Rendiconti del Seminario Matematico della Università di Padova

K. Peiffer

A stability criterion for periodic systems with first integrals

Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 165-178

<http://www.numdam.org/item?id=RSMUP_1994__92__165_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

A Stability Criterion for Periodic Systems with First Integrals.

K. PEIFFER (*)

ABSTRACT - The link between an extension of Routh's theorem and general systems with first integrals is briefly indicated. A criterion assuring stability in the presence of first integrals is established. More precisely, it is shown that under suitable hypotheses, conditional asymptotic stability implies unconditional stability. This extends a result of D. Aeyels and R. Sepulchre [1] to non autonomous periodic systems.

1. - Introduction.

The use of first integrals in order to get stability criteria has a long history (see for instance [5] or [6]). In classical mechanics, except for energy conservation, this idea appears probably for the first time in Routh's theorem. Here we deal with a system described by the Lagrange coordinates

$$q = (q_1, \ldots, q_m)^\top, \qquad r = (r_1, \ldots, r_k)^\top,$$

the variables r being ignorable. If we suppose the kinetic energy T and the potential energy V given by

$$T = T(q, \dot{q}, \dot{r}) = \frac{1}{2}\dot{q}^\top A(q)\,\dot{q} + \dot{r}^\top B(q)\,\dot{q} + \frac{1}{2}\dot{r}^\top C(q)\,\dot{r};\ V = V(q),$$

where the matrices A, C are symmetric and positive definite, then the

(*) Indirizzo dell'A.: Institut de Math. Pure et Appliqueé, Université Catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgio.

equations of motion are derived from the Lagrange function

$$L = L(q, \dot{q}, \dot{r}) = T(q, \dot{q}, \dot{r}) - V(q)$$

and given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,, \tag{1.1}$$

$$B\dot{q} + C\dot{r} = K\,. \tag{1.2}$$

Let us suppose that this system admits the «steady state» solution

$$\sigma = (q, \dot{q}, \dot{r}) = (0, 0, \beta)$$

with constant

$$K = \overline{K} = C(0)\beta\,.$$

Without loss of generality, we can assume that $\beta = 0$ and $\overline{K} = 0$. Let us now introduce the Routh function

$$\overline{L} = \overline{L}(q, \dot{q}, \dot{r}, K) = L(q, \dot{q}, \dot{r}) - \dot{r}^{\top} K$$

and note

$$\overline{L}^* = \overline{L}^*(q, \dot{q}, K) = \overline{L}(q, \dot{q}, C^{-1}\cdot(K - B\dot{q}), K)\,.$$

Putting

$$T_2 = T_2(q, \dot{q}) = \frac{1}{2}\dot{q}^{\top}(A - B^{\top}C^{-1}B)\dot{q}\,,$$

$$T_1 = T_1(q, \dot{q}, K) = \dot{q}^{\top}B^{\top}C^{-1}K\,,$$

$$T_0 = T_0(q, K) = \frac{1}{2}K^{\top}C^{-1}K\,,$$

we get

$$\overline{L}^* = T_2 + T_1 - (V + T_0)$$

and system (1) is clearly equivalent to

$$\frac{d}{dt}\frac{\partial \overline{L}^*}{\partial \dot{q}} - \frac{\partial \overline{L}^*}{\partial q} = 0\,, \tag{2.1}$$

$$\dot{K} = 0\,. \tag{2.2}$$

Moreover, stability of the steady state solution $\sigma = (0, 0, 0)$ of system

(1) amounts to stability of the solution $(q, \dot{q}, K) = (0, 0, 0)$ of system (2).

The classical Routh's theorem states that, if the modified potential $(T_0 + V)$ has a strict local minimum at $q = 0$ for $K = 0$, then the origin $q = 0$ of system (1) or (2) is stable for perturbations such that $K = 0$, i.e. the origin is conditionally stable.

In his paper [8] of 1953, L. Salvadori has shown that in this case, without supplementary hypotheses, the origin is stable for all perturbations. In other words, conditional stability implies unconditional stability.

A similar extension does not hold for the origin of a general parametrized system

$$\dot{y} = g(y, K), \tag{3.1}$$

$$\dot{K} = 0 \tag{3.2}$$

with $y \in \mathbb{R}^n$, $K \in \mathbb{R}^k$, $g \in \mathcal{C}^1$, $g(0, 0) = 0$. As a counterexamples it suffices to consider $g(y, K) = K^2 y$. This makes it clear that the «mechanical character of the equations is important in Salvadori's extension.

Considering Routh's hypotheses and assuming moreover that, for small K, $T_0(q, K) + V(q)$ has a local strict minimum at $q = \alpha(K)$ where α is continuous and $\alpha(0) = 0$, A. M. Lyapunov stated without proof that the steady state solution $\sigma = (0, 0, 0)$ is unconditionally stable. A proof is shortly indicated by V. V. Rumjantsev [7]. But even with a similar additional assumption, i.e. if for small K, we have a Lyapunov function for the solution $y = \alpha(K)$ of (3.1), conditional stability does not imply stability for system (3). This is clearly pointed out by the example

$$\dot{y} = K^2 y(y^2 - K^2), \tag{4.1}$$

$$\dot{K} = 0 \tag{4.2}$$

where $\alpha(K) \equiv 0$, $K \in \mathbb{R}$.

REMARK 1.1. It is worth to note here that, for any K, the origin $y = 0$ is stable for (4.1) and nevertheless, the origin $(y, K) = (0, 0)$ is unstable for (4). Moreover, for any $K \in \mathbb{R}$, the function $W(y) = y^2$ is a Lyapunov function for (4.1) and its derivative is even negative definite for $K \neq 0$. But there is no neighborhood of $y = 0$ in $\mathbb{R}$, independent of K, where $W' \leqslant 0$. That is the major reason why the auxiliary function

$$F = y^2 + K^2$$

is not a Lyapunov function for (4) and Salvadori's method can not be transposed to system (3).

REMARK 1.2. We notice that the origin of (3) can be stable even if, for any $K \neq 0$, the origin $y = 0$ of (3.1) is unstable. An example is given by $g(y, K) = K^2 y(K^2 - y^2)$.

For a general system (3), not necessarily of mechanical type, we can state the following, nowadays merely trivial proposition.

PROPOSITION 1.1. *If there exist* $\mathcal{C}^1$ *real functions* $W(y, K, t)$, $U(y, k)$ *and positive constants* $\varepsilon > 0$, $\eta > 0$ *such that*

(i) $W(y, K, t) \geqslant U(y, K)$,

(ii) $U(y, 0)$ *is positive definite in* y,

(iii) $W'(y, K, t) \leqslant 0$ *for* $\|y\| < \varepsilon$, $\|K\| < \eta$,

then the origin $(y, K) = (0, 0)$ *is stable for* (3).

PROOF. The function

$$F = \max\, (W(y, K, t), \|K\|)$$

is positive definite and non-increasing along the solutions of (3) in some neighborhood of the origin. Indeed, its Dini-derivative $D_+ F$ (see for instance [6]) is such that $D_+ F \leqslant 0$. This achieves the proof. ∎

REMARK 1.3. Proposition 1.1 holds also if equation (3.1) depends on time and if we suppose $g(y, K, t)$ continuous and locally lipschitzian in y. Moreover, the domains of y and K may be restricted to some open neighborhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^k$.

Recently, D. Aeyels and R. Sepulchre [1] stated a stability result for autonomous dynamical systems with a first integral

$$\dot{x} = f(x), \tag{5.1}$$

$$G(x) = h, \tag{5.2}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^k$, h constant, $f(0) = 0$ and $G(0) = 0$. They proved that, if $x = 0$ is asymptotically stable for perturbations x_0 such that $G(x_0) = 0$, then $x = 0$ is stable for all perturbations. In other terms, conditional asymptotic stability implies stability.

Of course, if the jacobian matrix of G is of maximal rank at the origin, the implicit function theorem assures that system (5) can be brought in the form (3). In this case, the theorem is only a particular case of well known total stability results (see for instance [2], p. 445) and in the corresponding mechanical case, i.e. Lagrange systems with ignorable coordinates and dissipation, many results have been obtained by C. Risito, V. V. Rumjantsev, L. Salvadori and others. But if the rank of the jacobian matrix of G is not maximal, it may happen that (5) can not be brought in form (3) and the theorem is of full interest.

In their proof, D. Aeyels and R. Sepulchre do not use any Lyapunov or Lyapunov-like function. Therefore, the proof is completely different from that used by Salvadori in the extension of Routh's theorem and also from that of Malkin's total stability theorem. It is based only on the consideration of topological flows and so, at a first look, the autonomous character of (5.1) seems to be important. Our purpose in the next section is to extend the latter result to non autonomous periodic systems (5.1) with time dependant first integral (5.2).

2. – A stability criterion.

Let us consider the continuous functions

$$f\colon \mathbb{R} \times \Omega \to \mathbb{R}^n, (t, x) \to f(t, x)$$

and

$$G\colon \mathbb{R} \times \Omega \to \mathbb{R}^k, (t, x) \to G(t, x)$$

where $\Omega \subset \mathbb{R}^n$ is an open connected set containing the origin. We assume that f is locally lipschitzian in x, T-periodic in t and that for every $t \in \mathbb{R}$,

$$f(t, 0) = 0\,. \tag{6}$$

Moreover, we assume that G is $\mathcal{C}^1$ and satisfies

$$\frac{\partial G}{\partial t}(t, x) + \frac{\partial G}{\partial x}(t, x)\cdot f(t, x) = 0 \tag{7}$$

on $\mathbb{R} \times \Omega$. Then the origin is an equilibrium point for the system

$$\dot{x} = f(t, x) \tag{8.1}$$

which admits the first integral

$$G(t, x) = h\,, \tag{8.2}$$

h being an arbitrary constant in $\mathbb{R}^k$. Without loss of generality, we can assume that

$$G(t, 0) = 0\,. \tag{9}$$

Now, we can state the following.

THEOREM 2.1. *If G is continuous in x, uniformly on $\mathbb{R}^- = (-\infty, 0]$, and if $x = 0$ is uniformly asymptotically stable for perturbations (t_0, x_0) such that $G(t_0, x_0) = 0$, then $x = 0$ is uniformly stable for (8.1).*

PROOF. We represent the solution of (8.1) issued from $(t_0, x_0) \in I \times \Omega$ by $x(t; t_0, x_0)$ and note $B_\varepsilon = \{x\colon \|x\| < \varepsilon\}$. As f is T-periodic, it suffices to show that $x = 0$ is stable. Let's procede ab absurdo and suppose that $x = 0$ is unstable. Then there exists some $\varepsilon > 0$ such that $\overline{B_\varepsilon} \subset \Omega$ and a sequence (t_i, x_i) with $t_i > 0$, $x_i \to 0$ as $i \to \infty$ such that

$$\|x(t_i;\, 0,\, x_i)\| = \varepsilon\,. \tag{10}$$

Without loss of generality, we may assume that

$$\forall t \in [0, t_i)\colon \|x(t;\, 0,\, x_i)\| < \varepsilon\,. \tag{11}$$

Once $\varepsilon > 0$ is fixed, the hypothesis of conditional asymptotic stability can be written as follows:

$$\exists \delta > 0,\ \forall \eta > 0,\ \exists \sigma > 0,\ \forall t'' \in \mathbb{R},\ \forall x_0'\colon \|x_0'\| \leq \delta \quad \text{and} \quad G(t'', x_0') = 0\,,$$

$$\forall t \geq t'',\ \|x(t;\, t'',\, x_0')\| < \varepsilon/2 \tag{12}$$

and

$$\forall t \geq t + \sigma,\ \|x(t;\, t'',\, x_0')\| < \eta\,. \tag{13}$$

Clearly $\delta < \varepsilon/2$ and we may assume $\eta < \delta/2$, $\sigma > T$. Consider now the sequence of solutions $x(t; 0, x_i)$. As $x_i \to 0$, because of (10) (11), there exists, for every i large enough, some $t_i'\colon 0 < t_i' < t_i$ such that

$$\|x_i'\| = \delta\,, \tag{14}$$

where $x_i' = x(t_i', 0, x_i)$, and

$$\forall t \in (t_i', t_i)\colon \delta < \|x(t;\, 0,\, x_i)\| < \varepsilon\,. \tag{15}$$

Obviously, $t_i' \to \infty$ as $i \to \infty$ and for i large enough, we have

$$t_i' = t_i'' + n_i T\,, \qquad n_i \in \mathbb{N},\ t_i'' \in [0, T]\,. \tag{16}$$

The sequence (t_i'', x_i') is bounded. Therefore, going to a subsequence if necessary, we get convergence and

(17) $$(t_i'', x_i') \to (t'', x_0'), \qquad t'' \in [0, T], \ \|x_0'\| = \delta.$$

Let's first show that $G(t'', x_0') = 0$. As G is continuous,

(18) $$G(t'', x_0') = \lim_{i\to\infty} G(t_i'', x_i')$$

and, as f is T-periodic,

$$x_i = x(0; t_i', x_i') = x(-n_i T; t_i'', x_i').$$

Thus, G being a first integral, we get

$$G(t_i'', x_i') = G(t_i'', x(t_i''; t_i'', x_i')) = G(-n_i T, x(-n_i T; t_i'', x_i')) =$$
$$= G(-n_i T, x_i).$$

By hypothesis, $G(t, x) \to 0$ as $x \to 0$ uniformly on $\mathbb{R}^-$. Thus,

$$\lim_{i\to\infty} G(-n_i T, x_i) = 0$$

and, because of (18),

(19) $$G(t'', x_0') = 0.$$

Consider now the solution $x(t; t'', x_0')$. Given (12), there exists some $\theta > 0$ such that, for any t in the compact interval $[t - \theta, t + \sigma + \theta]$,

(20) $$\|x(t; t'', x_0')\| < \varepsilon/2$$

and for large i, by continuity with respect to initial conditions,

(21) $$\|x(t; t_i'', x_i') - x(t; t'', x_0')\| < \eta/2.$$

Clearly, for i large enough, $t'' - \theta < t_i'' < t_i'' + \sigma < t'' + \sigma + \theta$, and we get

(22) $$\forall t \in [t_i'', t_i'' + \sigma], \qquad \|x(t; t_i'', x_i')\| < \varepsilon.$$

As

(23) $$x(t; t_i'', x_i') = x(t + n_i T; t_i', x_i') = x(t + n_i T; 0, x_i),$$

it follows from (22) that

$$\forall t \in [t_i', t_i' + \sigma], \qquad \|x(t; 0, x_i)\| < \varepsilon.$$

But $t_i' < t_i$ and therefore (11) implies

$$\|x(t; 0, x_i)\| < \varepsilon \quad \text{on } [0, t_i'].$$

Thus $\|x(t;\, 0,\, x_i)\| < \varepsilon$ on $[0,\, t_i' + \sigma]$ and (10) now implies

$$t_i' + \sigma < t_i.$$

Putting $t = t_i'' + \sigma$ in (23) and taking account of (15), we get

$$\|x(t_i'' + \sigma;\, t_i'',\, x_i')\| = \|x(t_i' + \sigma;\, t_i',\, x_i')\| = \|x(t_i' + \sigma;\, 0,\, x_i)\| > \delta.$$

Together with (21), the last relation yields

$$\|x(t_i'' + \sigma;\, t'',\, x_0')\| > \delta - \frac{\eta}{2} > \frac{3}{4}\delta.$$

On the other hand, because of (13),

$$\|x(t'' + \sigma;\, t'',\, x_0')\| < \eta < \frac{\delta}{2}.$$

The last inequalities lead to a contradiction for i large enough. This achieves the proof. ■

REMARK 2.2. Obviously, if the set $X_t = \{x\colon G(t, x)\} = 0$ does not depend on t, hypothesis of conditional uniform asymptotic stability can be replaced by conditional asymptotic stability.

REMARK 2.3. The hypothesis that f is T-periodic is restrictive but it can certainly not simply be dropped. This is made clear by the example

$$\dot{x} = -x + \sin(ty), \qquad \dot{y} = 0$$

given in [6]. Indeed, the origin $(x, y) = (0, 0)$ is unstable although it is asymptotically stable for perturbations such that $y = 0$.

Before giving several examples, we define a function of class $\mathcal{K}$ and establish an auxiliary proposition.

DEFINITION 2.1. *A real function* a, *defined on some interval* $[0, \rho]$, $\rho > 0$, *is said to be of class* $\mathcal{K}$ *if it is continuous, strictly increasing and if* $a(0) = 0$.

Consider the equation

$$\dot{x} = g(t, x, K) \tag{24}$$

where g is defined on $\mathbb{R} \times \Omega \times \mathbb{R}^k$, continuous, T-periodic in t, locally lipschitzian in x and $g(t, 0, K) = 0$. Then we get the following:

PROPOSITION 2.1. *If there exists a $\mathcal{C}^1$ function $V(t, x, K)$ and functions a, b of class $\mathcal{K}$ with*

(i) $a(\|x\|) \leqslant V(t, x, K) \leqslant b(\|x\|)$,

(ii) $\dot{V}(t, x, K) \leqslant 0$ *on* $\mathbb{R} \times \Omega \times \mathbb{R}^k$,

(iii) *for any* $K \in \mathbb{R}^k$, $x = 0$ *is asymptotically stable.*

Then, for any compact set $A \subset \mathbb{R}^k$, stability and asymptotic stability are uniform in $(t, K) \in \mathbb{R} \times A$.

PROOF. As g is T-periodic, uniformity on $t \in \mathbb{R}$ is obvious and it suffices to establish uniformity on $K \in A$.

Clearly, for any $\varepsilon > 0$ small enough such that $B_\varepsilon \subset \Omega$, there exists some $\delta > 0$ such that, for any (x_0, K): $\|x_0\| \leqslant \delta$, $K \in \mathbb{R}^k$ and any $t \geqslant 0$, $\|x(t; 0, x_0, K)\| < \varepsilon$. Indeed, it suffices to choose $\delta > 0$, $\delta < b^{-1}(a(\varepsilon))$.

Let us now show that, for any ν: $0 < \nu < \delta$, there exists some $\sigma(\delta, \nu) > 0$ such that for any (x_0, K): $\|x_0\| \leqslant \delta$, $K \in A$ and any $t \geqslant \sigma$, $\|x(t; 0, x_0, K)\| < \nu$. If we put $\eta = b^{-1}(a(\nu))$, it suffices to establish the existence of some $t^*(x_0, K)$: $0 \leqslant t^* \leqslant \sigma$ such that $\|x(t^*; 0, x_0, K)\| < \eta$. Obviously, because of (iii), for any (x_0, K): $\|x_0\| \leqslant \delta$, $K \in A$, there is some $t^*(x_0, K) > 0$ such that $\|x(t^*; 0, x_0, K)\| < \eta/2$. By continuity, there exists some open ball $B(x_0, \rho)$ in $\mathbb{R}^n$ and some open ball $B(K, \rho)$ in $\mathbb{R}^k$ with $\rho(x_0, K) > 0$ such that, for any $(x_0', K') \in B(x_0, \rho) \times B(K, \rho)$, we get $\|x(t^*; 0, x', K')\| < \eta$. The family $B(x_0, \rho) \times B(K, \rho)$: $(x_0, K) \in \overline{B}_\delta \times A$ is clearly an open covering of the compact set $\overline{B}_\delta \times A$. Taking a finite covering with the corresponding constants t_i^*, $1 \leqslant i \leqslant j$, we can choose $\sigma = \max\{t_1^*, \ldots, t_j^*\}$ and the proposition is proved. ■

3. – A few examples.

EXAMPLE 3.1. As a first example, we consider a spherical pendulum of mass m and length l, the connection point moving along the vertical upwards z axis like $z = a \cos t$, $a > 0$. We suppose the force of gravity $\boldsymbol{f} = -mg\boldsymbol{e}_3$ and some viscous friction $\boldsymbol{F} = -k\boldsymbol{v}$ acting on the particle. The spherical coordinates φ, θ are chosen such that for the mass m

$$x = l \cos\varphi \sin\theta, \qquad y = l \sin\varphi, \qquad z = -l \cos\varphi \cos\theta + a \cos t\,.$$

These generalized coordinates are regular at the origin $\varphi = \theta = 0$. Except for an additive function of time, the Lagrangian can be written

$$L = \frac{ml^2}{2}[\dot{\varphi}^2 + \dot{\theta}^2\cos^2\varphi - \frac{2a}{l}\sin t(\dot{\varphi}\sin\varphi\cos\theta + \dot{\theta}\cos\varphi\sin\theta)] + \\ + mgl\cos\varphi\cos\theta$$

and the generalized friction forces are given by

$$F_\theta = -kl^2\dot{\theta}\cos^2\varphi + kla\sin t\cos\varphi\sin\theta,$$

$$F_\varphi = -kl^2\dot{\varphi} + kla\sin t\sin\varphi\cos\theta.$$

Putting

$$\alpha = \frac{a}{l}, \qquad \beta = \frac{k}{m}, \qquad \omega^2 = \frac{g}{l},$$

we can write the equations of motion as follows

$$\begin{aligned} \ddot{\varphi} &= -\beta\dot{\varphi} - \sin\varphi\cos\theta(\omega^2 - \alpha\cos t - \alpha\beta\sin t) - \dot{\theta}^2\cos\varphi\sin\varphi, \\ \ddot{\theta}\cos\varphi &= -\beta\dot{\theta}\cos\varphi - \sin\theta(\omega^2 - \alpha\cos t - \alpha\beta\sin t) + 2\dot{\varphi}\dot{\theta}\sin\varphi. \end{aligned} \tag{25}$$

Using the auxiliary function

$$w(\varphi, \theta, \dot{\varphi}, \dot{\theta}) = \dot{\varphi}\sin\theta - \dot{\theta}\cos\varphi\sin\varphi\cos\theta$$

which, except for a constant, represents the third component of the angular momentum, we easily verify that

$$\dot{w} = -\beta w.$$

Hence, system (25) admits the first integral

$$G(\varphi, \theta, \dot{\varphi}, \dot{\theta}, t) \equiv (\dot{\varphi}\sin\theta - \dot{\theta}\cos\theta\sin\varphi\cos\varphi)e^{\beta t} = K. \tag{26}$$

Clearly, the jacobian matrix of G is not of maximal rank at the origin so that Theorem 2.1 can be useful. The first hypothesis of this theorem is satisfied. Indeed, $G \to 0$ as $(\varphi, \theta, \dot{\varphi}, \dot{\theta}) \to 0$ uniformly on $t \in \mathbb{R}^-$. Moreover, the set $\{(\varphi, \theta, \dot{\varphi}, \dot{\theta}) : G = 0\}$ is time independent and the condition $K = 0$ permits us to reduce the problem of a spherical pendulum to that of a plane one. More precisely, the condition $K = 0$ implies one of the equations

$$\theta = 0; \qquad \varphi = 0; \qquad c\,\mathrm{tg}\,\varphi = \sin\theta; \qquad \mathrm{tg}\,\varphi = c\sin\theta$$

where the constant c satisfies $|c| \leqslant 1$.

In the last case, near the origin, we get

$$\cos\varphi = (1 + c^2 \sin^2\theta)^{-1/2},$$

$$\sin\varphi = c \sin\theta(1 + c^2 \sin^2\theta)^{-1/2},$$

$$\dot\varphi = c\dot\theta \cos\theta(1 + c^2 \sin^2\theta)^{-1}$$

and (25) reduces to

$$\ddot\theta = -\beta\dot\theta - \sin\theta\sqrt{1 + c^2 \sin^2\theta}(\omega^2 - \alpha\cos t - \alpha\beta\sin t) + + 2c^2\dot\theta^2 \cos\theta \sin\theta/(1 + c^2 \sin^2\theta). \tag{27}$$

If α is small and satisfies $\alpha\sqrt{1+\beta^2} < \omega^2$, the auxiliary function

$$V(\theta, \dot\theta, t, c) = \frac{\dot\theta^2}{2(\omega^2 - \alpha\cos t - \alpha\beta\sin t)} + \int_0^\theta \sin u \sqrt{1 + c^2 \sin^2 u}\, du$$

is positive definite in $(\theta, \dot\theta)$ and such that

$$d_1(\|\theta, \dot\theta\|) \leqslant V(\theta, \dot\theta, t, c) \leqslant d_2(\|\theta, \dot\theta\|)$$

where the real functions d_1, d_2 do not depend on c and are of class $\mathcal{K}$. If moreover

$$\alpha\sqrt{\beta^4 + \frac{5}{4}\beta^2 + \frac{1}{4}} < \beta\omega^2,$$

the derivative $\dot V$, computed along (27), is given by

$$\dot V = -\frac{\dot\theta^2}{\omega^2 - \alpha\cos t - \alpha\beta\sin t} \cdot \left[\beta + \frac{1}{2}\,\frac{\alpha\sin t - \alpha\beta\cos t}{\omega^2 - \alpha\cos t - \alpha\beta\sin t} - \frac{2c^2\dot\theta\sin\theta\cos\theta}{1 + c^2\sin^2\theta}\right]$$

and satisfies, for θ and $\dot\theta$ small,

$$\dot V \leqslant -d_3(\|\dot\theta\|)$$

where d_3 is another function of class $\mathcal{K}$, independent of c. For any c, $|c| \leqslant 1$, uniform asymptotic stability of $(\theta, \dot\theta)$ can easily be deduced from a theorem of N. N. Krasovskii [3] or V. M. Matrosov [4]. As the functions d_1, d_2, d_3 do not depend on c, uniformity with respect to c can be deduced from Proposition 2.1. Finally, uniform asymptotic stability for $(\varphi, \dot{\varphi})$ follows directly from tg $\varphi = c \sin \theta$.

The other cases are treated in a similar way and Theorem 2.1 together with Proposition 2.1 assures that the origin of system (25) is uniformly stable.

We notice that this example is merely academic because a stronger result, i.e. uniform asymptotic stability, can be obtained using the linear approximation directly in (25). Nevertheless, it shows how Theorem 2.1 can work with a time dependent first integral and illustrates the somewhat strange condition $G \to 0$ as $x \to 0$ uniformly on $\mathbb{R}^-$.

EXAMPLE 3.2. Consider the system

$$\begin{aligned} \ddot{x} &= -h(t)\,x(x\dot{x} + y^3\dot{y}^3) - kx\,, \\ \ddot{y} &= -h(t)\,y(x\dot{x} + y^3\dot{y}^3) - ky\,, \end{aligned} \tag{28}$$

with $k > 0$, $h(t) > 0$, h continuous and periodic. Here, the friction force is acting only in the radial direction and its norm is chosen rather artificially. The energy

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + kx^2 + ky^2)$$

can not be used as a Lyapunov function for (28) because its time derivative

$$\dot{E} = -h(t)(x\dot{x} + y\dot{y})(x\dot{x} + y^3\dot{y}^3)$$

is not semi-negative definite. But, as all forces are central ones, the system admits the first integral

$$G(\dot{x}, \dot{y}, x, y) \equiv x\dot{y} - \dot{x}y = K\,.$$

The condition $K = 0$ yields $y = mx$ or $x = my$ with $|m| \leqslant 1$. Considering only the former case, we get

$$\ddot{x} = -h(t)\,\dot{x}x^2(1 + m^6x^2\dot{x}^2) - kx \tag{29}$$

and now the energy

$$E = \frac{1}{2}(\dot{x}^2 + kx^2)$$

satisfies

$$\dot{E} = -h(t)\,x^2\dot{x}^2(1 + m^6x^2\dot{x}^2)\,.$$

Asymptotic stability of the origin $(x, \dot{x}) = (0, 0)$ follows from [3] or [6].

The case $x = my$ is similar and Theorem 2.1 together with Proposition 2.1 assures stability of the origin $(\dot{x}, \dot{y}, x, y) = (0, 0, 0, 0)$ for (28).

EXAMPLE 3.3. As a last example, we consider the system

$$\ddot{x} + g(x, \dot{x})\dot{x} + (a + \varepsilon \cos t)x = 0\,, \tag{30.1}$$

$$\dot{\varepsilon} = 0\,, \tag{30.2}$$

with $a > 0$, $g(x, \dot{x}) \geqslant 0$.

If $g(x, \dot{x}) = 1$, the origin $x = 0$ is clearly asymptotically stable for $\varepsilon = 0$ and, by Theorem 2.1, the origin $(x, \dot{x}, \varepsilon) = (0, 0, 0)$ is stable for (30), even if a is a square integer, i.e. even if there is parametric resonance. Moreover, as (30.1) is linear, we can conclude that $x = \dot{x} = 0$ is stable for small ε. So we find again a result of the damped Mathieu equation which, of course, is well known in the literature. We note, that for small ε, asymptotic stability of $x = \dot{x} = 0$ in (30.1) can be established directly by using the Lyapunov function

$$V(\dot{x}, x, t) = \frac{1}{2}\left(\frac{\dot{x}^2}{a + \varepsilon \cos t} + x^2\right). \tag{31}$$

In case of nonlinear friction, for instance $g(x, \dot{x}) = \dot{x}^2$ or $g(x, \dot{x}) = x^2$, Theorem 2.1 still shows that the variables $(x, \dot{x}, \varepsilon)$ can be controlled by choosing x_0, $\dot{x}_0$ and ε small enough. But (31) is not more a Lyapunov function for (30.1) and we can not conclude, as above, that $(x, \dot{x}) = (0, 0)$ is stable for (30.1). This last observation becomes evident if we consider $g(x, \dot{x}) = x^2 - \varepsilon$. Indeed, Theorem 2.1 still applies and for $\varepsilon \neq 0$, the origin $(x, \dot{x}) = (0, 0)$ is clearly unstable for (30.1).

REFERENCES

[1] D. AEYELS - R. SEPULCHRE, *Stability for dynamical systems with first integrals: a topological criterion*, Systems and Control letters, North-Holland, **19** (1992), pp. 461-465.

[2] A. ISIDORI, *Nonlinear Control Systems*, Comm. and Control. Eng. Series, 2nd ed., Springer, New York, Heidelberg, Berlin (1989).

[3] N. N. KRASOVSKII, *Problems of the Theory of Stability of Motion*, Stanford Univ. Press, Stanford, California (1963); translation of the Russian edition, Moscow (1959).

[4] V. M. MATROSOV, *On the stability of motion*, J. Appl. Math. Mech., **26** (1962), pp. 1337-1353; transl. from P.M.M., **26** (1962), pp. 885-895.

[5] K. PEIFFER, *A remark on the stability of steady state motions*, Diff. Integ. Eqs., 4 (6), (1991) pp. 1195-1207.
[6] N. ROUCHE - P. HABETS - M. LALOY, *Stability Theory by Liapunov's Direct Method*, Applied Mathematical Sciences, 22, Springer, New York, Heidelberg, Berlin (1977).
[7] V. V. RUMJANTSEV, *On the stability of steady state motions*, J. App. Math. Mech., **32** (3) (1968), pp. 517-521; transl. from P.M.M., **32**(3) (1968), pp. 504-508.
[8] L. SALVADORI, *Un osservazione su un criterio di stabilità del Routh*, Rend. Ac. Sci. Fis.-Mat. Napoli, **20**, ser. 4 (1953), pp. 269-275.

Manoscritto pervenuto in redazione il 17 febbraio 1993.