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Some Elementary Remarks about n-Local Fields.

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Local fields (i.e. complete fields relative to a discrete rank one valuation) play a key role in the theory of algebraic functions of one variable and elsewhere. Local fields have been studied from many points of view (see [4], [5], [6], [12], [13], [14], [15]). n-local fields (see the definition below) are a natural generalization of local fields. These fields appear in the theory of algebraic functions of several variables and in algebraic geometry. In the last years many problems on n-local fields, as, for example, class-field theory, have developed (see [8], [9]).

The aim of this paper is to make out some elementary results about n-local fields. In the first section general notations and definitions are given. In the second section we remark that the n-local field of bounded Laurent power series in many variables over a perfect field (see the definition below), can be defined in the same way as the field of Witt's vectors, starting from a field of repeated formal power series in several variables over a perfect field. In this case the Witt's operations are applied only to coefficients in the same way as in the classical case. However, we give all the steps of that construction.

Let k be a field and $k(X, Y)$ the field of rational functions of two variables over k.

In the third section we give a description of the maximal completion of the field $k(X, Y)$ relative to a rank two and discrete valuation, trivial on k.

Finally in the last section is proved that if K is a n-local field such that its residue field has the property that for every natural number m has a finite set of separable extension of degree m, then K has, for every natural number m, a finite number of tamely ramified extensions of degree m.

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1. Definition.

We refer the reader to [3], [7], [11] and [12] for usual definitions. By *valued field* we mean a pair (K, v) where K is a field and v a valuation on K. As usual, we denote with O_v the valuation rings of v, M_v its maximal ideal, G_v the value group of v and k_v the residue field of v. If $x \in O_v$, x^* denotes the natural image of x into k_v. We shall say that (K, v) is a *local field* if v is discrete of rank one and K is complete with respect to v.

Let (K, v) be a valued field where v is discrete and of rank n.

Let O be the valuation ring of v and let $(0) = M_n \subset M_{n-1} \subset \ldots \subset M_0$ be all the prime ideals of O. For every $i < n$, denote O_i the ring of fractions of O with respect to the prime ideal M_i. In particular $O_0 = O$ and $O_n = K$. Denote v_i, $i = 0, \ldots, n-1$, the valuation on K associated to the valuation ring O_i. One has $v_0 = v$ and for every i, v_i is a valuation of rank $n - i$ on K. Denote $\bar{v}_i$, $i = 0, 1, \ldots, n-1$ the valuation on the field $K_{i+1} = O_{i+1}/M_{i+1}$, associated with the valuation ring O_i/M_{i+1}. It is easy to see that $\bar{v}_i$ is of rank one. Particularly one has $K_n = K$ and $\bar{v}_{i-1} = v_{n-1}$.

We shall say that (K, v) is *n-local* if $(K_{i+1}, \bar{v}_i)$ is local for all $i = 0, 1, \ldots, n-1$.

Let k be a field and n a natural number. Denote $k((t_1)) \ldots ((t_n))$ *the field of repeated power series in n indeterminates* $t_1, \ldots, t_n$ over k. (For example one has: $k((t_1))((t_2)) = (k((t_1)))((t_2))$, etc.). This field is in a natural way endowed with a rank n and discrete valuation v. Moreover $(k((t_1)) \ldots ((t_n)), v)$ is n-local.

Another example of n-local field is obtained as follows. Let (K, v) be a local field. Let us denote $K\{\{t\}\}$ the set of all formal Laurent series $\sum_{-\infty}^{+\infty} a_n t^n$ over K which verify the following two conditions (see [8]):

i) the set $\{v(a_n)\}$ is lower bounded,

ii) $v(a_{-n}) \to \infty$ if $n \to \infty$.

Define in $K\{\{t\}\}$ the following valuation w_1:

$$w_1\left(\sum_{-\infty}^{+\infty} a_n t^n\right) = \inf_n v(a_n).$$

It is easy to see that $(K\{\{t\}\}, w_1)$ is a local field whose residue field is just $k((t))$ where k is the residue field of (K, v).

We shall say that $(K\{\{t\}\}, w_1)$ is the *local field of bounded Laurent series over* (K, v). Moreover we can define on $K\{\{t\}\}$ a rank two valuation u_1 as follows: Let $\alpha = \sum_{-\infty}^{+\infty} a_n t^n$ and let n_1 be the last integer number such that $v(a_n) = w_1(\alpha)$. Now we shall define $u_1(\alpha) = (w_1(\alpha), n_1)$. The residue field of $(K\{\{t\}\}, u_1)$ is just k. In fact u_1 is the composite valuation of w_1, with the order valuation on $k((t))$. (See [11, pag. 43]).

Furthermore we can define the local field $(K\{\{t_1\}\}\{\{t_2\}\}, w_2)$ as the local field of bounded Laurent series over $(K\{\{t_1\}\}, w_1)$ and by recurrence we can define the local field $(K\{\{t_1\}\} \dots \{\{t_n\}\}, w_n)$ whose residue field is just $k((t_1)) \dots ((t_n))$. The field $K\{\{t_1\}\}, \dots, \{\{t_n\}\}$ is naturally endowed with a rank $n+1$ valuation u_n and $(K\{\{t_1\}\} \dots \{\{t_n\}\}, u_n)$ is a $n+1$ local field. Usually we shall say that $K\{\{t_1\}\}, \dots, \{\{t_n\}\}$ is the *field of bounded Laurent series in n variables over* (K, v).

2. Alternative definition of $K\{\{t_1\}\} \dots \{\{t_n\}\}$.

Let k be a perfect field of characteristic $p > 0$ and let $(T(k), v)$ be the Witt's local field associated to k (see [4], [14]). We remind that $T(k)$ is the quotient field of the ring of Witt's vectors and v the natural valuation. The aim of this section is to show that $T(k)\{\{t_1\}\} \dots \{\{t_n\}\}$ can be constructed in the same way as the field $T(k)$ starting with the field $k_n = k((t_1)) \dots ((t_n))$. In other words if k is a perfect field of characteristic $p > 0$, we shall build a rank one and discrete valuation ring of zero characteristic whose residue field is just k_n.

1) Let $A_0 = Q[X_i]_{i \in I}$ be the polinomial ring in a set X_i of indeterminates over Q, the field of rational numbers. Denote $S_n = A_0((Y_1)) \dots ((Y_n))$. For any natural number i, define inductively the mapping

$$(p^i)\colon A_n \to A_n \tag{1}$$

as follow: The mapping $(p^i)\colon A_0 \to A_0$ is just the raising at the p^i-power, i.e. $(p^i)(a) = a^{p^i}$ for all $a \in A_0$ and for $n \geq 1$, one has: if $a \in A_n$, $a = \sum_{k > -\infty} a_k Y_n^k$, $a_k \in A_{n-1}$ then: $(p^i)(a) \sum_{k > -\infty} a_{k,p^i} Y_n^k$, where $a_{k,p_i} = (p^i)(a_k)$.

Now let us define B_n to be the set of all sequences with entries in A_n: $\alpha = (\alpha_0, \alpha_1, \dots)$, $\alpha_i = \sum_{j > -\infty} a_{ij} Y_n^j$, $a_{ij} \in A_{n-1}$. The set B_n is a ring with operations defined component wise. Define the mapping $\varphi_n\colon B_n \to B_n$ by:

$$\varphi_n(\alpha) = \varphi_n(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \dots)$$

where

$$\alpha^{(0)} = \alpha_0, \quad \alpha^{(1)} = \alpha_{0,p} + p\alpha_1, \dots, \alpha^{(m)} = \alpha_{0,p^m} + p\alpha_{1,p^{m-1}} + \dots + p^m \alpha_m .$$

If we denote

$$P\alpha = (\alpha_{0,p}, \alpha_{1,p}, \dots, \alpha_{n,p}, \dots)$$

then as one easily sees:

$$\alpha^{(n)} = (P\alpha)^{(n-1)} + p^n \alpha_n .$$

Also it is obvious that: $\alpha_0 = \alpha^{(0)}$ and generally:

$$\alpha_n = \frac{1}{p^n} (\alpha^{(n)} - \alpha_{0,p^n} - \dots - p^{n-1} \alpha_{n-1,p})$$

and so the mapping φ_n is a bijection.

If $\alpha, \beta \in B_n$, then one defines:

$$\alpha + \beta = \varphi_n^{-1}(\varphi_n(\alpha) \oplus \varphi_n(\beta)),$$

$$\alpha\beta = \varphi_n^{-1}(\varphi_n(\alpha) \odot \varphi_n(\beta)),$$

where $\oplus$ and $\odot$ are the addition and multiplication (defined component wise) in B_n. It is clear that relative to the above defined operations, B_n is a commutative ring with identity $(1, 0, 0, \dots)$.

Furthermore, denote:

$$A'_0 = Z[X_i]_{i \in I}, \qquad A'_n = A'_0((Y_1))\dots((Y_n)), \qquad n \geqslant 1.$$

It is clear that A'_n is a subring of A_n.

If

$$\alpha_i = \sum_{k > -\infty} a_{ik} Y_n^k, \qquad \beta_i = \sum_{k > -\infty} b_{ik} Y_n^k$$

are two elements of A'_n, then we write:

$$\alpha_i \equiv \beta_i (\mathrm{mod}\, p^e), \qquad e \geqslant 0$$

if and only if $a_{ij} \equiv b_{ik} (\mathrm{mod}\, p^e)$ for all k. We remark that in A'_0 the congruence relation is defined in an obvious way: two polynomials with integral coefficients are congruent modulo p^e if and only if the coefficients of similar terms are congruent modulo p^e.

The following remarks are easy to prove and are left to the reader (see [4], pag. 157).

REMARK 2.1. Let a_i, $b_i \in A_n'$, $0 \leqslant i \leqslant m$. Then the system of congruences:

$$a_i \equiv b_i \pmod{p^e}, \qquad 0 \leqslant i \leqslant m$$

is equivalent to:

$$a^{(i)} \equiv b^{(i)} \pmod{p^{e+1}}, \qquad 0 \leqslant i \leqslant m$$

where $a^{(i)} = a_{0,\, p^i} + pa_{1,\, p^{i-1}} + \ldots + p^i a_i$ and similar $b^{(i)}$.

Let us denote B_n' the subset of B_n consisting of all elements $\alpha = (\alpha_1, \alpha_1, \alpha_2, \ldots)$ such that $\alpha_m \in A_n'$ for all $m \geqslant 0$.

REMARK 2.2. Let $\alpha, \beta \in B_n'$. If $\alpha \circ \beta$ means one of the elements $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$ defined as above, then for all $m \geqslant 0$, $(\alpha \circ \beta)_m$ is a polynomial with integral coefficients and without free terms in $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m$:

$$(\alpha + \beta)_m = s_m(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m),$$

$$(\alpha\beta)_m = p_m(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m).$$

2) Now let k be a perfect field of characteristic $p > 0$. Let us define

$$W(k_n) = \{\alpha / \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)\}$$

with $\alpha_i \in k((t_1)) \ldots ((t_n)) = k_n$.

If

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots),$$

$$\beta = (\beta_0, \beta_1, \ldots, \beta_m, \ldots),$$

are elements of $W(k_n)$, let us define:

$$\alpha + \beta = ((\alpha + \beta)_0, (\alpha + \beta)_1, \ldots, (\alpha + \beta)_m, \ldots),$$

$$\alpha\beta = ((\alpha\beta)_0, (\alpha\beta)_1, \ldots, (\alpha\beta)_m, \ldots),$$

where $(\alpha + \beta)_m$, $(\alpha\beta)_m$ are defined as above

REMARK 2.3. $W(k_n)$ endowed with these operations is an integral domain.

The mapping (1) can be defined for every $\alpha \in W(k_n)$ and one has:

$$p^i \alpha = (0, \ldots, 0, \alpha_{0,\, p^i}, \alpha_{1,\, p^i}, \ldots).$$

(If $\varepsilon \in k_n$ then ε_{p^i} is defined by raising the coefficient of ε to the power p^i. Since k is perfect i can be also negative) and so for every element $\alpha \in W(k_n)$ there exists an element $\alpha_{,p^{-i}} \in W(k_n)$ such that $(\alpha_{,p^{-i}})_{(p^i)} = \alpha$. So the ideal of $W(k_n)$ generated by p is the same to the set of all vectors $\alpha = (\alpha_0, \alpha_1, \dots)$ such that $\alpha_0 = 0$. But now it is clear that the residue ring $W(k_n)/(p)$ is canonically isomorphic to k_n.

Now we shall define a functions:

$$v_n\colon W(K_n) \to \mathbb{Z}$$

as follows if $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m, \dots)$, then $v_n(\alpha) = \infty$ if $\alpha = 0$ and $v_n(\alpha) = r$, where r is the greatest natural number such that $\alpha_i = 0$ for all $i < r$. Particularly $v_n(p) = v(p \cdot 1) = 1$. Now we can prove:

REMARK 2.4. $W(k_n)$ is a complete rank one and discrete valuation ring whose residue field is k_n.

For every $\alpha \in k_n$ denote

$$\{\alpha\} = (\alpha, 0, 0, \dots, 0, \dots) \in W(k_n).$$

If $\alpha \in k_n$ then

$$p^s\{\alpha\} = (0, \dots, 0, \alpha_{p^s}, 0, \dots).$$

Therefore, if $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m, \dots)$ is an element of $W(k_n)$ then it has the p-adic representation:

$$\alpha = \sum_{m=0} p^m \{\alpha_{m, p^{-m}}\}. \tag{2}$$

Let $T(k_n)$ be the quotient field of $W(K_n)$ and denote also by v_n the natural extension of v_n to $T(K_n)$. The (rank one and discrete) valuation v_n together with the natural valuation u of $k_n = k((t_1)) \dots ((t_n))$ define on $T(k_n)$ a rank $n+1$ and discrete valuation u_n, defined as follows: If $\alpha \in T(K_n)$, then $u_n(\alpha) = (v_n(\alpha), u(\alpha/p^{v_n(\alpha)})^*)$ (here ε^* means the image of $\varepsilon \in W(k_n)$ in the residue field). It is easy to see that $(T(k_n), u_n)$ is a $(n+1)$-complete field. We shall say that $(T(k_n), v_n)$ is the field of *Teichmuler-Witt vectors.*

THEOREM 2.5. Let k be a perfect field of characteristic $p > 0$ and let $(T(k), v)$ be the field of Witt's vectors associated to k. Then for every natural number $n \geqslant 1$, there exists a natural isomorphism of local fields φ_n between $(T(k_n), v_n)$ and $(T(k)\{\{t_1\}\} \dots \{\{t_n\}\}, w_n)$.

PROOF. For the sake of simplicity we shall define only φ_1; the general definition of φ_n will be left to the reader.

Let $\alpha \in T(k_1)$ be given by (2). Let us denote $\beta_n = \alpha_{m \cdot p^{-m}}$. Then $\beta_m = \sum_{s > -\infty} a_{m,s}^{p^{-m}} t_1^s$. Since $b_s = \sum_{m=0} \{a_{m,s}^{p^{-m}}\} p^m$ is an element of $T(k)$ for all s, we can assign to α the element $\varphi_1(\alpha) = \sum_{-\infty}^{+\infty} b_s t_1^s$ of $T(k)\{\{t_1\}\}$. That is the definition of φ_1.

3. Some rank two completions.

Let (K, v) be a valued field where v is a rank one and discrete valuation. Let $(K(X), u)$ be an extension of v to $K(X)$ (X indeterminate over K) such that u is of rank two. In this section we are dealing with the description of the maximal completion of $(K(X), u)$ (see [12], Ch. II for the notion of maximal completion).

1) Now we shall make some considerations about the valuation u (see [10]).

Let O_u be the valuation ring of u and let $(0) \subset M_1 \subset M_2$ be all the prime ideals of O_u. Let O_w be the quotient ring of O_u relative to the complement of M_1 and let w be the valuation of the $K(X)$ associated to O_w. Denote also by v' the valuation of the field $k_w = O_w / M_1$ associated to the valuation ring O_u / M_1. Since one has $G_w \simeq Z \simeq G_{v'}$ we can assume that $G_u \simeq G_w \times G_{v'}$, ordered lexicographically. Moreover the valuation u can be defined (up to equivalence) as follows. Let t be an element of $K(X)$ such that $u(t) = (1, 0)$. Then $w(t) = 1$. If $x \in K(X)$ then $u(x) = (a, b)$ where $a = w(x)$ and $b = v'((xt^{-a})^*)$, (y^* being the image in k_w of $y \in O_w$).

With the above notations, two cases are possible (see [10]): $O_w \cap K = K$ (then u is called of the *first kind*) and $O_w \cap K = O_v$ (we shall that u is of *second kind*). We shall describe the maximally completion of u in both cases, separately.

2) Let us assume that u is of first kind. Then w is trivial on K and so it is defined by an irreductible polynomial of $K[X]$ or is the valuation at the infinity. Since in the case when w is the valuation at the infinity the things are quite similarly to the case $f = X$, we can consider only the case when w is defined by an irreductible polynomial f. In this case $k_w \simeq K(\alpha)$ where α is a suitable root of f. Then v' is an extension of v to $K(\sigma)$. According to [10], u can be defined as follows: if $F \in K[X]$, let us write: $F = F_0 + F_1 f + \ldots + F_r f^r$ where $\deg F_i < \deg f$, $0 \leq i \leq r$. Then one has

$$u(F) = \inf \left(i, v'(F_i(\alpha))\right).$$

Let $(K_2', \widetilde{w}')$ be the completion of $K(X)$ relative to w. It is well known that K_2' is canonically isomorphic to the field of formal power series in one variable t' over the field $K(\alpha)$. By this isomorphism we identify the polynomial $F = F_0 + F_1 f + \ldots + F_r f^r$, $\deg F < \deg f$ to $F_0(\alpha) + F_1(\alpha)t' + \ldots + F_r(\alpha)t'^r$.

Let $(K_1, \widetilde{v}')$ be the completion of $(K(\alpha), v')$ and let $K_2 = K_1((t))$. Define on K_2 a valuation $\widetilde{u}$ as follows: if $\alpha = \sum_{n > -\infty} a_n t^n \in K_2$, then $\widetilde{u}(\alpha) = (n, \widetilde{v}(a_n))$ where n is the last integer such that $a_n \neq 0$. It is clear that $(K_2', \widetilde{w}')$ is contained naturally as a subfield in $(K_2, \widetilde{u})$. Moreover $\widetilde{u}$ is a rank two discrete valuation whose residue field is just k_u. If $\widetilde{w}$ is the rank one valuation on K_2, associated to $\widetilde{u}$ then $O_{\widetilde{w}} = K_1[[t]]$ and its residue field is just $(K_1, \widetilde{v}_1')$. Finally it is clear that $(K_2, \widetilde{u})$ is a maximally completion of $(K(X), u)$.

3) Let us assume that $O_w \cap K = O_v$ and $M_1 \cap K = M_v$ (i.e. u is of second kind). Clearly one has $O_u \cap K = O_v$, $M_2 \cap K = M_v$ and $O_{v'} = O_u / M_1$. Since $O_{v'} \cap k_v = k_v$ it follows that it exists a valuation on k_w which is trivial on k_v. This means that k_w / k_v is a transcendental extension and so w is a r.t. extension (see [11]) of v to $K(X)$. Then according to [1, Theorem 2.1], w is defined by a minimal pair (a, s) where a is algebraic and separable over K and v has a unique extension, denoted v_1, to $K(a)$ (see [2, Theorem 3.8]). Let f be the minimal polynomial of a over K, $\gamma = w(f)$ and e the smallest natural number such that $e\gamma \in G_{v_1}$. Let $h \in K[X]$ be such that $\deg h < \deg f$ and that $w(h(X)) = v_1(h(a)) = e\gamma$, and $r = f^e/h$. According to [1, Theorem 2.1] one has: $w(r) = 0$ and r^* is transcendental over k_v. Moreover $k_w = k_{v_1}(r^*)$.

Now since v' is trivial over K_{v_1} it follows that it is defined by an irreducible polynomial $G(r^*)$ or is the valuation at the infinity. Let g be a lifting in $K[X]$ of $G(r^*)$ (see [10]). Then by [10] the valuation u is defined as follows. Let $F \in K[X]$ and let $F = F_0 + F_1 g + \ldots + F_s g^s$, $\deg F_i < \deg g$, $0 \leqslant i \leqslant s$ be the g-expansion of F. Then one has:

$$u(F) = \inf_i \left((w(F_i), 0) + iu(g)\right).$$

Before to the maximal completion of $(K(X), u)$ we shall make some comments. Let $n = \deg G(r^*)$ (relative to the variable r^*). Since $((g/h^n)^*) = G(r^*)$ and is transcendental over k_v, then according to [2, Proposition 1.1] there exists a root b of g such that $v(b - a) \leqslant \delta$. It is easy to see that for any $F \in K[X]$, $\deg G < \deg f$, one has: $(F(b)/F(a))^* = 1$, and that $(f(b)^e/h(b))^* = c$ is a root of $G(r^*)$, i.e. $G(c) = 0$. Moreover if v_2 denotes a suitable extension of v to $K(b)$ (it may be proved that v_2 is in fact the unique extension of v to $K(b)$) then $k_{v_2} = k_{v_1}(c)$.

Let $(\widetilde{K(b)}, \tilde{v}_2)$ be the completion of $(K(b), v_2)$ and let $(k_{v_2}((t)), \tilde{v}')$ be the completion of $(k_{v_1}(r^*), v')$ (in this last completion the element $G(r^*)$ goes onto t). Let $(\widetilde{K(b)}\{\{t\}\}, \tilde{u})$ be the valued field where as usual $(\widetilde{K(b)}\{\{t\}\}, w_1)$ is the local field of bounded Laurent series over $(K(b), \tilde{v}_2)$ and where $\tilde{u}$ is the rank two valuation defined by $\tilde{u}\left(\sum_{-\infty}^{+\infty} a_n t^n\right) = (\inf_n \tilde{v}_2(a_n), n_0)$. Here n_0 is the smallest integer number such that the inf on the first component is reached. The reader is refered to [12, Ch. II], to prove that $(\widetilde{K(b)}\{\{t\}\}, \tilde{u})$ is the maximally completion of $(K(X), u)$, above defined.

4) Let k be a field. We can apply the above observations to the field $K = k(X)$, X an indeterminate. Then v will be a valuation on K trivial over k and u a rank two extension of v to $K(Y) = k(X, Y)$. We leave to the reader the task to describe the maximally completion of $(k(X, Y), u)$ in both cases when u is of first or second kind.

4. Finiteness of the number of extensions of given degree.

In what follows the expression «a finite number of extensions of degree n with a given property $\mathcal{P}$, of a field K» means: there exists a finite set $\mathcal{L}$ of extension of degree n, with the property $\mathcal{P}$, of the field K such that every extension of K, with the property $\mathcal{P}$ is K-isomorphic to an element of $\mathcal{L}$.

1) Lemma 4.1. Let (K, v) be a local field. Assume that the residue field k_v is such that for any natural number m it has only a finite number of separable extensions of degree m. Then for every natural number n the field K has a finite number of extension of degree n which are *tamely ramified* (see [4], pag. 248).

Proof. Let n be a fixed natural number and let L/K be a tamely ramified extension of degree $\leqslant n$. Let p and π be respectively fixed uniformisants in K and L. Denote also v the unique extension of v to L and let l_v the residue field of (L, v). Let K_1 be the maximal unramified extension of K included in L. Then $L = K_1(\sqrt[e]{p\varepsilon})$, where ε is an unity of K_1 and $e = e(K/L)$ (see [15], pag. 89) is the ramification indes of L/K. By hypothese one has:

$$(e, q) = 1, \qquad \text{where} \qquad q = \operatorname{char} k_v .$$

Let us consider all separable extensions l_v of k_v when L runs over the set $\mathcal{L}$ of all tamely ramified extensions of K of degree $\leqslant n$. By hypothesis and by obvious condition $[l_v : k_v] \leqslant n$ it follows that there exists a finite separable extension l of k_v such that $k_v \subseteq l_v \subseteq l = k_v(b)$ for all l_v. Let N be an unramified extension of k, $N = K(a)$, $v(a) = 0$, $a^* = b$ and whose residue field is just l. Let $l \in \mathcal{L}$. Since $l_v \subseteq l$ then L_{ur}, the unramified part of L is contained in N. One has: $L = L_{ur}(\sqrt[e]{p\varepsilon})$, ε an unity of L_{ur}, $e \leqslant n$ and $(e, q) = 1$.

We want to find a finite extension of N which contains $\sqrt[e]{\varepsilon}$ for all units ε of N, and $e \leqslant n$, $(e, q) = 1$. For unit ε on N one has $\varepsilon^* \in l$ and by hypothesis there exists a finite extension Σ of l which contains all the radicals $\sqrt[e]{\varepsilon^*}$, when ε runs all unites of N and e all natural numbers smaller than n and relatively prime to q. Let $\Sigma = l(b')$ and let $S = N(c')$ be the unique unramified extension of N whose residue field is just Σ. It is clear that for every unit $\varepsilon \in N$ and every e, $e \leqslant n$ and $(e, q) = 1$, in S there exists the element $\sqrt[e]{\varepsilon}$. Let f be the smallest common multiple of all numbers $e \leqslant n$ and $(e, q) = 1$. It is clear that $S(\sqrt[f]{p}) = T$ contains all extensions L in the set $\mathcal{L}$ above considered and since T/K is a separable extension, then it has only a finite number of subfields which are extensions of K. Particularly the set $\mathcal{L}$ has only a finite number of elements, as claimed.

Let (k, v) be an n-local field. We shall say that a finite extension (L, w) of (K, v) is *tamely ramified* if $[G_w : G_v]$ the order of the quotient group G_w/G_v is relatively prime to the characteristic of k_v and the residue field l_w is a separable extension of k_v. Look at the notation in the first section. Then for every t, $0 \leqslant t \leqslant n-1$, L_{t+1}/K_{t+1} is a tamely ramified extension of local fields. One has the following result.

THEOREM 4.2. Let (K, v) be a n-local field. Assume that for every natural number m the residue field k_v has a finite number of separable extensions of degree m. Then for every natural number m', K has a finitely many extensions of degree m' which are tamely ramified.

PROOF. The proof follows by induction after n. The case $n = 1$ was treated in Lemma 4.1. Let us assume that $n > 1$ and the result is true for every $n' < n$. Look at the notation in the first section. Denote K' the residue field of (K, v_{n-1}) and v' the valuation on K' defined by the valuation ring O/M_{n-1}. It is clear that (K', v') is a $n-1$ local field whose residue field is just k_v. Hence by inductive hypothesis (K', v') has for every m' a finite number of tamely ramified extensions of degree at most m'.

Let $\mathcal{L}$ be the set of all tamely ramified extensions of (K, v) of degree at most m'. Let L'' be the composite over K' of all extension L', when (L, w) runs $\mathcal{L}$ (as above L' is the residue field of (L, w) relative to v_{n-1}).

According to the inductive hypothesis L''/K' is a finite (separable) extension. Let F be the composite of all unramified extensions L of K such that $(L, w) \in \mathcal{L}$ (we are saying that (L, w) is *unramified* if (L, w_{n-1}) is an unramified extension of (K, v_{n-1})).

Furthermore, let $L^{(1)}, w_{n-1}^{(1)})$ be the unramified extension of (K, v_{n-1}) whose residue field is just L''. It is clear that $F \subsetneq L^{(1)}$. Let $(L, w) \in \mathcal{L}$, then $\bar{L} = L^{(1)}L$ is a totally ramified extension of $(L^{(1)}, w_{n-1}^{(1)})$. Hence on has: $\bar{L} = L^{(1)}(\sqrt[e]{\pi\varepsilon})$, where Π is an uniformising element of (K, v_{n-1}), ε a unity of $L^{(1)}$ and $(e, p) = 1$ $(p = \operatorname{char} k_v)$.

If ε is unit of L'' and e is a natural number relative prime to p, then by a slight computation one see that $L''(\sqrt[e]{\varepsilon})$, is a tamely extension of (L'', w''), where w'' is the unique extension of v' to L''. Hence, according to the inductive hypothesis, the composite of all extensions of L'' of the form $L''(\sqrt[e]{\varepsilon})$, where $(e, p) = 1$, $e \leqslant m'$, ε a unit of L, is a (finite) tamely ramified extension of (L'', w''). Then one has $L''' = L''(\gamma)$ and let $L^{(1)}(y)$ be an extension of $(L^{(1)} \cdot w_{n-1}^{(1)})$ such that $\bar{y}$ (the residue of y relative to the unique extension of $w_{n-1}^{(1)}$ to $L^{(1)}(y)$) is just γ and that $[L^{(1)}(y) : L^{(1)}] = [L''(\gamma) : L'']$. According to Hensel' Lemma, it follows that for every $\varepsilon \in L^{(1)}$ and every natural e, $e \leqslant m'$ and relatively prime to p, one has: $\sqrt[e]{\varepsilon} \in L^{(1)}(y)$. But then we can deduce that for every $(L, w) \in \mathcal{L}$ one has: $K \subsetneq L \subsetneq L^{(1)}(y, \sqrt[i]{\pi})$ Since $L^{(1)}(y)/K$ is a finite separable extension, then $\mathcal{L}$ is a finite set, as claimed.

COROLLARY 4.3. Let (K, v) be a n-local field. Assume that its residue field k_v is of zero characteristic and for every natural number m has only a finite number of extensions of degree m. Then for every natural number n', K has a finite number of extensions of degree n'.

Let (K, v) be a n-local field. We utilise the notations of § 1. Assume that the residue field k_v is finite and the local field $(K_1, \bar{v}_0)$ is of characteristic 0. Then (K, v_1) is $n - 1$ local field whose residue field is just K_1. Since K_1 is a finite extension of a p-adic field it has for every number n only a finite number of extensions of degree m. Therefore by Corollary 4.3, for every natural number n', K has only a finite number of extensions of degree n', hence one has the following result:

COROLLARY 4.4. Let (K, v) be n local field. Assume that its residue field k_v is finite and the local field (K_1, v_0) (see § 1) is of characteristic zero. Then for every natural number n', K has a finite number of extensions of degree n'.