac{dx^r}{dt}\right), \qquad i = 1, \dots, r. \tag{3.12}$$

Now System (3.12) has one and only one solution with initial conditions $x^i(0) = x_0^i$, $i = 1, \dots, r$ and 3.10.

So Lemma 3.2 is proved.

From now on we only consider symmetric generalized ruled submanifolds of a standard space $\overline{M} = \overline{M}(c)$ of constant curvature c. If R is such a submanifold of R, we denote by U the vector subbundle of TR whose fiber, at $x \in R$, is $U_x = T_x F_x$, being F_x the ruling of R through the point x.

The orthogonal subbundle $U^\perp$ is also called the distribution orthogonal to the foliation of R.

(3.13) LEMMA. *Each geodesic γ of the distribution $U^\perp$ orthogonal to the foliation of a symmetric generalized ruled submanifold R of $\overline{M}$ is* *locally mapped into itself by the geodesic reflection of $\overline{M}$ with respect to each ruling of R which intersects γ.*

PROOF. If $y \in \gamma$, let F_y be the ruling passing through y and let S_y be the geodesic reflection of $\overline{M}$ with respect to F_y. We know that S_y is an isometry of $\overline{M}$. Then dS_y, mapping the subbundle U into itself, it will also map the subbundle U into itself. Moreover it will be, for $Y \in \Gamma(U^\perp)$,

$$dS_y(\nabla_Y Y) = \nabla_{dS_y(Y)} dS_y(Y). \tag{3.14}$$

Therefore it is

$$dS_y(\nabla_Y Y) = P_U(\nabla_{dS_y(Y)} dS_y(Y)) + P_{U^\perp}(\nabla_{dS_y(Y)} dS_y(Y)).$$

On the other hand we have.

$$dS_y(\nabla_Y Y) = dS_y(P_U(\nabla_Y Y)) + dS_y(P_{U^\perp}(\nabla_Y Y)).$$

But we have already observed that dS_y maps U and $U^\perp$ into themselves, so it will be

$$dS_y(P_U(\nabla_Y Y)) \in U \quad \text{and} \quad dS_y(P_{U^\perp}(\nabla_Y Y)) \in U^\perp .$$

Then we have, in particular,

$$P_{U^\perp}(\nabla_{dS_y(Y)} dS_y(Y)) = dS_y(P_{U^\perp}(\nabla_Y Y)). \tag{3.15}$$

Now let γ' be the curve image of γ by S_y.

Then, if $Y \in \Gamma(U^\perp)$ is the generic vector tangent to the curve γ, $Y' = dS_y(Y)$ is a vector tangent to γ' which must be a vector of $U^\perp$.

So, for (3.15), we have

$$0 = dS_y(\overset{U^\perp}{\nabla}_Y Y) = dS_y(P_{U^\perp}(\nabla_Y Y)) =$$

$$= P_{U^\perp}(\nabla_{dS_y(Y)} dS_y(Y)) = \overset{U^\perp}{\nabla}_{dS_y(Y)} dS_y(Y) = \overset{U^\perp}{\nabla}_{Y'} Y' ,$$

i.e. also γ' is a geodesic of $U^\perp$.

On the other hand S_y fixes y and dS_y changes Y_y in $-Y_y$. Then γ' must pass through y with tangent vector $-Y_y$. So for the uniqueness of the geodesics of $U^\perp$, it follows that $\gamma' = \gamma$.

Now we come to our main result.

(3.16) THEOREM. *If* R *is a symmetric generalized ruled submanifold of* $\overline{M}$, *then* R *is a minimal submanifold of* $\overline{M}$.

PROOF. Let x be a point of R. Moreover let $X_x^1, \ldots, X_x^p$ be an orthonormal basis of the tangent space U_x at x of the ruling F_x of R passing through x and let $Y_x^{p+1}, \ldots, Y_x^r$ be an orthonormal basis of $U_x^\perp$, where $r = \dim R$.

Then the mean curvature vector of R at x, $H_x(R)$, will be given by

$$H_x(R) = = \frac{1}{r}\left[\sum_1^p {}_i \overline{P}_{(T_x R)^\perp} \overline{\nabla}_{x_x^i} X^i + \sum_{p+1}^r {}_j \overline{P}_{(T_x R)^\perp} (\overline{\nabla}_{Y_x^j} Y^j)\right], \tag{3.17}$$

where by $\overline{\nabla}$ we denote the Levi-Civita connection on $\overline{M}$ and by $\overline{P}_{(T_x R)^\perp}$ the orthogonal projection of $T_x \overline{M}$ onto $(T_x R)^\perp$.

But, being the ruling F_x a totally geodesic submanifold of $\overline{M}$, it is

$$\overline{\nabla}_{X_x^i} X^i \in T_x F_x \subset T_x R .$$

So it results

$$\overline{P}_{(T_x R)^\perp} \overline{\nabla}_{X_x^i} X^i = 0 .$$

Therefore we have simply

$$H_x(R) = \frac{1}{r} \sum_{p+1}^r {}_j \overline{P}_{(T_x R)^\perp} (\overline{\nabla}_{Y_x^j} Y^j) . \tag{3.18}$$

Now we consider, for $j = p+1, \ldots, r$, the unique geodesic γ^j of $U^\perp$ passing through x with tangent vector Y_x^j (Lemma 3.2). We recall that we have $T_x \gamma^j \subset U_x^\perp$ and $P_{U_x^\perp}(\nabla_{Y_x^j} Y^j) = 0$.

Then it also results

$$\overline{P}_{T_x \gamma^j} (\overline{\nabla}_{Y_x^j} Y^j) = 0 . \tag{3.19}$$

In fact it is

$$\overline{P}_{T_x \gamma^j} (\overline{\nabla}_{Y_x^j} Y^j) = P_{T_x \gamma^j}(P_{U_x^\perp}(\overline{P}_{T_x R}(\overline{\nabla}_{Y_x^j} Y^j))) = P_{T_x \gamma^j}(P_{U_x^\perp}(\nabla_{Y_x^j} Y^j)) .$$

For Lemma 3.13, γ^j is a curve of $R \subset \overline{M}$ locally mapped into itself by the geodesic reflection of $\overline{M}$ with respect to each ruling F_y of R passing through a point $y \in \gamma^j$.

Then γ^j is a 2-symmetric submanifold of $\overline{M}$ and hence (see [CD-M-R]) we have that $\overset{1}{N}_y \gamma_1^j \subset U_y$, for each $y \in \gamma^j$.

In particular it is $N_x \gamma^j \subset U_x$.

From (3.19) it results

$$\overline{\nabla}_{Y_x^j} Y^j = \overline{P}_{T_x\gamma^j}(\overline{\nabla}_{Y_x^j} Y^j) + \overline{P}_{(T_x\gamma^j)^\perp}(\overline{\nabla}_{Y_x^j} Y^j) = \overline{P}_{(T_x\gamma^j)^\perp}(\overline{\nabla}_{Y_x^j} Y^j).$$

Observing that

$$\overline{P}_{(T_x\gamma^j)^\perp}(\overline{\nabla}_{Y_x^j} Y^j) \in \overset{1}{N}_x \gamma^j \subset U_x \subset T_x R,$$

we conclude that

$$\overline{P}_{(T_x R)^\perp}(\overline{\nabla}_{Y_x^j} Y^j) = 0.$$

From this equality and from (3.15) we have finally $H_x(R) = 0$; i.e. R is a minimal submanifold of $\overline{M}$ as desired.

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Manoscritto pervenuto in redazione il 27 marzo 1992.