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A Collar Neighborhood Theorem for a Complex Manifold.

C. DENSON HILL - MAURO NACINOVICH (*)

SUMMARY - For a real paracompact smooth manifold D with smooth boundary M the collar neighborhood theorem is well known. But for an intrinsically defined complex manifold D with a smooth boundary M , there is no such analogous theorem (see [7],[8]). This is closely related to the failure, in general, of the Newlander-Nirenberg theorem up-to-the boundary (see [9],[10]); which can occur in the presence of some pseudoconcavity of M . However the up-to-the boundary version of the Newlander-Nirenberg theorem is valid if the boundary M is strictly pseudoconvex (see [5]), or even when M is weakly pseudoconvex (see [3]). This is of course a local result near a boundary point $p \in M$. Thus the question arises as to when these local extensions, of the complex structure of D across M , can be pieced together to give a global collar neighborhood whose complex structure is an extension of the complex structure from D . We show here that it can be done when the boundary M is strictly pseudoconvex. When $\dim_{\mathbb{C}} D = 1$, there is no condition at all required on M . Of course when D is a real analytic manifold with real analytic boundary M , and the integrable almost complex structure on D is also real analytic up-to-the boundary M , then the collar neighborhood exists without any assumption about the Levi convexity of M . This follows by the identity theorem from complex analysis.

1. Existence of the collar neighborhood.

Let Ω be a paracompact (*i.e.* countable at infinity) smooth manifold of dimension $2n$, $n \geq 2$, and let D be an open domain in Ω , with a smooth boundary $M = \partial D$.

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We assume tht M is a closed connected differentiable real submanifold of Ω , of dimension $2n - 1$ and countable at infinity.

Let $J_0: T\Omega \rightarrow T\Omega$ be a smooth almost complex structure on Ω , formally integrable on $\bar{D}$. Then we have the following.

THEOREM 1. *Assume that $M = bD$ is strictly pseudoconvex for the structure J_0 . Then we can find an open submanifold ω of Ω , containing $\bar{D}$, and a complex structure $J: T\omega \rightarrow T\omega$, such that $J|_D = J_0|_D$.*

PROOF. The statement follows by an argument which uses Zorn's lemma and the local Newlander-Nirenberg theorem up-to-the boundary (see [5], [3]).

We introduce the family $\mathfrak{X}$ of pairs (X, J) , where X is an open submanifold of Ω containing D , and $J: TX \rightarrow TX$ an integrable almost complex structure on X such that $J|_D = J_0|_D$. As $(D, J_0|_D) \in \mathfrak{X}$, the family $\mathfrak{X}$ is non-empty.

On $\mathfrak{X}$ we define an equivalence relation by setting

$$(X_1, J_1) \sim (X_2, J_2)$$

iff

$$(i) \quad X_1 \cap M = X_2 \cap M;$$

(ii) there is an open neighborhood G_{X_1, X_2} of $X_1 \cap M = X_2 \cap M$ in Ω such that $J_1|_{G_{X_1, X_2}} = J_2|_{G_{X_1, X_2}}$.

We denote by $\tilde{\mathfrak{X}}$ the quotient $\mathfrak{X}/\sim$ and by $[X, J]$ the equivalence class of $(X, J) \in \mathfrak{X}$ in $\tilde{\mathfrak{X}}$.

In $\tilde{\mathfrak{X}}$ we define an order relation $<$ by setting

$$[X_1, J_1] < [X_2, J_2]$$

iff:

$$(a) \quad X_1 \cap M \subsetneq X_2 \cap M;$$

(b) J_1 and J_2 agree on an open neighborhood G_{X_1, X_2} of $X_1 \cap M$ in $X_1 \cap X_2$.

We want to show that $\tilde{\mathfrak{X}}$ is inductive; i.e. that every chain in $\tilde{\mathfrak{X}}$ has an upper bound in $\tilde{\mathfrak{X}}$. Let $\mathcal{C}$ be a chain in $\tilde{\mathfrak{X}}$ for the ordering $<$. If $\mathcal{C}$ is finite, it has a maximum, which is therefore a majorant of $\mathcal{C}$. Assume now that $\mathcal{C}$ is infinite. Let

$$M_0 = \cup \{X \cap M \mid [X, J] \in \mathcal{C}\}.$$

This is an open subset of M . Let $\mathfrak{W} = \{W_\nu \mid \nu \in N\}$ be a countable open covering in Ω of M_0 which is locally finite, and with $\overline{W}_\nu$ compact for every ν , and $\overline{W}_\nu \cap M \subset M_0$. We define by recurrence

$$\mathfrak{W}_0 = \{W_0\}$$

$$\mathfrak{W}_{k+1} = \left\{ W_\nu \in \mathfrak{W} \mid W_\nu \cap \bigcup \mathfrak{W}_k \neq \emptyset, W_\nu \notin \bigcup_{j \leq k} \mathfrak{W}_j \right\}.$$

For

$$\tilde{W}_k = \bigcup \mathfrak{W}_k$$

we have

- (1) $\overline{\tilde{W}_k}$ is compact;
- (2) $\tilde{W}_k \cap \tilde{W}_{k+2} = \emptyset, \forall k$;
- (3) $\bigcup \tilde{W}_k \cap M = M_0$.

To construct a majorant for the chain $\mathcal{C}$ we proceed in the following way. We set

$$U_k = \tilde{W}_k \cap M.$$

We note that $\overline{U}_k$ is a compact subset of M_0 for every k and then we can find $(X_1, J_1) \in \mathfrak{X}$ with $[X_1, J_1] \in \mathcal{C}$ such that

$$\overline{U}_0 \cup \overline{U}_1 \subset X_1 \cap M.$$

Let $V_0 \subset \tilde{W}_0 \cap X_1$ be an open neighborhood of U_0 in Ω and let us set $\omega_1 = D \cup V_0$. We define an integrable almost complex structure on ω_1 by $\tilde{J}_1: T\omega_1 \rightarrow T\omega_1$ being the restriction of J_1 to $D \cup V_0 \subset X_1$.

Next we choose $(X_2, J_2) \in \mathfrak{X}$ with $[X_2, J_2] \in \mathcal{C}$ and $[X_1, J_1] < [X_2, J_2]$ such that

$$\overline{U}_0 \cup \overline{U}_1 \cup \overline{U}_2 \subset X_2 \cap M.$$

By point (b) in the definition of the order relation, we can find an open neighborhood G_{X_1, X_2} of $X_1 \cap M$ in Ω such that $G_{X_1, X_2} \subset X_1 \cap X_2$ and $J_1|_{G_{X_1, X_2}} = J_2|_{G_{X_1, X_2}}$.

Then we can find an open neighborhood V_1 of U_1 in $\tilde{W}_1 \cap G_{X_1, X_2}$. We set $\omega_2 = \omega_1 \cup V_1$ and we can define on ω_2 an integrable almost complex structure $\tilde{J}_2: T\omega_2 \rightarrow T\omega_2$ by $\tilde{J}_2|_{\omega_1} = \tilde{J}_1$ and $\tilde{J}_2|_{V_1} = J_2|_{V_1} = J_1|_{V_1}$.

By recurrence we prove the following: for every ν we can find

- (α) an open neighborhood V_ν of U_ν in $\tilde{W}_\nu$;
 (β) an element $(X_\nu, J_\nu) \in \mathfrak{X}$ with $[X_\nu, J_\nu] \in \mathcal{C}$, such that $[X_{\nu-1}, J_{\nu-1}] < [X_\nu, J_\nu]$;
 (γ) an open neighborhood $G_{X_{\nu-1}, X_\nu}$ of $M \cap X_{\nu-1}$ in $X_{\nu-1} \cap X_\nu$ on which $J_{\nu-1} = J_\nu$ such that

$$V_\nu \subset G_{X_{\nu-1}, X_\nu},$$

$$\overline{U}_0 \cup \dots \cup \overline{U}_{\nu+1} \subset X_\nu.$$

Because $V_\nu \cap V_{\nu+j} = \emptyset$ for $j \geq 2$, if we set

$$X = D \cup V_0 \cup V_1 \cup \dots$$

and we define $J: TX \rightarrow TX$ by

$$J|D_0 = J_0,$$

$$J|V_\nu = J_\nu|V_\nu,$$

we obtain an integrable almost complex structure on X .

We have $(X, J) \in \mathfrak{X}$, $X \cap M = M_0$ and for each $[Y, J_Y] \in \mathcal{C}$, the structures J and J_Y agree by construction on a neighborhood of $Y \cap M$.

Hence $[X, J]$ is a majorant of $\mathcal{C}$.

By Zorn's lemma, $\tilde{\mathfrak{X}}$ contains a maximal element $[\omega, J]$. We need to prove that $\omega \supset M$.

Let $M_0 = \omega \cap M$ and suppose to the contrary, that $M_0 \neq M$. Let $\rho: \Omega \rightarrow \mathbf{R}$ be a defining function for D in Ω , i.e. we assume that $\rho < 0$ on D , $\rho = 0$ on M , $d\rho \neq 0$ on M and $\rho > 0$ on $\Omega - \overline{D}$.

Let $\{\varphi_\nu\}$ be a partition of unity on a neighborhood of M_0 in ω , with $\varphi_\nu \geq 0$, and $\text{supp } \varphi_\nu$ compact for every ν . Then, for a suitable choice of a sequence $\{\varepsilon_\nu\}$ of positive real numbers,

$$\tilde{D} = \{\rho < \sum \varepsilon_\nu \varphi_\nu\}$$

is an open neighborhood of D in Ω , with $M_0 \subset \tilde{D}$ and $b\tilde{D}$ smooth and strictly pseudoconvex for the extension of the integrable almost complex structure J to $\tilde{D}$.

It $p \in M - M_0$, then p is a boundary point of $\tilde{D}$ and then, by the Newlander-Nirenberg theorem up to the boundary, we can find an open submanifold B of Ω , containing $\tilde{D} \cup \{p\}$, on which a complex structure J' is defined, extending the complex structure J on $\tilde{D}$. But

then $[\omega, J] = [\tilde{D}, J|_{\tilde{D}}] < [B, J']$ and this gives a contradiction to the fact that $[\omega, J]$ was maximal in $\tilde{\mathcal{X}}$. Therefore $\omega \supset M$ and the proof is complete.

2. Remarks.

1) When $\dim_{\mathbb{C}} D = 1$, so $\dim_{\mathbb{R}} M = 1$, one can take any smooth extension J of the almost complex structure $J_0|_{\tilde{D}}$ some open neighborhood $\omega \supset \tilde{D}$. This J is then a complex structure on ω since there is no formal integrability requirement in complex dimension one. Thus Theorem 1 holds without any condition on M .

2) Suppose $\dim_{\mathbb{C}} D \geq 2$ and, instead of assuming that M is strictly pseudoconvex, we assume that at every point of M the Levi form has at least one negative eigenvalue. Then we cannot appeal to the up-to-the-boundary version of the Newlander-Nirenberg theorem, because there are known counterexamples (see [8]). So let us assume instead that M is locally embeddable at each point. The existence of a collar neighborhood (ω, J) of $(D, J_0|_D)$, as in Theorem 1, then follows by a result of Dłwilewicz [4].

3) In fact, we can do away with the global hypothesis that M be the abstract boundary of a complex manifold D , as in Theorem 1, and replace it by a microlocal hypothesis: let M be a smooth paracompact (*i.e.* countable at infinity) abstract strictly pseudoconvex CR manifold (of hypersurface type). Consider the following condition:

(A) For every p on M , the given CR structure on M has a local extension to the germ of a complex structure on the pseudoconvex side of M . Here the extension is intended in the sense of an abstract boundary; *i.e.*, there is a local smooth integrable almost complex structure which extends the CR structure to the pseudoconvex side near each point.

We ask the question: does M have a global embedding as a closed CR hypersurface in some open complex manifold X ? Assume M has a real dimension $2n - 1$ with $n > 1$.

THEOREM 2. *M has such a global embedding if and only if the microlocal condition (A) is satisfied.*

PROOF. The condition is obviously necessary. To show it is also sufficient, first note that by the Newlander-Nirenberg result up to the

boundary (see Hanges-Jacobowitz[5]), we have that M is locally embeddable at each point. Hence we have the Hans Lewy local extension of CR functions to the pseudoconvex side of M . It follows that the local extensions of the CR structure piece together, in effect, producing our D from Theorem 1. Rather than go into details, we refer the reader to Dwilewicz[4], where this type of argument is treated very explicitly. Then we apply our Theorem 1.

4) Going back to the situation of section 1, suppose that our strictly pseudoconvex M is compact, and forms the abstract boundary of an open Stein manifold D . Then it follows from the work of Andreotti and Grauert[1] that $\bar{D}$ has a Stein neighborhood in the collar. A related result, for the case where the boundary M of D is assumed in the concrete sense, was found by Heunemann[6]. Let us for convenience now take $\dim M = 2n + 1$ with $n > 0$. We may then apply the well-known results (see Narasimhan[11] and Bishop[2]) and conclude that M has a global closed CR embedding in $\mathbb{C}^{2n+3}$ and a global closed CR immersion in $\mathbb{C}^{2n+2}$. But this does not give the best result. Indeed we have

THEOREM 3. *Let M be a smooth compact strictly pseudoconvex CR manifold (of hypersurface type) with $\dim M = 2n + 1$, $n \geq 1$. Then*

($n \geq 2$): M has a global closed CR embedding in $\mathbb{C}^{2n+2}$ and a global closed CR immersion in $\mathbb{C}^{2n+1}$.

($n = 1$): M^3 has a global closed CR embedding in $\mathbb{C}^4$ and a global closed CR immersion in $\mathbb{C}^3$ provided that M^3 forms the abstract boundary of an open Stein surface D^2 .

PROOF. When $\dim M \geq 5$, it follows by the theorem of Boutet de Monvel(*) that M has a global closed CR embedding in $\mathbb{C}^n$, for some N . When $\dim M \geq 3$, we get such an embedding into $\mathbb{C}^N$ by using the remark just before Theorem 3. So in any case we get an embedding in $\mathbb{C}^N$, for some N . It then suffices to take a generically chosen holomorphic projection into $\mathbb{C}^{2n+2}$ or $\mathbb{C}^{2n+1}$. For further details, see the more general Proposition at the end of Hill-Nacinovich(**).

(*) L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles*, Sem. Goulaouic-Lions-Schwartz, 9 (1975).

(**) Hill-Nacinovich, *The Topology of Stein CR Manifolds and the Lefschetz Theorem*, Ann. Inst. Fourier, Grenoble, 43, 2 (1993), pp. 459-468.

3. On the non-uniqueness of the collar neighborhood.

Let us first consider the case $n = 1$. On $C - [1, \infty]$ we denote by $\alpha(z)$ the branch of $\sqrt{1 - z}$ with positive real part.

Then we consider on the closed unit disc $\bar{D} = \{|z| \leq 1\}$ the function

$$\varphi(z) = \begin{cases} Az + \exp\left(-\frac{1}{\alpha(z)}\right) & \text{for } z \neq 1, \\ A & \text{for } z = 1. \end{cases}$$

For every $A \in C$ this defines a Whitney function on the closed disc. For $|A|$ large it is a biholomorphism of the open disc D onto an open domain G of C . By Whitney's theorem, for large A , φ extends to a diffeomorphism $\tilde{\varphi}$ of a neighborhood U of $\bar{D}$ in C onto a neighborhood V of $\bar{G}$ in C .

Then we consider the two complex structures on U defined by the single coordinate patch (U, z) and $(U, \tilde{\varphi}(z))$ respectively. We claim that the two structures do not agree on any neighborhood of 1, while they obviously agree on the open disc D and hence on $\bar{D}$. Indeed, $\varphi|_D$ is holomorphic on D for both structures, but has no analytic extension beyond 1 for the first one, as $Az - \varphi$ would then extend to a non-zero analytic function flat at 1. It obviously extends for the second structure, being the restriction to D of the coordinate function.

We can now easily construct examples of non-uniqueness in several variables. If $n > 1$, denoting by e_1 the vector $(1, 0, \dots, 0)$ in C^n , we consider $\Omega = \{z \in C^n \mid |z - e_1/2| < 1/2\}$. With $D = \{t \in C \mid |t| < 1\}$ and $U, \tilde{\varphi}$ as above we realize that the two structures defined on the neighborhood $\tilde{\Omega} = U \times D^{n-1}$ of $\bar{\Omega}$ by the single coordinate patch $(\tilde{\Omega}, z^1, \dots, z^n)$ and $(\tilde{\Omega}, \tilde{\varphi}(z^1), z^2, \dots, z^n)$ respectively cannot possibly agree on an open neighborhood of e_1 .

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