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On the Location of Zeros of Solutions of Non-homogeneous Linear Differential Equations.

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ABSTRACT - For any non-homogeneous linear differential equation of arbitrary order, having rational functions for coefficients, we determine precisely, those rays $\operatorname{Arg} z = \phi$ which have the property that for any $\varepsilon > 0$, there is a solution of the equation which has infinitely many zeros tending to ∞ in the sector $|\operatorname{Arg} z - \phi| < \varepsilon$. We remark that it is possible for there to be infinitely many such for a given equation.

1. Introduction.

For a non-homogeneous linear differential equation of arbitrary order n, having polynomial coefficients,

$$w^{(n)} + R_{n-1}(z) w^{(n-1)} + \ldots + R_0(z) w = Q(z), \tag{1.1}$$

the following result was proved in [2] concerning the possible location of the zeros of the solutions of (1.1):

THEOREM 1.1. Given an equation (1.1) where $n \geqslant 2$, and where $R_0(z), \ldots, R_{n-1}(z)$, and $Q(z)$ are polynomials with $Q(z) \not\equiv 0$. Then there exist a nonnegative integer t and real numbers $s_0, s_1, \ldots, s_{t+1}$ with $-\pi = s_0 < s_1 \ldots < s_{t+1} = \pi$, such that for each $k \in \{0, 1, \ldots, t\}$, one of the following two properties holds:

(*a*) For any $\theta \in (s_k, s_{k+1})$ and any $\varepsilon > 0$, there is a solution of (1.1) having infinitely many zeros in $|\operatorname{Arg} z - \theta| < \varepsilon$.

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(b) For any $\varepsilon > 0$, any solution of (1.1) can have at most finitely many zeros in the closed sector $s_k + \varepsilon \leqslant \operatorname{Arg} z \leqslant s_{k+1} - \varepsilon$.

We remark that the results in [2] (which are reviewed in § 3 below for the reader's convenience) provide a method for the explicit calculation of the numbers s_k from the equation, and also for determining which of the two possibilities (a) or (b) in Theorem 1.1 holds for a given k. (Examples produced in [2] show that for a given equation (1.1), it is possible for property (a) to hold for some k, while property (b) holds for other k. For example, if we consider the third-order equation,

$$(1.2) \qquad w''' + (z^2+3)w'' + (2z^2+z+3)w' + (z^2+z+2)w = Q(z),$$

where $Q(z)$ is any polynomial which is not identically zero, it is shown in [2; § 9] that the set $\{s_0, \ldots, s_{t+1}\}$ consists of $s_0 = -\pi$, $s_1 = -5\pi/6$, $s_2 = -\pi/2$, $s_3 = -\pi/6$, $s_4 = \pi/6$, $s_5 = \pi/2$, $s_6 = 5\pi/6$, and $s_7 = \pi$, and that property (a) holds for $k = 0, 1, 5, 6$, while property (b) holds for $k = 2, 3, 4$.) We also remark that analogous results to Theorem 1.1 hold for equations (1.1) having more general coefficients than polynomials, such as rational functions or algebraic functions (see [2; Theorem 6.1] or § 3 below).

For a given equation (1.1) and a value s_k (where $k \in \{1, \ldots, t\}$), it is obvious that if property (a) holds for either k or $k-1$, then for any $\varepsilon > 0$, there is a solution of (1.1) having infinitely many zeros in $|\operatorname{Arg} z - s_k| < \varepsilon$. (This follows easily by applying property (a) to a ray $\operatorname{Arg} z = \theta$ where θ is sufficiently close to s_k.) However, if for this given s_k, property (b) holds for both k and $k-1$, then for a given $\varepsilon > 0$, no information is provided by Theorem 1.1 on whether or not there exists a solution of (1.1) which has infinitely many zeros in $|\operatorname{Arg} z - s_k| < \varepsilon$. If for some $\varepsilon > 0$ no such solution exists, then s_k is really extraneous since property (b) could then be improved to actually assert that for any $\varepsilon > 0$, any solution of (1.1) can have at most finitely many zeros on $s_{k-1} + \varepsilon \leqslant \operatorname{Arg} z \leqslant s_{k+1} - \varepsilon$. In fact, it is easy (see § 3) to give simple examples of equations (1.1) which possess such extraneous s_k. In this paper, we develop a simple method for determining whether or not a given s_k is extraneous, and this method is given in Theorem 4.1 in § 4. In order to develop this method, we must review how the s_k are produced and this is done in § 3. There are several reasons why methods in [2] can produce extraneous s_k. One reason is that we first deal with the homogeneous equation corresponding to (1.1), and we may find an s_k for which the homogeneous equation has solutions $f \not\equiv 0$ having infinitely many zeros around $\operatorname{Arg} z = s_k$ but for which s_k is extraneous for the non-homogeneous equation. (This is the case in the example in § 3). Another reason for extraneous s_k is that our asymptotic existence theorem produces solutions having prescribed asymptotic behaviour in sectors, and the extraneous s_k can occur as the boundary rays of these sectors. In this paper (see Theorem 6.1 in § 6 below), we deal with this problem of boundary rays by developing a process of continuing the solutions over these rays, while at the same time preserving the asymptotic properties of the solutions in the enlarged domain. This process is based on Phragmen-Lindelöf principles.

It should be noted that the problem of extraneous rays arose earlier in [5] in connection with the oscillation properties of solutions of homogeneous linear differential equations of arbitrary order n, having polynomial coefficients,

$$w^{(n)} + R_{n-1}(w)^{(n-1)} + \dots + R_0(z)w = 0 . \tag{1.3}$$

The problem for (1.3) is considerably simpler than for (1.1) because for homogeneous equations there cannot be the «mixture» of properties (a) and (b) as the following result [5; Theorem 1] shows:

THEOREM 1.2. Given an equation (1.3) where $n \geqslant 2$ and where $R_0(z), \dots, R_{n-1}(z)$ are polynomials. Then one of the following two properties holds:

(A) For any $\theta \in (-\pi, \pi]$ and any $\varepsilon > 0$, there is a solution $f \not\equiv 0$ of (1.3) having infinitely many zeros in the sector $|\mathrm{Arg}\, z - \theta| < \varepsilon$.

(B) There exist a nonnegative integer t and real numbers s_0, $s_1, \dots, s_{t+1}$ with $-\pi = s_0 < s_1 < \dots < s_{t+1} = \pi$, such that for any $\varepsilon > 0$, any solution $f \not\equiv 0$ of (1.3) has at most finitely many zeros in the closed sector $s_k + \varepsilon \leqslant \mathrm{Arg}\, z \leqslant s_{k+1} - \varepsilon$ for each $k \in \{0, 1, \dots, t\}$.

It was shown in [1] that a simple example of an equation (1.3) which possesses property (A) in Theorem 1.2 is,

$$w^{(n)} + z^2 w'' + zw' + w = 0 \quad \text{for any } n \geqslant 3 . \tag{1.4}$$

However for second-order equations (1.3), a classical result (see Hille [7; § 5.6] or R. Nevanlinna [9; p. 345] or Wittich [16; p. 282]) shows that all second-order equations (1.3) possess property (B) in Theorem 1.2.

For a given homogeneous equation (1.3), the results in [5] provided a method for determining which property (A) or (B) in Theorem 1.2 holds. If property (B) holds, the results in [5] give a method for explicitly calculating the numbers $s_1, s_2, \dots, s_t$ and will also determine if an s_k is extraneous in the sense that for some $\varepsilon > 0$, no solution $f \not\equiv 0$ of (1.3) has infinitely many zeros in $|\mathrm{Arg}\, z - s_k| < \varepsilon$. The problem of determining which s_k (if any) are extraneous for (1.3) was solved in [5] by also using a continuation process for the solutions. However, for the reasons indicated earlier, the process in [5] for (1.3) is of a much simpler nature than that required in this paper for (1.1). For completeness, the results from [5] are reviewed in § 10 below.

2. Concepts from the Strodt theory [10].

In this paper, we will consider equations (1.1) whose coefficients are of a more general nature than just polynomials or rational functions. We will actually treat equations (1.1) whose coefficients belong to a certain type of function field (i.e. a logarithmic differential field of rank zero) consisting of functions which are analytic in some element of a neighborhood system of sectorial regions. This neighborhood system was introduced by W. Strodt in [10; § 94], and is denoted $F(a, b)$. It has the following property (see [2; p. 269]):

LEMMA 2.1. Let V be an element of $F(a, b)$, and let $\varepsilon > 0$ be arbitrary. Then there is a constant $R_0(\varepsilon) > 0$ such that V contains the set, $a + \varepsilon \leqslant \operatorname{Arg} z \leqslant b - \varepsilon$, $|z| \geqslant R_0(\varepsilon)$.

In a logarithmic differential field of rank zero over $F(a, b)$, every element $f(z)$ is *admissible* in $F(a, b)$ (i.e. is analytic in some element of $F(a, b)$) and if $f(z)$ is not identically zero, then $f(z)$ is asymptotically equivalent to a function of the form cz^{α} (where c and α are constants with c nonzero and α real) as z tends to infinity over elements of $F(a, b)$. (The concept of a logarithmic differential field of rank zero over $F(a, b)$ was introduced by W. Strodt in [12; p. 244].) We are using the strong relations of asymptotic equivalence, $f \sim g$, and of asymptotic smallness, $f \ll g$, which were introduced by Strodt in [10; § 13]. (The relation, $f \ll g$, is a generalization of the usual relation, $f = o(g)$. The strong relation of asymptotic equivalence defined in [10; § 13] is designed to ensure that if $M(z)$ is a nonconstant logarithmic monomial of rank $\leqslant p$ (i.e. a function of the form,

$$M(z) = Kz^{a_0}(\operatorname{Log} z)^{a_1} \dots (\operatorname{Log}_p z)^{a_p} \tag{2.1}$$

for real a_j, and complex $K \neq 0$), then $f \sim M$ implies $f' \sim M'$ in $F(a, b)$ (see [10; § 28]). As usual, z^{α} and $\operatorname{Log} z$ will denote the principal branches of these functions on $|\operatorname{Arg} z| < \pi$). If $f \sim M$ in $F(a, b)$ where M is given by (2.1), then we will denote a_0 by $\delta_0(f)$, a_1 by $\delta_1(f)$ etc. If $f \equiv 0$, we will set $\delta_0(f) = -\infty$.

The following two facts are proved in [6; p. 309] and [10; § 28] respectively:

LEMMA 2.2. Let $f(z)$ be admissible in $F(a, b)$. Then:

(*A*) If $f \to 0$ in $F(a, b)$, then $zf'(z) \to 0$ in $F(a, b)$.

(*B*) If $f \ll 1$ in $F(a, b)$, then $\theta_j f \ll 1$ in $F(a, b)$ for each $j = 1, 2, \ldots$ where θ_j denotes the operator $\theta_j f = z(\operatorname{Log} z)(\operatorname{Log}\operatorname{Log} z)\ldots \ldots(\operatorname{Log}_{j-1} z) f'(z)$.

We will write $f_1 \approx f_2$ in $F(a, b)$ to mean that $f_1 \sim cf_2$ for some nonzero constant c. An admissible function $g(z)$ in $F(a, b)$ is called *trivial* in $F(a, b)$ if $g \ll z^{-\alpha}$ in $F(a, b)$ for every $\alpha > 0$. If $f \sim cz^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d > 0$, then the *indicial function* of f is the function $IF(f, \phi)$ defined by,

$$IF(f, \phi) = \operatorname{Cos}(d\phi + \operatorname{Arg} c) \quad \text{for } a < \phi < b. \tag{2.2}$$

(It is obvious that $IF(f, \phi)$ has at most finitely many zeros on (a, b).) If g is any admissible function in $F(a, b)$, we will denote by $\int g$ any primitive of g in an element of $F(a, b)$. We will require the following two facts (see [2; p. 270]):

LEMMA 2.3. Let $f \sim cz^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d > 0$. If (a_1, b_1) is any subinterval of (a, b) on which $IF(f, \phi) < 0$ (respectively, $IF(f, \phi) > 0$), then for all real α, $\exp \int f \ll z^{\alpha}$ (respectively, $\exp \int f \gg z^{\alpha}$) in $F(a_1, b_1)$.

LEMMA 2.4. Let $\alpha = a + bi$ be a complex number. Then for any $\varepsilon > 0$, we have $z^{a-\varepsilon} \ll z^{\alpha}$ and $z^{\alpha} \ll z^{a+\varepsilon}$ in $F(-\pi, \pi)$.

We will also require the following facts. The first is obvious and the second follows from [10; Lemma 30]:

LEMMA 2.5. (*a*) If b is a real number, then on $|\operatorname{Arg} z| < \pi$, we have $|z^{bi}| \leqslant e^{|b|\pi}$ and $|z^{bi}| \geqslant e^{-|b|\pi}$.

(*b*) If f is a trivial function in $F(a, b)$, then f' is also a trivial function in $F(a, b)$.

3. Preliminaries and results from [2] and [5].

Given a homogeneous equation (1.3) where the $R_j(z)$ are functions which belong to a logarithmic differential field of rank zero over some

$F(a, b)$, we can follow the procedure developed in [5; pp.6-8] (or see [2; pp. 271-274]), and thereby determine for (1.3) the critical degree, the critical equation, the logarithmic set, the full factorization polynomial, the exponential set, and the transition set on (a, b) as defined in [5]. As defined in [5; p. 8], if $M_1, M_2, \dots M_p$ are the elements of the logarithmic set for (1.3), and if $\psi_1, \psi_2, \dots \psi_p$ are admissible functions in some $F(c, d)$ such that for each j, the function ψ_j solves (1.3) and satisfies $\psi_j \sim M_j$ in $F(c, d)$ for $j = 1, \dots, p$, then we will call $\{\psi_1, \dots, \psi_p\}$ a *complete logarithmic set of solutions* of (1.3) in $F(c, d)$.

Using the above concepts, we recall the following result from [2; p. 274] or [5; p. 10-11]:

THEOREM 3.2. Given the equation (1.3) where $n \geqslant 1$ and where the functions $R_0(z), \dots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. When written in terms of the operator θ (where $\theta w = zw'$) let (1.3) have the form $\Omega(w) = 0$, where $\Omega(w) = \sum_{j=0}^{n} B_j(z) \theta^j w$. Let p be the critical degree, let $F^*(\alpha) = 0$ be the critical equation, and let $N_1, \dots, N_s$ be the distinct elements (if any) of the exponential set for (1.3). Then there exist a nonnegative integer d, with $d \leqslant n - p$, and a set $\{V_1, \dots, V_d\}$ of d distinct functions such that all of the following hold:

(*a*) For each j, the function V_j belongs to a logarithmic differential field of rank zero over $F(a, b)$, and there exists $k \in \{1, \dots, s\}$ such that $V_j \sim N_k$ over $F(a, b)$.

(*b*) If $j \neq m$, then there exists a strictly positive real number $c = c(j, m)$ such that $V_j - V_m \approx z^{-1+c}$ over $F(a, b)$.

(*c*) For each $j \in \{1, \dots, d\}$, the equation $\Omega_j(u) = 0$, where,

$$\Omega_j(u) = \Omega\left(\left(\exp \int V_j\right) u\right) \Big/ \left(\exp \int V_j\right), \tag{3.8}$$

has coefficients belonging to a logarithmic differential field of rank zero over $F(a, b)$, and has a strictly positive critical degree t_j;

(*d*) $t_1 + \dots + t_d = n - p$;

(*e*) if $k \in \{1, \dots, s\}$ and if N_k has multiplicity m as a critical monomial of the full factorization polynomial for (1.3), then $\sum_{j \in J_k} t_j = m$, where J_k is the set of all $j \in \{1, \ldots, d\}$ for which $V_j \sim N_k$ over $F(a, b)$;

(f) let Ω_0 denote Ω, and let E_1 denote the union (for j in $\{0, 1, \ldots,, d\}$) of the transition sets of the equations $\Omega_j(u) = 0$ on (a, b), say $E_1 = \{r_1, \ldots, r_q\}$ where $r_1 < r_2 < \ldots < r_q$. Let $r_0 = a$ and $r_{q+1} = b$. (If E_1 is empty, set $q = 0$.) Then in each of $F(r_0, r_1)$, $F(r_1 r_2), \ldots, F(r_q, r_{q+1})$ separately, the following hold:

(i) the equation $\Omega(w) = 0$ possesses a complete logarithmic set of solutions $\{\phi_1, \ldots, \phi_p\}$, and for each $j \in \{1, \ldots,, d\}$, the equation $\Omega_j(u) = 0$ possesses a complete logarithmic set of solutions, $\{\phi_{j,1}, \ldots, \phi_{j,t_j}\}$;

(ii) if we set $\Delta_0 = \{\phi_1, \ldots, \phi_p\}$, and

$$(3.9) \qquad \Delta_j = \left\{\left(\exp \int V_j\right) \phi_{j,1}, \ldots, \left(\exp \int V_j\right) \phi_{j,t_j}\right\}$$

for $j \in \{1, \ldots, d\}$ then the set $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_d$ is a fundamental set of solutions of (1.3).

REMARK. The functions V_j can be explicitly calculated from (1.3) (see [5; Theorem 3]), and hence the set E_1 can be explicitly calculated from (1.3).

The previous theorem can be used to discuss the oscillation properties of solutions of an equation (1.3). We now recall the following concepts which were introduced in [2; p. 276] for an equation (1.1) whose coefficients are admissible functions in some $F(a, b)$:

(A) We say that (1.1) has the *global oscillation property in* $a < \operatorname{Arg} z < b$ if the following holds: For any $\theta \in (a, b)$ and any $\varepsilon > 0$, there exist strictly positive constants δ and K, and a solution $f \not\equiv 0$ of (1.1), such that $\delta < \min\{\theta - a, b - \theta, \varepsilon\}$, and such that f is analytic and has infinitely many zeros $z_1, z_2, \ldots$, with $\lim_{m \to \infty} |z_m| = +\infty$, on the region defined by $|\operatorname{Arg} z - \theta| < \delta$ and $|z| > K$.

(B) We say that (1.1) has the *non-oscillation property in* $a < \operatorname{Arg} z < b$, if for any solution $f \not\equiv 0$ of (1.1) which is admissible in $F(a, b)$, and any $\varepsilon > 0$, there exists a constant $K = K(\varepsilon, f) > 0$ such that f is analytic and has no zeros on the set defined by,

$$(3.10) \qquad a + \varepsilon \leqslant \operatorname{Arg} z \leqslant b - \varepsilon \quad \text{and} \quad |z| > K.$$

Using Theorem 3.2, we can now deduce the following oscillation result for equations (1.3) as stated in [2; p. 277]:

THEOREM 3.3. Assume the hypothesis and notation of Theorem 3.2. Let E_2 denote the union (over all j and k in $\{1, \ldots, d\}$ with $j \neq k$) of the sets of zeros on (a, b) of the functions $IF(V_j - V_k, \theta)$. Let $E_3 = E_1 \cup E_2$ (where E_1 is defined in Part (f) of Theorem 3.2), say $E_3 = \{s_1, \ldots, s_t\}$, where $s_1 < s_2 < \ldots < s_t$. Set $s_0 = a$ and $s_{t+1} = b$. (If E_3 is empty set $t = 0$.) Then the following hold:

(*A*) If for some $j \in \{0, 1, \ldots, d\}$, the critical equation of the equation $\Omega_j(u) = 0$ possesses two distinct roots having the same real part, then (1.3) has the global oscillation property in $a < \operatorname{Arg} z < b$.

(*B*) If (1.3) does not satisfy the condition in Part (*A*), then (1.3) possesses the non-oscillation property in $s_k < \operatorname{Arg} z < s_{k+1}$ for each $k \in \{0, 1, \ldots, t\}$.

REMARKS. (*a*) In view of the Remark after Theorem 3.2, it easily follows that the set $E_3 = \{s_1, \ldots, s_t\}$ can be explicitly calculated from (1.3). It will be seen later that the numbers $s_1, \ldots, s_t$ in Theorem 1.1 for equation (1.1) are precisely the numbers $s_1, \ldots, s_t$ obtained in Theorem 3.3 for the corresponding homogeneous equation (1.3), and so they can be explicitly calculated from (1.1).

(*b*) It is easy to see that Theorem 1.2 follows immediately from Theorem 3.3 since we may take $(a, b) = (-\pi, \pi)$ when the coefficients of (1.3) are polynomials.

For non-homogeneous equations (1.1), the following result was proved in [2; p. 278]:

THEOREM 3.4. Given the equation (1.1) where $n \geqslant 2$ and the $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and where $Q(z)$ is an admissible function in $F(a, b)$ for which there is a real number β such that $Q(z) \approx z^{\beta}$ over $F(a, b)$. Using (3.1) and dividing the equation (1.1) through by a suitable real power of z, we can rewrite (1.1) in the form $\Omega(w) = \psi$, where $\Omega(w)$ is of the form (3.3), whose coefficients satisfy (3.4), and where for some complex $c \neq 0$ and some real number σ, we have $\psi \sim cz^{\sigma}$ over $F(a, b)$. We define a function $M(z)$ as follows: If σ is not a root of the critical equation, $F^*(\alpha) = 0$, of $\Omega(w) = 0$, we set $M = (F^*(\sigma))^{-1} cz^{\sigma}$. If σ is a root, say of multiplicity r, of $F^*(\alpha) = 0$, we set,

$$M(z) = K_0^{-1} cz^{\sigma} (\operatorname{Log} z)^r , \tag{3.11}$$

where K_0 is the value at $\alpha = \sigma$ of the r^{th} derivative of $F^*(\alpha)$. Let $r_1 < r_2 < \ldots < r_q$ be the points of the transition set for $\Omega(w) = 0$ on (a, b), and set $r_0 = a$ and $r_{q+1} = b$ (If the transition set is empty, set $q = 0$). Then in each of $F(r_0, r_1)$, $F(r_1, r_2)$, ..., $F(r_q, r_{q+1})$ separately, the equation (1.1) possesses a solution $w_0 \sim M$.

For a non-homogeneous equation (1.1), the function $M(z)$ in (3.11) plays a key role in the oscillation properties of (1.1). Therefore, the following concepts were introduced in [2; p. 278]:

Under the hypothesis of Theorem 3.4, the number σ will be called the *principal exponent* of (1.1). If σ is a root of $F^*(\alpha) = 0$, we will call r the *multiplicity* of σ. If σ is not a root of $F^*(\alpha) = 0$, we will say that it has multiplicity $r = 0$. Following the terminology of [10; §§ 67, 69], the function $M(z)$ will be called the *principal monomial* of (1.1), and any solution $w(z)$ of (1.1) which satisfies $w \sim M$ over some $F(c, d)$, will be called a *principal solution* of (1.1) in $F(c, d)$.

Finally, as in [2; p. 280], the oscillation properties of the solutions of (1.1) can be deduced from Theorem 3.4 as follows:

THEOREM 3.5. Given an equation (1.1), where $n \geqslant 2$, where the coefficients $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and where $Q(z)$ is an admissible function in $F(a, b)$ for which there exists a real number β such that $Q(z) \approx z^{\beta}$ over $F(a, b)$. Let σ be the principal exponent of (1.1) with multiplicity r. Applying Theorem 3.2 to the homogeneous equation (1.3) corresponding to (1.1), let p, $F^*(\alpha)$, $N_1, \ldots, N_s$, $V_1, \ldots, V_d$, and $\Omega_0, \ldots, \Omega_d$ be as defined in Theorem 3.2 and let $s_0, \ldots, s_{t+1}$ be as in Theorem 3.3. Then:

(*A*) Assume that at least one of the following two conditions is satisfied: (i) the equation $F^*(\alpha) = 0$ possesses two distinct roots α and λ such that $\operatorname{Re}(\alpha) = \operatorname{Re}(\lambda)$ and $\sigma < \operatorname{Re}(\alpha)$; (ii) the equation $F^*(\alpha) = 0$ possesses a root α of multiplicity at least $r + 1$ such that $\operatorname{Re}(\alpha) = \sigma$. Then, (1.1) possesses the global oscillation property in $a < \operatorname{Arg} z < b$.

(*B*) Let $k \in \{0, 1, \ldots, t\}$, and assume that the following condition is satisfied: (iii) there exists $j \in \{1, \ldots, d\}$ such that $IF(V_j, \theta) > 0$ on (s_k, s_{k+1}), and the critical equation of $\Omega_j(u) = 0$ possesses at least two distinct roots having the same real part. Then equation (1.1) possesses the global oscillation property in $s_k < \operatorname{Arg} z < s_{k+1}$.

(*C*) Let $k \in \{0, 1, \ldots, t\}$, and assume that neither condition (i) or (ii) in Part (*A*) holds. Assume further that the following condition holds: (iv) for every $j \in \{1, 2, \ldots, d\}$ such that the critical equation of $\Omega_j(u) = 0$ has two distinct roots having the same real part, we have

$IF(V_j, \theta) < 0$ on (s_k, s_{k+1}). Then equation (1.1) possesses the non-oscillation property in $s_k < \text{Arg } z < s_{k+1}$.

REMARKS. (*a*). It is easy to see (see [2; p. 290]) that Theorem 1.1 follows from Theorem 3.5 when we take $(a, b) = (-\pi, \pi)$. We note that the points s_k in Theorem 1.1 are the points s_k given in Theorem 3.5.

(*b*) It is easy to give simple examples of equations (1.1) for which some of the points s_k which are produced by Theorem 3.5 are extraneous in the sense described in § 1. For example, we may take the simple equation,

$$w'' + (1 - i)w' - iw = 1. \tag{3.12}$$

Of course, this equation can be explicitly solved and possesses the general solution,

$$w = i + c_1 e^{-z} + c_2 e^{iz} \quad \text{for arbitrary } c_1, c_2. \tag{3.13}$$

Since e^{-z} and e^{iz} are trivial functions in $F(0, \pi/2)$ (e.g., see Lemma 2.3), it follows that any solution of (3.12) satisfies $w \sim i$ on $F(0, \pi/2)$. It follows from Lemma 2.1, that no solution of (3.12) can have infinitely many zeros on a sector $|\text{Arg } z - (\pi/4)| < \varepsilon$ where $\varepsilon < \pi/4$. However, we now show that $\pi/4$ is an s_k produced by our method and so extraneous. To see this, we first use (3.1) to write (3.12) in the form $\Omega(w) = \psi$, where $\Omega(w)$ is of the form (3.3) whose coefficients satisfy (3.4). This form is,

$$z^{-2}\theta^2 w + ((1 - i)z^{-1} - z^{-2})\theta w - iw = 1. \tag{3.14}$$

For the homogeneous equation corresponding to (3.14), the critical degree is clearly zero, and the full factorization polynomial $H(z, v)$ in (3.6) is,

$$H(z, v) = v^2 + (1 - i - z^{-1})v - i. \tag{3.15}$$

Using the remarks at the end of § 2, we find that $H(z, v)$ has two critical monomials, $N_1 = -1$ and $N_2 = i$ and both have multiplicity 1. By definition, the set $\{N_1, N_2\}$ is the exponential set for the homogeneous equation corresponding to (3.14), and so in Theorem 3.2 (using Parts (*a*) and (*e*)), the set of functions $\{V_1, \ldots, V_d\}$ consists of two functions, $V_1 \sim N_1$ and $V_2 \sim N_2$ over $F(-\pi, \pi)$. Since $IF(V_1 - V_2, \phi)$ vanishes at $\phi = \pi/4$, it follows from Theorem 3.3 that $\pi/4$ is an s_k as we claimed. We note that by the same reasoning, $-3\pi/4$ is also an s_k, but our main result (which will be stated in the next section) will show that $-3\pi/4$ is not extraneous because both of the functions e^{-z} and e^{iz} have large growth in $F(-\pi, -\pi/2)$.

4. Main result.

We now state one of our main results. The proof will be concluded in § 8.

THEOREM 4.1. Given an equation (1.1), where $n \geqslant 2$, where the coefficients $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and where $Q(z)$ is an admissible function in $F(a, b)$ for which there exists a real number β such that $Q(z) \approx z^\beta$ over $F(a, b)$. Applying Theorem 3.2 to the homogeneous equation (1.3) corresponding to (1.1), let p, $F^*(\alpha)$, $N_1, \ldots, N_s$, $V_1, \ldots, V_d$, and $\Omega_0, \ldots, \Omega_d$ be as defined in Theorem 3.2, and let $s_0, \ldots, s_{t+1}$ be as in Theorem 3.3. Let $k \in \{1, \ldots, t\}$ and assume that equation (1.1) possesses the non-oscillation property in both $s_{k-1} < \operatorname{Arg} z < s_k$ and $s_k < \operatorname{Arg} z < s_{k+1}$ (see Theorem (3.5)). Then:

(A) Assume that at least one of the following two conditions is satisfied: (i) There exists $j \in \{1, \ldots, d\}$ such that $IF(V_j, s_k) = 0$; (ii) There exist distinct elements j and m in the set $\{1, \ldots, d\}$ such that $IF(V_j - V_m, s_k) = 0$ while $IF(V_j, s_k) > 0$ and $IF(V_m, s_k) > 0$. Then, there is an admissible solution $f(z)$ of (1.1) in $F(s_{k-1}, s_{k+1})$ such that for any $\varepsilon > 0$, $f(z)$ possesses infinitely many zeros $z_1, z_2, \ldots,$ with $\lim_{n \to \infty} |z_n| = +\infty$, which lie in the sector $|\operatorname{Arg} z - s_k| < \varepsilon$.

(B) Assume that neither condition (i) or (ii) in Part (A) holds. Then (1.1) has the non-oscillation property in $s_{k-1} < \operatorname{Arg} z < s_{k+1}$.

REMARKS. 1) Theorem 4.1 gives a simple criterion for determining when an s_k, for $k \in \{1, \ldots, t\}$, is extraneous, namely when Part (B) holds. In the case where the coefficients of (1.1) are rational functions (so that $(a, b) = (-\pi, \pi)$), one can ask if the negative real axis (i.e., $s_{t+1} = \pi$) is also extraneous in the sense described in § 1. Clearly, Theorem 4.1 does not apply directly to $k = t + 1$, but one can use it indirectly to test $s_{t+1} = \pi$ by the following simple device: We make the change of independent variable $\zeta = -z$ in (1.1) which converts the negative real axis into $\operatorname{Arg} \zeta = 0$. We then determine whether the value 0 is an s_k for the transformed equation (using Theorem 3.3), and, if so, we then use Theorems 3.5 and 4.1 to test it.

2) The method of proof of Theorem 4.1 will show that if either (i) or (ii) in Part (A) holds, then the sequence of zeros of the solution $f(z)$ in

Part (A) will have a strictly positive exponent of convergence (see [14; p. 50]).

5. Continuation theorems.

DEFINITION 5.1. Given an equation (1.3) where we assume the hypothesis and notation of Theorem 3.2. Let $\{M_{0,1}, \ldots, M_{0,t_0}\}$ be the logarithmic set of $\Omega(w) = 0$ (so that $t_0 = p$) and for $j \in \{1, \ldots, d\}$, let $\{M_{j,1}, \ldots, M_{j,t_j}\}$ denote the logarithmic set for the equation $\Omega_j(u) = 0$. Setting $V_0 \equiv 0$, and letting $V_1, \ldots, V_d$ be as in Theorem 3.2, the set of n functions

$$M_{j,q} \exp \int V_j \quad \text{for } 0 \leqslant j \leqslant d, \quad 1 \leqslant q \leqslant t_j, \tag{5.1}$$

will be called the *asymptotic set for* (1.3). (We note that the elements of the asymptotic set are admissible in $F(a, b)$.) If $\{G_1, \ldots, G_n\}$ denotes the asymptotic set for (1.3), and $\{f_1, \ldots, f_n\}$ is a fundamental set of solutions of (1.3) consisting of functions which are admissible in $F(c, d)$, where (c, d) is a subset of (a, b), and satisfying $f_j/G_j \to 1$ in $F(c, d)$ for $1 \leqslant j \leqslant n$, then we will call $\{f_1, \ldots, f_n\}$, a *basic fundamental set for* (1.3) in $F(c, d)$. (Thus, Theorem 3.2 asserts the existence of a basic fundamental set for (1.3) in each of $F(r_0, r_1), \ldots, F(r_q, r_{q+1})$ separately, and hence in each $F(s_k, s_{k+1})$ separately, for $0 \leqslant k \leqslant t$, where the s_k are as in Theorem 3.3.)

DEFINITION 5.2. Given an equation (1.1) where we assume the hypothesis and notation of Theorem 3.4. Let M denote the principal monomial of (1.1). Any solution $w_0(z)$ of (1.1) which is admissible in $F(c, d)$, where (c, d) is a subset of (a, b), and which satisfies $w_0/M \to 1$ in $F(c, d)$ will be called a *solution of principal type* for (1.1) in $F(c, d)$. (Thus, as in Definition 5.1, the result in Theorem 3.4 asserts the existence of a solution of principal type for (1.1) in each $F(s_k, s_{k+1})$ separately, for $0 \leqslant k \leqslant t$, where the s_k are the points obtained by applying Theorem 3.3 to the homogeneous equation corresponding to (1.1).)

We now state our continuation theorems. The proofs will be given in § 7.

THEOREM 5.1. Given the equation (1.3) where $n \geqslant 1$ and where the functions $R_0(z), \ldots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let $s_0, \ldots, s_{t+1}$ be as in Theorem 3.3. Then for any $k \in \{1, \ldots, t\}$, the equation (1.3) possesses a basic fundamental set in $F(s_{k-1}, s_{k+1})$.

THEOREM 5.2. Given the equation (1.1) where $n \geqslant 2$ and the $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and where $Q(z)$ is an admissible function in $F(a, b)$ for which there is a real number β such that $Q(z) \approx z^{\beta}$ over $F(a, b)$. Let $s_0, \ldots, s_{t+1}$ be the points obtained by applying Theorem 3.3 to the homogeneous equation corresponding to (1.1). Then, for any $k \in \{1, \ldots, t\}$, the equation (1.1) possesses a solution of principal type in $F(s_{k-1}, s_{k+1})$.

6. Preliminary results for continuation theorems.

We will require several preliminary results. The first is a combination of several Phragmen-Lindelöf principles whose proofs can be found in [14; pp. 176-180].

LEMMA 6.1. Let $f(z)$ be analytic and of finite order of growth in a closed sectorial region of the form $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant K$. Then there exists $\delta > 0$ such that for any real numbers c and d, with $\alpha \leqslant c < d \leqslant \beta$ and $d - c < \delta$, for which the limits,

$$(6.1) \qquad L_1 = \lim_{r \to +\infty} f(re^{ic}) \quad \text{and} \quad L_2 = \lim_{r \to +\infty} f(re^{id}),$$

exist and are finite, the following conclusions hold: $L_1 = L_2$, and $f(z) \to L_1$ as $z \to \infty$ in $c \leqslant \arg z \leqslant d$.

LEMMA 6.2. Given the equation (1.1) where the coefficients $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and where either $Q(z) \equiv 0$ or $Q(z)$ is an admissible function in $F(a, b)$ for which there is a real number β such that $Q(z) \approx z^{\beta}$ over $F(a, b)$. Let $f(z)$ be a solution of (1.1) which is admissible in $F(a, b)$. Then, for any real numbers c and d, with $a < c < d < b$, there exists $K > 0$ such that f is analytic and of finite order in the closed sectorial region defined by, $c \leqslant \operatorname{Arg} z \leqslant d$, $|z| \geqslant K$.

PROOF. For homogeneous equations (i.e., $Q(z) \equiv 0$) the conclusion was proved in [5; Lemma 8.2] by using the method developed in [3] for transforming the equation to the unit disk and applying the Valiron-Wiman theory (see [3; p. 149]). The same proof also works in the non-homogeneous case because the non-homogeneous term $Q(z)$, when transformed to the unit disk, becomes a function whose growth is negligible with respect to the growth of a solution of positive order.

LEMMA 6.3. Let $f(z)$ be analytic and of finite order of growth in a closed sectorial region of the form $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant K$. Then, for any real numbers α_1 and β_1 with $\alpha < \alpha_1 < \beta_1 < \beta$, there exists $K_1 > 0$ such that $f'(z)$ is analytic and of finite order on $\alpha_1 \leqslant \arg z \leqslant \beta_1$, $|z| \geqslant K_1$.

PROOF. This follows immediately from Cauchy's formula for derivatives.

LEMMA 6.4. Let $f(z)$ be an admissible function in $F(a, b)$ and let $c \in (a, b)$. Assume that there exist real numbers a_1 and b_1 with $a < a_1 < c < b_1 < b$ such that f is analytic and of finite order of growth on the closed sectorial region $a_1 \leqslant \operatorname{Arg} z \leqslant b_1$, $|z| \geqslant K$, for some $K > 0$. Assume that for some complex number σ, we have $f \to \sigma$ over $F(c, b)$. Assume also that for some positive integer k, there exist complex constants $c_1, \ldots, c_k$, and distinct real numbers $\lambda_1, \ldots, \lambda_k$ such that,

$$f = \sum_{j=1}^{k} c_j z^{i\lambda_j}(1 + E_j) + o(1) \quad \text{over } F(a, c), \tag{6.2}$$

where $E_1, \ldots, E_k$ are admissible functions in $F(a, c)$ which are all $o(1)$ in $F(a, c)$. Then the following two conclusions hold:

(A) $f \to \sigma$ over $F(a, b)$.

(B) If $\sigma = 0$ then all $c_j = 0$ for $1 \leqslant j \leqslant k$.

PROOF. We first prove Part (B) by induction on k. We assume that $\sigma = 0$. If $k = 1$ in (6.2), we set $g = f/z^{i\lambda_1}$. Since $z^{i\lambda_1}$ is bounded from below on $|\operatorname{Arg} z| < \pi$ by a nonzero constant (see Lemma 2.5), it follows from the hypothesis that $g \to 0$ over $F(c, b)$ and $g \to c_1$ over $F(a, c)$. Since g is clearly of finite order on $a_1 \leqslant \operatorname{Arg} z \leqslant b_1$, for $|z|$ sufficiently large, it easily follows from Lemmas 2.1 and 6.1 that $c_1 = 0$ proving that Part (B) holds for $k = 1$.

We now assume that $k > 1$ and that Part (B) holds for $k - 1$. If f satisfies (6.2), we set $g = f/z^{i\lambda_k}$, so that over $F(a, c)$, we have

$$g = c_k(1 + E_k) + \sum_{j=1}^{k-1} c_j z^{i(\lambda_j - \lambda_k)}(1 + E_j) + E, \tag{6.3}$$

where $E \to 0$ by Lemma 2.5. We now compute zg' from (6.3), and observe that zE' and all zE_j' tend to zero over $F(a, c)$ by Part (A) of Lemma 2.2. Since the functions $z^{i(\lambda_j - \lambda_k)}$ are all bounded on $|\mathrm{Arg}\, z| < \pi$ by Lemma 2.5, the resulting relation has the form,

$$(6.4)\quad zg' = \sum_{j=1}^{k-1} c_j i(\lambda_j - \lambda_k) z^{i(\lambda_j - \lambda_k)}(1 + E_j) + o(1) \qquad \text{on } F(a, c).$$

Again by Lemma 2.5, we have $g \to 0$ over $F(c, b)$ so that by Lemma 2.2, we have $zg' \to 0$ over $F(c, b)$. Finally, we note that zg' is of finite order on some closed region $a_2 \leqslant \mathrm{Arg}\, z \leqslant b_2$, $|z| \geqslant K_1$ where $a_1 < a_2 < c < b_2 < b_1$, by Lemma 6.3, and we note also that the numbers $\lambda_j - \lambda_k$ are all distinct for $1 \leqslant j \leqslant k - 1$. Thus, by the induction hypothesis, it follows from (6.4) that $c_j i(\lambda_j - \lambda_k) = 0$ for $1 \leqslant j \leqslant k - 1$. Since the λ_j are distinct, we have $c_j = 0$ for $1 \leqslant j \leqslant k - 1$, and so from (6.2) we have

$$(6.5)\qquad f = c_k z^{i\lambda_k}(1 + E_k) + o(1) \qquad \text{over } F(a, c).$$

Since we have proved the result for $k = 1$, we can now conclude that $c_k = 0$ also, proving Part (B) by induction.

Proof of Part (A). We set $f_1 = f - \sigma$ so that $f_1 \to 0$ over $F(c, b)$. We distinguish two cases.

Case I. The set $\{\lambda_1, \ldots, \lambda_k\}$ in (6.2) does not contain zero. Thus, writing f_1 as $f - \sigma z^{i0}$ and using (6.2), we can apply Part (B) to f_1 to conclude that $\sigma = 0$ and $c_j = 0$ for $1 \leqslant j \leqslant k$. Thus $f_1 \to 0$ in $F(a, c)$ also. Using Lemmas 2.1 and 6.1, it follows that there is a $\delta > 0$ such that $f_1 \to 0$ as $z \to \infty$ in $|\mathrm{Arg}\, z - c| \leqslant \delta/3$ and thus $f_1 \to 0$ over $F(c - (\delta/3), c + (\delta/3))$. From [10; Lemma 97], we can conclude that $f_1 \to 0$ over $F(a, b)$ and Part (A) is proved in this case.

Case II. The set $\{\lambda_1, \ldots, \lambda_k\}$ in (6.2) contains zero, say, $\lambda_m = 0$. Again we set $f_1 = f - \sigma$, and we have from (6.2) that in $F(a, c)$,

$$(6.6)\qquad f_1 = \sum_{j \neq m} c_j z^{i\lambda_j}(1 + E_j) + (c_m - \sigma) z^{i\lambda_m} + o(1).$$

Since $f_1 \to 0$ over $F(c, b)$, we can conclude from Part (A) that $c_j = 0$ for $j \neq m$ and $c_m = \sigma$. Thus $f_1 \to 0$ over $F(a, c)$ also, and as in *Case I*, we again obtain $f_1 \to 0$ over $F(a, b)$ proving Part (A) completely.

7. Proofs of Theorems 5.1 and 5.2.

We begin with some notation:

Definition 7.1. Let $M_1 = z^{\alpha}(\mathrm{Log}\, z)^m$ and $M_2 = z^{\sigma}(\mathrm{Log}\, z)^n$, where

α and σ are complex numbers, and m and n are nonnegative integers. We write $M_1 \approxeq M_2$ if $\operatorname{Re}(\alpha) = \operatorname{Re}(\sigma)$ and $m = n$. (Thus, $M_1 \approxeq M_2$ if and only if $M_1/M_2 = z^{i\lambda}$ for some real number λ).

DEFINITION 7.2. Given a homogeneous equation (1.3) whose coefficients $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let q be the critical degree of (1.3), and let $\{P_1, \ldots, P_q\}$ be the logarithmic set for (1.3), say,

$$(7.1) \qquad P_j = z^{\alpha_j}(\operatorname{Log} z)^{n_j} 1 \quad \text{for } j = 1, \ldots, q,$$

where α_j is complex and n_j is a nonnegative integer. Let $\alpha_j = \sigma_j + i\lambda_j$ where σ_j and λ_j are real, and consider all the triples $A_j = (\sigma_j, n_j, \lambda_j)$. We arrange these triples in decreasing lexicographic order, say, $A_{j_1}, \ldots, A_{j_q}$ (so that for each k, we have $\sigma_{j_k} \geqslant \sigma_{j_{k+1}}$, and if equality holds, we have $n_{j_k} \geqslant n_{j_{k+1}}$, and if equality holds again, we have $\lambda_{j_k} \geqslant \lambda_{j_{k+1}}$. Since the triples are all distinct, this order is a total order). The corresponding q-tuple $(P_{j_1}, \ldots, P_{j_q})$ will be called the *ordered logarithmic system* for (1.3). By partitioning this system by keeping together only those triples which have the same pairs (σ_j, n_j) and if we set $Q_m = P_{j_m}$ for $m = 1, \ldots, q$, then we can assume that the ordered logarithmic system has the form $(Q_1, \ldots, Q_q)$ and can be partitioned as follows:

$$(7.2) \qquad (Q_1, Q_2, \ldots, Q_{d_1}), \quad (Q_{d_1+1}, \ldots, Q_{d_2}), \ldots, (Q_{d_r+1}, \ldots, Q_q),$$

where by Definition 7.1 and Lemmas 2.4 and 2.5(a), we have on $F(-\pi, \pi)$,

$$(7.3) \qquad Q_1 \approxeq \ldots \approxeq Q_{d_1} \quad \text{and} \quad Q_m = o(Q_1) \quad \text{for } m > d_1,$$

and in general (setting $d_{r+1} = q$),

$$(7.4) \quad Q_{d_j+1} \approxeq \ldots \approxeq Q_{d_{j+1}} \quad \text{and} \quad Q_m = o(Q_{d_j+1}) \quad \text{for } m > d_{j+1},$$

over $F(-\pi, \pi)$ for $1 \leqslant j \leqslant r$. We will call (7.2) the *canonical partition* of $(Q_1, \ldots, Q_q)$.

PROOF OF THEOREM 5.1. We are given an equation (1.3), and we assume the notation in Theorems 3.2 and 3.3 when we apply these results to (1.3). Let $k \in \{1, \ldots, t\}$, and set $H_m = \exp \int V_m$ for $1 \leqslant m \leqslant d$, and set $H_0 \equiv 1$. In each of (s_{k-1}, s_k) and (s_k, s_{k+1}), none of the indicial functions $IF(V_j, \theta)$ or $IF(V_j - V_m, \theta)$ for $j \neq m$, have any zeros and so must be of constant sign. It follows from Lemma 2.3 that if j and m are distinct elements of $\{0, 1, \ldots, d\}$ then

(7.5) $\quad$ either H_j/H_m or H_m/H_j is trivial in $F(s_{k-1}, s_k)$,

and

(7.6) $\quad$ either H_j/H_m or H_m/H_j is trivial in $F(s_k, s_{k+1})$.

It easily follows (e.g., see [2; Lemma 7.2]) that there exist permutations $\{m_0, \ldots, m_d\}$ and $\{q_0, \ldots, q_d\}$ of $\{0, 1, \ldots, d\}$ such that both of the following hold:

$$H_{m_0} \gg H_{m_1} \gg \ldots \gg H_{m_d} \quad \text{in } F(s_{k-1}, s_k); \tag{7.7}$$

$$H_{q_0} \gg H_{q_1} \gg \ldots \gg H_{q_d} \quad \text{in } F(s_k, s_{k+1}). \tag{7.8}$$

Let $\Delta_0, \ldots, \Delta_d$ denote the sets of solutions in (3.9) of equation (1.3) in $F(s_k, s_{k+1})$, and let $\bar{\Delta}_0, \ldots, \bar{\Delta}_d$ denote the corresponding sets in $F(s_{k-1}, s_k)$. We begin with Δ_{m_0}, and we let $(Q_1,, \ldots, Q_q)$ denote the ordered logarithmic system for the equation $\Omega_{m_0}(u) = 0$, with canonical partition (7.2). By Theorem 3.2, there are admissible functions $\psi_1, \ldots, \psi_q$ in $F(s_k, s_{k+1})$ and there are admissible functions $g_1, \ldots, g_q$ in $F(s_{k-1}, s_k)$ such that

$$\Delta_{m_0} = \{\psi_1 H_{m_0}, \ldots, \psi_q H_{m_0}\} \quad \text{and} \quad \psi_j \sim Q_j \quad \text{in } F(s_k, s_{k+1}), \tag{7.9}$$

and

$$\bar{\Delta}_{m_0} = \{g_1 H_{m_0}, \ldots, g_q H_{m_0}\} \quad \text{and} \quad g_j \sim Q_j \quad \text{in } F(s_{k-1}, s_k). \tag{7.10}$$

We begin with solution $\psi_1 H_{m_0}$ of (1.3). By basic existence theory (e.g. [15; Theorem 2.2]), this solution has an extension $G_1(z)$ which is admissible in $F(a, b)$. Hence in $F(s_{k-1}, s_k)$, the solution G_1 can be written as a linear combination of the elements of $\bar{\Delta}_0 \cup \ldots \cup \bar{\Delta}_d$. Thus in view of (7.5), (7.7) and (7.10), there exist constants $c_1, \ldots, c_q$ such that,

$$G_1/H_{m_0} = \sum_{j=1}^{q} c_j g_j + T_1 \quad \text{on } F(s_{k-1}, s_k), \tag{7.11}$$

where T_1 is trivial in $F(s_{k-1}, s_k)$. Thus from (7.10) and (7.3), and Lemma 2.4, we have,

$$G_1/Q_1 H_{m_0} = \sum_{j=1}^{d_1} c_j (g_j/Q_1) + o(1) \quad \text{on } F(s_{k-1}, s_k). \tag{7.12}$$

But from (7.2) and (7.10) clearly for $1 \leqslant j \leqslant d_1$, there is a real number λ_j such that,

$$g_j/Q_1 = z^{i\lambda_j}(1 + o(1)) \quad \text{over } F(s_{k-1}, s_k), \tag{7.13}$$

and the numbers $\lambda_1, \ldots, \lambda_{d_1}$ are all distinct since the functions $Q_1, \ldots, Q_{d_1}$ are distinct. In addition, we note that on $F(s_k, s_{k+1})$, we have $G_1/Q_1 H_{m_0} \to 1$ since $G_1 = \psi_1 H_{m_0}$ and $\psi_1 \sim Q_1$. Finally, we note that $G_1/Q_1 H_{m_0}$ is of finite order of growth on any sector $a_1 \leqslant \operatorname{Arg} z < b_1$ where $a < a_1 < b_1 < b$, for $|z|$ sufficiently large, since G_1 has this property by Lemma 6.2. Thus, we can apply Lemma 6.4 to conclude that,

$$G_1/Q_1 H_{m_0} \to 1 \quad \text{over } F(s_{k-1}, s_{k+1}). \tag{7.14}$$

Since $Q_1 H_{m_0}$ is in the asymptotic set for (1.3), we now have constructed the first element, namely G_1, in the basic fundamental set for (1.3) in $F(s_{k-1}, s_{k+1})$. The same proof is obviously valid for the extension G_j of $\psi_j H_{m_0}$ for each $j = 1, \ldots, d_1$, so now have solutions $G_1, \ldots, G_{d_1}$ of (1.3) such that for $j = 1, \ldots, d_1$

$$G_j/Q_j H_{m_0} \to 1 \quad \text{over } F(s_{k-1}, s_{k+1}). \tag{7.15}$$

We now consider the solution $\psi_{d_1+1} H_{m_0}$, where d_1 is as in (7.2). As above, this solution has an extension G_{d_1+1} to $F(a, b)$, which is of finite order of growth, and as in (7.11) we have,

$$G_{d_1+1}/H_{m_0} = \sum_{j=1}^{q} c_j g_j + T_{d_1+1} \quad \text{on } F(s_{k-1}, s_k), \tag{7.16}$$

where T_{d_1+1} is trivial. Dividing (7.16) by Q_1, we obtain (7.12) with G_1 replaced by G_{d_1+1}. Of course, (7.13) is still valid. But now, $G_{d_1+1}/Q_1 H_{m_0}$ tends to zero over $F(s_k, s_{k+1})$ by (7.9) and (7.3), since it agrees with ψ_{d_1+1}/Q_1. Thus by Lemma 6.4, we must have $c_j = 0$ for $1 \leqslant$ $\leqslant j \leqslant d_1$. Thus, when we divide (7.16) by Q_{d_1+1}, we are now back in the situation we had in (7.12), and we can argue as before to show $G_{d_1+1}/Q_{d_1+1} H_{m_0} \to 1$ over $F(s_{k-1}, s_{k+1})$. Similarly, all the extensions G_j of $\psi_j H_{m_0}$ for $d_1 + 1 \leqslant j \leqslant d_2$ yield solutions satisfying $G_j/Q_j H_{m_0} \to 1$ over $F(s_{k-1}, s_{k+1})$. If $r > 1$ in (7.2), we consider the extension G_{d_2+1} of $\psi_{d_2+1} H_{m_0}$. As above G_{d_2+1}/H_{m_0} can be written as a linear combination (7.16). As above, by dividing by Q_1 and using Lemma 6.4, we obtain $c_j = 0$ for $1 \leqslant j \leqslant d_1$. Then dividing the relation by Q_{d_1+1}, we obtain using Lemma 6.4 that $c_j = 0$ for $j \leqslant d_2$. Then we are back in the situation in (7.12) and we show that $G_{d_2+1}/Q_{d_2+1}H_{m_0} \to 1$ over $F(s_{k-1}, s_{k+1})$. In this way, we obtain solutions $G_1, \dots, G_q$, which are admissible and satisfy

$$G_j/Q_j H_{m_0} \to 1 \quad \text{in } F(s_{k-1}, s_{k+1}). \tag{7.17}$$

We also note that $\{G_1, \dots, G_q\}$ together with the union of the sets $\bar{\Delta}_{m_j}$ for $j = 1, \dots, d$, form a fundamental set for (1.3) in $F(s_{k-1}, s_k)$. This can be seen as follows: A dependence relation would obviously take the form,

$$\sum_{j=1}^{q} c_j (G_j/H_{m_0}) + T_2(z) \equiv 0\,, \tag{7.18}$$

where T_2 is trivial in $F(s_{k-1}, s_k)$. Dividing (7.18) by Q_1, and noting that $G_j/Q_1 H_{m_0}$ is of the form $z^{i\lambda_j}(1 + o(1))$ for $1 \leq j \leq d_1$, we can apply Lemma 6.4 to the zero function (where we take $(a, b) = (s_{k-1}, s_k)$ and c to be any point in (s_{k-1}, s_k)) to yield $c_j = 0$ for $1 \leq j \leq d_1$. We then divide (7.18) by Q_{d_1+1} and repeat the argument. In this way we obtain $c_j = 0$ for $1 \leq j \leq q$ and independence is now clear since the union of the sets $\bar{\Delta}_{m_j}$ consists of linearly independent solutions.

We now proceeded by induction. We assume that r is a nonnegative integer less than d, and we assume that we have constructed solutions $G_{j,l}$ of (1.3) for $j = m_0, \dots, m_r$ and $1 \leq l \leq t_j$, which are admissible in $F(s_{k-1}, s_{k+1})$ and satisfy

$$G_{j,l}/M_{j,l}H_j \to 1 \quad \text{over } F(s_{k-1}, s_{k+1}), \tag{7.19}$$

(where the $M_{j.l}$ are as in (5.1)) and have the property that together with the sets $\bar{\Delta}_{m_j}$ for $j = r + 1, \dots, d$, form a fundamental set for (1.3) in $F(s_{k-1}, s_k)$.

We use induction on r, and for ease of notation, let $(Q_1, \dots, Q_q)$ denote the ordered logarithmic system for $\Omega_{m_{r+1}}(u) = 0$, with canonical partition (7.2). For this equation, let $\psi_1, \dots, \psi_q$ and $g_1, \dots, g_q$ be the complete logarithmic sets of solutions in $F(s_k, s_{k+1})$ and $F(s_{k-1}, s_k)$ respectively, with $\psi_j \sim Q_j$ in $F(s_k, s_{k+1})$ and $g_j \sim Q_j$ in $F(s_{k-1}, s_k)$. We let Γ denote the subset of $\{m_0, \dots, m_r\}$ consisting of those m_j for which $H_{m_{r+1}} \ll H_{m_j}$ in $F(s_k, s_{k+1})$. For ease of notation, we denote the elements of Γ by m_α, $m_\beta, \dots$ where $\alpha < \beta < \dots$. Let Φ denote $\{m_0, \dots, m_r\} - \Gamma$, so that by the total ordering (7.8), we have $H_{m_j} \ll H_{m_{r+1}}$ in $F(s_k, s_{k+1})$ for $m_j \in \Phi$. As before, the solution $\psi_1 H_{m_{r+1}}$ has an extension G_1 to $F(a, b)$, and so can be written as a linear combination of the $G_{j,l}$ for $j = m_0, \ldots, m_r$, and the elements of $\bar{\Delta}_{m_j}$ for $j \geq r + 1$, in $F(s_{k-1}, s_k)$. Setting $U_{j,l} = G_{j,l}/H_j$ (so that $U_{j,l}/M_{j,l} \to 1$ over $F(s_{k-1}, s_{k+1})$ by (7.19)) this combination for G_1 in $F(s_{k-1}, s_k)$ takes the form,

$$(7.20) \qquad G_1 = W_{m_0} H_{m_0} + \ldots + W_{m_r} H_{m_r} + W H_{m_{r+1}} + \sum_{j=r+2}^{d} W_{m_j} H_{m_j},$$

where for $j \leq r$, W_{m_j} is a linear combination of the functions $U_{m_j,l}$, while W is a linear combination of $g_1, \ldots, g_q$, and where for $j > r + 1$, W_{m_j} is a linear combination of a complete logarithmic set of solutions of $\Omega_{m_j}(u) = 0$. We now set,

$$(7.21) \qquad G_1^* = G_1 - \sum_{m_j \in \Phi} W_{m_j} H_{m_j} \qquad \text{on } F(s_{k-1}, s_{k+1}),$$

so we have in $F(s_{k-1}, s_k)$,

$$(7.22) \qquad G_1^* = W_{m_\alpha} H_{m_\alpha} + W_{m_\beta} H_{m_\beta} + \ldots + W H_{m_{r+1}} + \sum_{j=r+2}^{d} W_{m_j} H_{m_j}.$$

On $F(s_{k-1}, s_k)$, we have from (7.22), (7.5) and (7.7) that $G_1^*/H_{m_\alpha} = W_{m_\alpha}^* + T$ where $T = o(z^c)$ for all $c < 0$. However, on $F(s_k, s_{k+1})$, it follows from (7.21) and the definitions of Γ, Φ and G_1 that $G_1^*/H_{m_\alpha} = o(z^c)$ for every $c < 0$. This is the same situation that we had in (7.16) and we argue the same way to prove all coefficients in W_{m_α} are zero. That is, we canonically partition the ordered logarithmic set for $\Omega_{m_\alpha}(u) = 0$ as in (7.2), say $(D_1, D_2, \ldots), (D_e, \ldots), \ldots$, and we use Lemma 6.4 successively on $G_1^*/D_1 H_{m_\alpha}$, $G_1^*/D_e H_{m_\alpha}$, etc. to conclude that all coefficients in W_{m_α} are zero. We then do the same for W_{m_β} and so on, so we obtain from (7.22) that

$$(7.23) \qquad G_1^* = W H_{m_{r+1}} + \sum_{j=r+2}^{d} W_{m_j} H_{m_j} \qquad \text{on } F(s_{k-1}, s_k).$$

Again from (7.21), we have $G_1^*/Q_1 H_{m_{r+1}} \to 1$ in $F(s_k, s_{k+1})$ while from (7.23), we have on $F(s_{k-1}, s_k)$,

$$(7.24) \qquad G_1^*/Q_1 H_{m_{r+1}} = \sum_{j=1}^{q} c_j (g_j/Q_1) + T_2,$$

where T_2 is trivial. This is the same situation as we had in (7.12) and by using Lemma 6.4 again, we obtain $G_1^*/Q_1 H_{m_{r+1}} \to 1$ over $F(s_{k-1}, s_{k+1})$. By replacing Q_1 in this argument by Q_j for any $j =$

$= 1, \ldots, d_1$, we obtain a solution G_j^* of (1.3) such that $G_j^* / Q_j H_{m_{r+1}} \to 1$ over $F(s_{k-1}, s_{k+1})$.

We then consider the extension G_{d_1+1} of $\psi_{d_1+1} H_{m_{r+1}}$ to $F(a, b)$, so that G_{d_1+1} is given by a linear combination as in (7.20). We then subtract $\sum_{m_j \in \Phi} W_{m_j} H_{m_j}$ from G_{d_1+1}, and call the resulting function $G_{d_1+1}^*$ which is given by the right side of (7.22) in $F(s_{k-1}, s_k)$. As above, we show $W_{m_\alpha} = 0$, $W_{m_\beta} = 0$, etc. so that $G_{d_1+1}^*$ is given by the right side of (7.23). This is the same situation as in (7.16), and by dividing the relation (7.23) by $Q_1 H_{m_{r+1}}$ and using Lemma 6.4, we find that all coefficients of $g_1, \ldots, g_{d_1}$ in W are zero. Then dividing (7.23) by Q_{d_1+1}, we find as in (7.16) that by Lemma 6.4 we obtain $G_{d_1+1}^* / Q_{d_1+1} H_{m_{r+1}} \to 1$ in $F(s_{k-1}, s_{k+1})$. We continue this process, and we have now constructed solutions $G_{j,l}$ of (1.3) for $j = m_{r+1}$ and $1 \leqslant l \leqslant t_j$ such that

(7.25) $$G_{j,l}/M_{j,l} H_j \to 1 \quad \text{over } F(s_{k-1}, s_{k+1}).$$

To complete the induction, we must show that these $G_{j,l}$, for $j = m_0, \ldots, m_{r+1}$, together with $\bar{\Delta}_{m_j}$ for $j \geqslant r+2$ form a fundamental set for (1.3) in $F(s_{k-1}, s_k)$. This is easily proved exactly as was done for the case $r = 0$ in (7.18). If we have a dependence relation, we divide it by H_{m_0} yielding a relation (7.18), which was shown to imply $c_j = 0$ for $1 \leqslant j \leqslant q$. We then divide the dependence relation by H_{m_1} and the same argument shows that all coefficients of the functions $G_{m_1, l}$ will be zero. We then continue and we obtain the desired conclusion that all coefficients in the dependence relation are zero.

8. Proof of Theorem 5.2.

Let σ be the principal exponent of (1.1) with multiplicity r, so that the principal monomial $M(z)$ of (1.1) is given by (3.11). By Theorem 3.4, the equation (1.1) possesses a principal solution w_1 in $F(s_{k-1}, s_k)$ and a principal solution w_2 in $F(s_k, s_{k+1})$, so,

(8.1) $$w_1 \sim M \quad \text{in} \quad F(s_{k-1}, s_k) \quad \text{and} \quad w_2 \sim M \quad \text{in} \quad F(s_k, s_{k+1}).$$

For the homogeneous equation (1.3) corresponding to (1.1) we adopt the notation developed in the proof of Theorem 5.1. In particular, we assume (7.5)-(7.8) hold, and we let $G_{j,l})$ be the elements in the basic fundamental set for (1.3) in $F(s_{k-1}, s_{k+1})$ satisfying the relation (7.19). We also note that from Lemma 2.3, in each of $F(s_{k-1}, s_k)$ and

$F(s_k, s_{k+1})$ separately, we have,

(8.2) for each $m = 1, \ldots, d$, either H_m or $1/H_m$ is trivial.

By basic existence theory, the solution w_2 has an extension D_2 which is admissible (and of finite order of growth by Lemma 6.2) in $F(a, b)$. Hence on $F(s_{k-1}, s_k)$ we can write (as in (7.20)),

(8.3) $D_2 = w_1 + W_{m_0}H_{m_0} + W_{m_1}H_{m_1} + \ldots + W_{m_d}H_{m_d}$ on $F(s_{k-1}, s_k)$,

where W_{m_j} is a linear combination of the functions $U_{m_j, l} = G_{m_j, l}/H_{m_j}$ for $l = 1, \ldots, t_{m_j}$. We now distinguish two cases.

Case I. The critical degree of (1.3) is zero. In this case the term in (8.3) for which $m_j = 0$, vanishes identically. From (8.2), we distinguish two subcases. First, if all H_{m_j} (for $m_j \neq 0$), are trivial in $F(s_{k-1}, s_k)$, then $D_2/M \to 1$ in both $F(s_k, s_{k+1})$ and in $F(s_{k-1}, s_k)$ (by (8.3)). It now follows from Lemmas 6.1 and 2.1, and [10; Lemma 97], that $D_2/M \to 1$ over $F(s_{k-1}, s_{k+1})$, and so D_2 is the desired solution. Second, if for some $m_j \neq 0$, the function H_{m_j} is not trivial in $F(s_{k-1}, s_k)$, then referring to (7.7), let q be the largest element of $\{0, 1, \ldots, d\}$ for which $m_q \neq 0$ and H_{m_q} is not trivial in $F(s_{k-1}, s_k)$. Then, from (8.3), we can write,

(8.4) $D_2 = w_1 + W_{m_0}H_{m_0} + \ldots + W_{m_q}H_{m_q} + T_1$ on $F(s_{k-1}, s_k)$,

where $T_1 = o(z^c)$ for all $c < 0$. We now let Φ denote the subset of $\{m_0, \ldots, m_q\}$ consisting of those m_j for which H_{m_j} is trivial in $F(s_k, s_{k+1})$, and we let Γ denote $\{m_0, \ldots, m_q\} - \Phi$. Let D_2^* denote the function $D_2 - \sum_{m_j \in \Phi} W_{m_j}H_{m_j}$ on $F(s_{k-1}, s_{k+1})$, so we have,

(8.5) $D_2^* = w_1 + W_{m_\alpha}H_{m_\alpha} + W_{m_\beta}H_{m_\beta} + \ldots + T_1$ on $F(s_{k-1}, s_k)$,

where we are denoting the elements of Γ by $m_\alpha, m_\beta, \ldots$, where $\alpha < \beta < \ldots$. Of course, on $F(s_k, s_{k+1})$, $D_2^*/M \to 1$, so that $D_2^*/H_{m_\alpha} = o(z^c)$ for all $c < 0$. This is exactly the same situation we had in (7.21) and (7.22), and, as shown there, we find $W_{m_\alpha} = 0$, $W_{m_\beta} = 0, \ldots$, and we obtain from (8.5) that $D_2^*/M \to 1$ over $F(s_{k-1}, s_k)$ also. As in the first subcase, D_2^* is then the desired solution.

Case II. The critical degree of (1.3) is not zero. In this case, the term in (8.3) with $m_j = 0$ need not vanish. For definiteness, let l denote the element of $\{0, 1, \ldots, d\}$ for which $m_l = 0$ (and so $H_{m_l} \equiv 1$). Again we distinguish two subcases. First, assume that for all $m_j \neq 0$, the function H_{m_j} is trivial in $F(s_{k-1}, s_k)$. Of course, by (7.7), then $l = 0$, and so by (8.3), we have

(8.6) $$D_2 = w_1 + W_{m_0} + T_2 \quad \text{on } F(s_{k-1}, s_k),$$

where $T_2 = o(z^c)$ for all $c < 0$. Let $(P_1, \dots, P_q)$ be the ordered logarithmic system for (1.3), where the P_j are given by (7.1). Then, of course, for some constants c_j, we have

(8.7) $$W_{m_0} = \sum_{j=1}^{q} c_j P_j (1 + E_j) \quad \text{where} \quad E_j \to 0 \text{ over } F(s_{k-1}, s_{k+1}).$$

We can write $W_{m_0} = u_1 + u_2 + u_3$, where u_1 consists of the terms in (8.7) where $\operatorname{Re}(\alpha_j) < \sigma$, where u_2 consists of the terms where $\operatorname{Re}(\alpha_j) = \sigma$, and where u_3 consists of the terms where $\operatorname{Re}(\alpha_j) > \sigma$. By Lemma 2.4, we have from (8.6) that,

(8.8) $$D_2 = w_1 + u_2 + u_3 + o(w_1) \quad \text{in } F(s_{k-1}, s_k).$$

Let $(P_1, \dots, P_{d_1})$, $(P_{d_1+1}, \dots, P_{d_2})$, ..., be those portions of the canonical partition of $(P_1, \dots, P_q)$ consisting of those P_j for which $\operatorname{Re}(\alpha_j) > \sigma$. Now $D_2/P_1 \to 0$ in $F(s_k, s_{k+1})$ while from (8.8) we have in $F(s_{k-1}, s_k)$,

(8.9) $$D_2/P_1 = \sum_{j=1}^{d_1} c_j (P_j/P_1)(1 + E_j) + o(1).$$

Since $P_j/P_1 = z^{i\lambda_j}$ for some real λ_j if $1 \leqslant j \leqslant d_1$, we can conclude from Lemma 6.4 that $c_j = 0$ for $1 \leqslant j \leqslant d_1$. We can then repeat the argument using P_{d_1+1} etc. instead of P_1, and we conclude that $u_3 \equiv 0$. Thus,

(8.10) $$D_2 = w_1 + u_2 + o(w_1) \quad \text{in } F(s_{k-1}, s_k).$$

We note that if $u_2 \equiv 0$, then D_2 is the desired solutions as in *Case I*. Thus we may assume that there are P_j for which $\operatorname{Re}(\alpha_j) = \sigma$, and we let $(P_\alpha, \dots, P_{e_1})$, $(P_{e_1+1}, \dots, P_{e_2})$, ..., be those portions of the canonical partition having $\operatorname{Re}(\alpha_j) = \sigma$. Noting that within each of these portions, the exponent n_j of Log z in (7.1) is the same, let these exponents in these portions be $\beta_1 > \beta_2 > \dots$ respectively. We note first that if $\beta_1 < r$ (where r is as in (3.11)), then by Lemma 2.5, clearly $u_2 = o(w_1)$ in $F(s_{k-1}, s_k)$ and again by (8.10), the desired solution is D_2 as before. Thus we may assume that $\beta_1 \geqslant r$. Assume first that $\beta_1 > r$, and let u_4 be the sum of the terms in u_2 corresponding to β_1 (that is, corresponding to $(P_\alpha, \dots, P_{e_1})$). From (8.10), we obtain,

(8.11) $$D_1/P_\alpha = \sum_{j\alpha}^{e_1} c_j (P_j/P_\alpha)(1 + E_j) + o(1) \quad \text{in } F(s_{k-1}, s_k).$$

But since $\beta_1 > r$, clearly $D_2/P_\alpha \to 0$ in $F(s_k, s_{k+1})$, so from Lemma 6.4, we conclude $c_j = 0$ for $j = \alpha, \ldots, e_1$. Thus $u_4 \equiv 0$. If $\beta_2 > r$, we apply the same argument, and we show that all terms in u_2 corresponding to those β_j for which $\beta_j > r$, are all zero. Thus we may write $u_2 = u_5 + u_6$, where u_5 consists of those terms (if any) where $\beta_j = r$, and u_6 contains the terms where $\beta_j < r$. Clearly $u_6 = o(w_1)$ in $F(s_{k-1}, s_k)$ so we have,

$$D_2 = w_1 + u_5 + o(w_1) \quad \text{in } F(s_{k-1}, s_k). \tag{8.12}$$

If $u_5 \equiv 0$, we are done as in *Case I*. Hence we may assume that in $F(s_{k-1}, s_k)$,

$$u_5 = \sum_{j \in J} c_j z^{\sigma + i\lambda_j} (\operatorname{Log} z)^r (1 + E_j) \quad \text{where } E_j \to 0, \tag{8.13}$$

for some nonempty set J, and some distinct real numbers λ_j. We note that $\lambda_j \neq 0$ if $c_j \neq 0$ since $\lambda_j = 0$ would imply that $z^\sigma (\operatorname{Log} z)^r$ belongs to the logarithmic set for (1.3). But then σ would be a root of the critical equation of (1.3) having multiplicity at least $r + 1$ which contradicts the definition of r. Thus we have in $F(s_{k-1}, s_k)$ from (8.12) and (8.13),

$$D_2/M = (w_1/M) + \sum_{j \in J} c_j z^{i\lambda_j} (1 + E_j) + o(1), \tag{8.14}$$

where we note that $w_1/M = z^{i0}(1 + E)$, where $E \to 0$, since $w_1 \sim M$ in $F(s_{k-1}, s_k)$. Since $D_2/M \to 1$ over $F(s_k, s_{k+1})$ (since $D_2 \equiv w_2$), we can conclude from Lemma 6.4 that $D_2/M \to 1$ over $F(s_{k-1}, s_{k+1})$. Thus D_2 is the desired solution.

The second subcase in *Case II*, is the case where not all H_{m_j} (for $m_j \neq 0$) are trivial in $F(s_{k-1}, s_k)$. Thus clearly from (7.7), $l > 0$. Let Φ denote the subset of $\{m_0, \ldots, m_{l-1}\}$ consisting of those m_j for which H_{m_j} is trivial in $F(s_k, s_{k+1})$, and let Γ denote $\{m_0, \ldots, m_{l-1}\} - \Phi$. Letting D_2^* denote the function $D_2 - \sum_{m_j \in \Phi} W_{m_j} H_{m_j}$, we have from (8.3),

$$D_2^* = w_1 + W_{m_l} + W_{m_\alpha} H_{m_\alpha} + W_{m_\beta} H_{m_\beta} + \ldots + T_3 \quad \text{on } F(s_{k-1}, s_k), \tag{8.15}$$

where $T_3 = o(z^c)$ for all $c < 0$, and where we are denoting the elements of Γ by $m_\alpha, m_\beta, \ldots$ where $\alpha < \beta < \ldots$. This is identical to the situation in (8.5), and we showed that $W_{m_\alpha} \equiv 0$, $W_{m_\beta} \equiv 0, \ldots$. Thus, we have

$$D_2^* = w_1 + W_{m_l} + T_3 \quad \text{on } F(s_{k-1}, s_k). \tag{8.16}$$

Since $D_2^*/M \to 1$ over $F(s_k, s_{k+1})$, this is exactly the same situation (with D_2 replaced by D_2^*) as we had in (8.6), and the same argument shows that $D_2^*/M \to 1$ over $F(s_{k-1}, s_{k+1})$. Thus D_2^* is the desired solution, and the proof is complete.

9. Proof of Theorem 4.1.

We assume the hypothesis of the theorem, and we adopt the notation developed in the previous two proofs. In particular, we let G_j, l be the elements of the basic fundamental set for (1.3) in $F(s_{k-1}, s_{k+1})$, satisfying (7.19) $\left(\text{and where } H_m = \exp\int V_m \text{ and } H_0 \equiv 1\right)$. Let W denote the solution of principal type of (1.1) in $F(s_{k-1}, s_{k+1})$, so that $W/M \to 1$ where M is the principal monomial (3.11).

PROOF OF PART (A). Assume condition (i) in Part (A) holds. For this j, the function $f = W - G_{j,1}$ is a solution of (1.1), and in view of (7.19), the equation $f(z) = 0$ can be written in the form,

$$z^{\alpha}(\operatorname{Log} z)^{\beta}(1 + E_1)\exp\int V_j = 1, \tag{9.1}$$

where $E_1 \to 0$ over $F(s_{k-1}, s_{k+1})$, and where $z^{\alpha}(\operatorname{Log} z)^{\beta}$ is $M_{j,1}/M$. This is the same equation as was obtained in [5; Formula (10.5)], and as shown in [5], since $IF(V_j, s_k) = 0$, the solution $f(z)$ satisfies the conclusion of Part (A).

Now assume (ii) holds, and consider the solution $f(z)$ of (1.1) defined by $W - G_{j,1} - G_{m,1}$. The equation $f(z) = 0$ can be written in the form,

$$\left(U_{j,1}\Big/\left(\left(W\exp\int -V_m\right) - U_{m,1}\right)\right)\exp\int(V_j - V_m) = 1, \tag{9.2}$$

where $U_{j,1} = G_{j,1}/H_j$ and $U_{m,1} = G_{m,1}/H_m$. Since $U_{j,1}/M_{j,1} \to 1$ and $U_{m,1}/M_{m,1} \to 1$ over $F(s_{k-1}, s_{k+1})$ by (7.19), and since $W\exp\int -V_m = o(U_{m,1})$ by Lemmas 2.3 and 2.4, it is clear that (9.2) has the form (9.1) with V_j replaced by $V_j - V_m$, and where $z^{\alpha}(\operatorname{Log} z)^{\beta}$ is $-M_{j,1}/M_{m,1}$. Thus f has the desired property as before. This proves Part (A).

PROOF OF PART (B). We now assume that neither (i) nor (ii) in Part (A) holds. We will show that for any admissible solution f of (1.1), there is an element of $F(s_{k-1}, s_{k+1})$ on which f has no zeros. This will yield the conclusion of Part (B) by Lemma 2.1. Letting $U_{j,l}$ denote $G_{j,l}/H_j$, so that $U_{j,l}/M_{j,l} \to 1$ over $F(s_{k-1}, s_{k+1})$, the solution f can be written,

$$(9.3) \qquad f = W + W_0 + W_1 \exp \int V_1 + \ldots + W_d \exp \int V_d ,$$

where W_j is a linear combination of the functions $U_{j,l}$. We now distinguish two cases.

Case I. For each $j \in \{1, \ldots, d\}$ for which $W_j \not\equiv 0$, we have $IF(V_j, s_k) < 0$. Since $IF(V_j, \theta)$ has no zeros on (s_{k-1}, s_k) or (s_k, s_{k+1}), we must have $IF(V_j, \theta) < 0$ on (s_{k-1}, s_{k+1}). Thus by Lemmas 2.3 and 2.4, we can write (9.3) as,

$$(9.4) \qquad f = W + W_0 + T_1 ,$$

where $T_1 = o(z^{-c})$ for any $c > 0$ over $F(s_{k-1}, s_{k+1})$. If $W_0 \equiv 0$, then (9.4) shows that f has no zeros on some element of $F(s_{k-1}, s_{k+1})$, since $f/M \to 1$, and so we are done. If $W_0 \not\equiv 0$, we observe that by the hypothesis of the theorem, clearly neither of the conditions (i) nor (ii) of Part (*A*) of Theorem 3.5 can hold (or we would have global oscillation). This is the same hypothesis as we had in [2; Lemma 7.5], and the same arguments shows that $W + W_0$ is either of the form $W(1 + o(1))$ or $K_0 U_{0,l}(1 + o(1))$ in $F(s_{k-1}, s_{k+1})$ for some $l \in \{1, \ldots, t_0\}$ and some nonzero constant K_0. In either case, (9.4) shows that f has no zeros on some element of $F(s_{k-1},s_{k+1})$. Thus we are done in Case I.

Case II. There exists $j \in \{1, \ldots, d\}$ for which $W_j \not\equiv 0$ and $IF(V_j, s_k) \geq 0$. Let J denote the set of all $j \in \{1, \ldots, d\}$ with this property. In view of the assumption that condition (i) does not hold, we must have $IF(V_j, s_k) > 0$ for $j \in J$, and hence we have $IF(V_j, \theta) > 0$ on (s_{k-1}, s_{k+1}) since $IF(V_j, \theta)$ has constant sign on both (s_{k-1}, s_k) and (s_k, s_{k+1}).

Subcase A. J has only one element, m. Then the relation (9.3) takes the form,

$$(9.5) \qquad f = W + W_0 + W_m \exp \int V_m + T_2 ,$$

where $T_2 = o(z^{-c})$ for all $c > 0$ over $F(s_{k-1}, s_{k+1})$. In view of the hypothesis that (1.1) does not have the global oscillation property in $s_k < \operatorname{Arg} z < s_{k+1}$, we have by *Part (B)* of Theorem 3.5, that the critical equation of $\Omega_m(u) = 0$ does not possess two distinct roots having the same real part. In view of Lemma 2.4, one term in the linear combination W_m dominates the rest, so that for some $l \in \{1, \ldots, t_m\}$ we have for some nonzero constant K_O, the relation

$$(9.6) \qquad f = K_O M_{m,l}\left(\exp\int V_m\right)(1 + o(1)) \quad \text{in } F(s_{k-1}, s_{k+1}),$$

which shows that f has no zeros on some element of $F(s_{k-1}, s_{k+1})$,

Subcase B. J has more than one element. Since we are assuming (ii) does not hold, if j and m are distinct elements of J, then $IF(V_j - V_m, s_k) \neq 0$. Since $IF(V_j - V_m, \theta)$ has constant sign on each of (s_{k-1}, s_k) and (s_k, s_{k+1}) by definition of the s_k, it follows that $IF(V_j - V_m, \theta)$ has constant sign on (s_{k-1}, s_{k+1}) for any two distinct elements j and m in J. Thus by Lemma 2.3, the set $Y = \left\{\exp\int V_j : j \in J\right\}$ has the property that the ratio of two distinct elements of Y is either trivial or its reciprocal is trivial in $F(s_{k-1}, s_{k+1})$. It easily follows that for some element $m \in J$, we have

$$(9.7) \qquad \left(\exp\int V_j\right)\Big/\left(\exp\int V_m\right) \quad \text{is trivial in } F(s_{k-1}, s_{k+1}),$$

for any $j \in J - \{m\}$. It easily follows from (9.3) that

$$(9.8) \qquad f = W_m\left(\exp\int V_m\right)(1 + o(1)) \quad \text{over } F(s_{k-1}, s_{k+1}).$$

However, by the same reasoning as in *Subcase A*, $W_m = K_O M_{m,l} \cdot$ $\cdot (1 + o(1))$ for some l and now it is clear that f has no zeros on some element of $F(s_{k-1}, s_{k+1})$. This concludes the proof.

10. The homogeneous case [5].

For a homogeneous equation (1.3) whose coefficients belong to a logarithmic differential field of rank zero over $F(a, b)$, we known from Theorem 3.3 that either (1.3) has the global oscillation property in $a < \operatorname{Arg} z < b$, or it possesses the non-oscillation property in all the sectors $s_k < \operatorname{Arg} z < s_{k+1}$ for $k = 0, 1, \ldots, t$. In the latter case, we can ask whether a given s_k (for $k \in \{1, \ldots, t\}$) is extraneous in the sense that there is an $\varepsilon > 0$ such that for any solution $f \not\equiv 0$ of (1.3), f has no zeros on a set of the form, $|\operatorname{Arg} z - s_k| < \varepsilon$, $|z| \geq K(f)$ for some $K(f) > 0$. The question of determining whether an s_k is extraneous for (1.3) was completely solved in [5; §7] as follows: Under the given hypothesis, Theorem 3.3 shows that the condition in *Part (A)* of Theorem 3.3 does not hold. It easily follows from Lemmas 2.3 and 2.4, that in each of $F(s_{k-1}, s_k)$ and $F(s_k, s_{k+1})$ separately, the whole asymptotic set $\{G_1, \ldots, G_n\}$ for (1.3) (see § 5) is totally ordered by the relation «$\ll$». That is, there exist permutations $\{m_1, \ldots, m_n\}$ and $\{q_1, \ldots, q_n\}$ of $\{1, \ldots, n\}$ such that

$$G_{q_1} \gg G_{q_2} \gg \ldots \gg G_{q_n} \quad \text{in } F(s_k, s_{k+1}), \tag{10.1}$$

and

$$G_{m_1} \gg G_{m_2} \gg \ldots \gg G_{m_n} \quad \text{in } F(s_{k-1}, s_k). \tag{10.2}$$

It is proved in [5; § 7] that s_k is extraneous if and only if $G_{q_j} = G_{m_j}$ for all $j = 1, \ldots, n$. Hence an s_k is not extraneous if and only if the ordering of the asymptotic set changes as we pass the ray $\operatorname{Arg} z = s_k$.

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