RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Cohomological interpretation of hypergeometric series

Rendiconti del Seminario Matematico della Università di Padova, tome 90 (1993), p. 239-263

<http://www.numdam.org/item?id=RSMUP_1993__90__239_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 90 (1993)

Cohomological Interpretation of Hypergeometric Series.

B. DWORK (*)

Introduction.

In joint work with F. Loeser [D-L1, 2] we have given a cohomological interpretation of generalized hypergeometric series by means of exponential modules. In this note we give a new explanation of this relation. This new exposition involves § 5, 6 and in particular Proposition 5.6. This article is based on lectures given at Oklahoma State University during the fall of 1992. We take this opportunity to thank the Mathematic Department of OSU for its hospitality.

1. The arithmetic gamma function.

For $l \in \mathbb{Z}$ we define $(z)_l \in \mathbb{Q}(z)$ to be $(\Gamma(z+l))/\Gamma(z)$. The following properties are trivial:

(1.1) $\quad (z)_0 = 1\,;$

(1.2) $\quad (z)_l = z(z+1)\ldots(z+l-1) \quad \text{if } l \geq 1\,;$

(1.3) $$(z)_{-l} = \frac{1}{(z-1)(z-2)\ldots(z-l)} \quad \text{if } l \geq 1\,.$$

We conclude that:

(1.4) the function $(z)_l$ takes finite values in $\mathbb{C}$ if we insist that for $z \in \mathbb{N}^{\times}$, l should be in $\mathbb{N}$;

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(1.5) the function $(z)_l$ takes values in $\mathbb{C}^\times$ if in addition we insist that for $z \in \mathbb{N}$, l should lie in $-\mathbb{N}$;

(1.6) $(z)_l = (-1)^l/(1-z)_{-l}$;

(1.7) if $l \in \mathbb{N}^\times$, $z \in \mathbb{N}$, then $(z)_l \neq 0$ if $z \leq -l$.

(1.8) if $y \in \mathbb{N}$, $z \in \mathbb{Z}$ then as elements of $\mathbb{Q}(x)$,

$$(x+z)_y (x)_z = (x+y)_z (x)_y$$

and each factor takes values in $\mathbb{C}$ if we insist that $x + y \in -\mathbb{N}$ whenever $x \in \mathbb{Z}$ and $z \in -\mathbb{N}^\times$.

2. Hypergeometric series.

Let A be an $m \times n$ matrix with coefficients in $\mathbb{Z}$ and let $l_1, \ldots, l_m$ be $\mathbb{Z}$-linear forms in $(s_1, \ldots, s_n)$ defined by

$$\begin{pmatrix} l_1(s) \\ \vdots \\ l_m(s) \end{pmatrix} = \begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \cdot & \ldots & \cdot \\ A_{m1} & \ldots & A_{mn} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}.$$

Let $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$ satisfy the condition (cf. (1.4)):

(2.1) $$\text{if } a_i \in \mathbb{N}^\times \text{ then } A_{ij} \in \mathbb{N} \quad (1 \leq j \leq n).$$

Subject to this condition we define a formal power series in $t = (t_1, \ldots t_n)$ with coefficients in $\Omega = \mathbb{Q}(a)$

(2.2) $$y(a, t) = \sum_{s \in \mathbb{N}^n} \frac{(-t_1)^{s_1} \ldots (-t_n)^{s_n}}{s_1! \ldots s_n!} \prod_{i=1}^m (a_i)_{l_i(s)}.$$

For comparison with classical formulae it is sometimes convenient to let $\mathcal{F}_1 \cup \mathcal{F}_2$ be a partition of $\{1, 2, \ldots, m\}$ and rewrite this last factor by means of

(2.3) $$\prod_{i=1}^m (a_i)_{l_i(s)} = \frac{\prod_{i \in \mathcal{F}_2} (a_i)_{l_i(s)}}{\prod_{i \in \mathcal{F}_1} (1-a_i)_{-l_i(s)} (-1)^{l_i(s)}}.$$

Let $\delta_i = t_j(\partial/\partial t_j)$, $1 \leq j \leq n$. Let $\mathcal{R} = \Omega(t)[\delta_1, \ldots, \delta_n]$. We define $\mathfrak{A}(a)$ to be the left ideal of $\mathcal{R}$ containing all $\theta \in \mathcal{R}$ such that $\theta y(a, t) = 0$.

3. Exponential modules.

We associate with hypergeometric series two exponential modules. Let

(3.1) $$R' = \Omega(t)[X_1, \ldots, X_m, X_1^{-1}, \ldots, X_m^{-1}].$$

Let $g \in R'$

$$-g(t, X) = x_1 + \ldots + X_m + \sum_{j=1}^{n} t_j X^{A^{(j)}},$$

where for $1 \leq j \leq n$, $X^{A^{(j)}} = \prod_{i=1}^{m} X^{A_{i,j}}$. Let $E_i = X_i(\partial/\partial X_i)$, $g_i = E_i g$ ($1 \leq$ $\leq j \leq m$) and let $D_{a,i,t} = E_i + a_i + g_i$, a differential operator on R'. We define an $\Omega(t)/\Omega$-connection σ on R' by

$$\sigma\left(\frac{\partial}{\partial t_j}\right) = \sigma_j = \frac{\partial}{\partial t_j} + \frac{\partial g}{\partial t_j}, \qquad \text{for } 1 \leq j \leq n.$$

The operators $D_{a,1,t}, \ldots, D_{a,m,t}, \sigma_1, \ldots \sigma_n$ commute. We define $\wp'_{a,t} =$ $= R' / \sum_{i=1}^{m} D_{a,i,t} R'$, an $\Omega(t)$-space with connection induced by σ. Then $\wp'_{a,t}$ is a left ($\mathcal{R}_1 = \Omega(t)[\sigma_1, \ldots, \sigma_n]$)-module. The non-commutative ring $\mathcal{R}_1$ is isomorphic to the ring $\mathcal{R}$ of § 2 under the mapping

$$\sigma: \left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}\right) \mapsto (\sigma_1, \ldots, \sigma_n).$$

Let $\mathfrak{A}_1(a)$ be the annihilator in $\mathcal{R}_1$ of [1], the class of 1 in $\wp'_{a,t}$. The object of this note (cf. Corollary 6.5) is to give a new, possibly more elementary, proof of Theorem C of [D-L2] which shows that under certain conditions $\mathfrak{A}_1(a)$ is isomorphic via σ to $\mathfrak{A}(a)$.

3.2. To construct the second exponential module associated with hypergeometric series, let $\mathfrak{S}_1 \cup \mathfrak{S}_2$ be a partition of $\{1, 2, \ldots, m\}$ satisfying the condition:

(3.2.1) $$\text{if } i \in \mathfrak{S}_2 \text{ then } A_{i,j} \in \mathbb{N} \text{ for } j = 1, 2, \ldots, n.$$

(Thus if $A_{i,j} \in -\mathbb{N}^{\times}$ for some j then $i \in \mathfrak{S}_1$). Let $H_0 = \mathbb{Z}^m$ (the support of R'). Let $\tilde{H}_0$ be the subset

$$\tilde{H}_0 = \{u \in H_0 \mid u_i \in \mathbb{N} \text{ if } i \in \mathfrak{S}_2\}.$$

Let $\tilde{R}$ be the subring of R' consisting of the $\Omega(t)$-span of $\{X^u \mid u \in \tilde{H}_0\}$.

By (3.2.1), $g \in \tilde{R}$ and the differential operators $\{D_{a,i,t}\}_{1 \le i \le m}$, $\{\sigma_j\}_{1 \le j \le n}$ are stable on $\tilde{R}$. We define $\tilde{\mathcal{W}}_{a,t} = \tilde{R} / \sum_{i=1}^{m} D_{a,i,t} \tilde{R}$ which is again a left $(\mathcal{R}_1 = \Omega(t)[\sigma_1, \ldots, \sigma_n])$-module.

Let $\tilde{1}$ be the class of 1 in $\tilde{\mathcal{W}}_{a,t}$. Let $\mathfrak{A}_1(a)$ denote the annihilator of $\tilde{1}$ in $\mathcal{R}_1$.

4. Dual modules.

4.1. We construct a space adjoint to R'. Let

$$R'^* = \left\{ \sum_{u \in H_0} B_u \frac{1}{X^u} \,|\, B_u \in \Omega(t) \,\forall u \in H_0 \right\}$$

an $\Omega(t)$-space (not a ring) whose elements include infinite sums over $H_0 (= \mathbb{Z}^m)$. We have a pairing $R'^* \times R' \to \Omega(t)$ given by

$$(\xi^*, \xi) \mapsto \langle \xi^*, \xi \rangle \overset{\text{def}}{=} \text{the coefficient of } X^0 \text{ in } \xi^* \xi .$$

By this pairing we identify R'^* with $\operatorname{Hom}(R', \Omega(t))$ and adjoint to $D_{a,i,t}$ we have

$$D^*_{a,i,t} = -E_i + a_i + g_i \qquad (1 \le i \le m).$$

The connection on R'^* takes the form

$$\sigma^* \left(\frac{\partial}{\partial t_j} \right) = \sigma_j^* = \frac{\partial}{\partial t_j} - \frac{\partial g}{\partial t_j} \qquad (1 \le j \le n)$$

and we have the basic relation

$$\frac{\partial}{\partial t_j} \langle \xi^*, \zeta \rangle = \langle \sigma_j^* \xi^*, \xi \rangle + \langle \xi^*, \sigma_j \xi \rangle . \tag{4.1.1}$$

We define $\mathcal{K}'_{a,t}$ to be the annihilator of $\sum_{i=1}^{m} D_{a,i,t} R'$ in $\mathcal{R}'^*$, i.e.

$$\mathcal{K}'_{a,t} = \{ \xi^* \in R'^* \,|\, D^*_{a,i,t} \xi^* = 0, \ 1 \le i \le m \} .$$

We have a connection on $\mathcal{K}'_{a,t}$ induced by the restriction of $\{\sigma_j^*\}_{1 \le j \le n}$.

It is known [D, chap. 9] that $\mathcal{K}'_{a,t}$ is a finite $\Omega(t)$-space and if $\xi^* = \sum_{u \in \mathbb{Z}^m} B_u (1/X^u)$, $B_u \in \Omega[\![t]\!] \,\forall u$, satisfies the conditions that $D^*_{a,i,t} \xi^* = 0$, $1 \le i \le m$, then $\zeta^* \in \mathcal{K}'_{a,t} \otimes_{\Omega(t)} \Omega((t))$.

4.2. The Ω-space $\mathcal{K}'_{a,0}$ is easily described. If is of dimension 1; we describe a basis element $\xi^*_{a,0}$. If a satisfies the condition

(4.2.1) $$a_i \notin \mathbb{N}^\times \text{ for any } i \in \{1, \dots, m\}$$

then we may take

(4.2.2) $$\zeta^*_{a,0} = \sum_{u \in H_0} \frac{1}{X^u} \prod_{i=1}^{m} (a_i)_{u_i}.$$

If on the contrary $\mathfrak{S}$ is the set of all i such that $a_i \notin \mathbb{N}^\times$ and $\mathfrak{S}'$ is the complementary set in $\{1, 2, \dots, m\}$ then a basis is given by

(4.2.3) $$\zeta^*_{a,0} = \prod_{i \in \mathfrak{S}} \left(\sum_{u_i = -\infty}^{\infty} \frac{(a)_{u_i}}{X_i^{u_i}} \right) \cdot \prod_{i \in \mathfrak{S}'} X_i^{a_i} \cdot \exp\left(- \sum_{i \in \mathfrak{S}'} X_i \right).$$

We now put

$$\xi^*_{a,t} = \zeta^*_{a,0} \exp(g(t, X) - g(0, X)).$$

We conclude that

(4.2.4.1) $$\xi^*_{a,t} \in \sum_{u \in \mathbb{Z}^m} \frac{1}{X^u} \Omega[\![t]\!],$$

(4.2.4.2) $$D^*_{a,i,t} \xi^*_{a,t} = 0, \qquad 1 \leq i \leq m,$$

(4.2.4.3) $$\sigma^*_j \xi^*_{a,t} = 0, \qquad 1 \leq j \leq n.$$

Therefore $\xi^*_{a,t}$ is a horizontal element of $\mathcal{K}'_{a,t} \otimes_{\Omega(t)} \Omega((t))$.

PROPOSITION 4.2.5.
(4.2.5.1) *If a satisfies* (4.2.1) *then*

$$y(a, t) = \langle \xi^*_{a,t}, 1 \rangle.$$

(4.2.5.2) *If a satisfies* 2.1 *but not* (4.2.1) *then*

$$0 = \langle \xi^*_{a,t}, 1 \rangle.$$

PROOF. The first assertion follows by a routine calculation using (4.2.2) The second assertion follows from the fact that if $a_1 \in \mathbb{N}^\times$ then by (4.2.3) the support of $\xi^*_{a,0}$ lies in $u_1 \geqslant 1$ while by (2.1) the support of $g(t, X) - g(0, X)$ lies in $u_1 \geqslant 0$ and hence the same holds for $\exp(g(t, X) - g(0, X))$. The assertion now follows from the definitions.

PROPOSITION 4.2.6. *If a satisfies* (4.2.1) *then* $\mathfrak{A}_1(a) \subset \mathfrak{A}(a)$.

PROOF. If $\theta \in \mathcal{R}$ then by (4.1.1), (4.2.5.1)

$$\theta(t, \delta)\, y = \langle \xi^*_{a,t}\, \theta(t, t\sigma)\, 1 \rangle$$

and so if $\theta(t, t\sigma)[1] = 0$ in $\mathcal{W}'_{a,t}$ then $\theta(t, \delta)\, y = 0$.

REMARK 4.2.6.1. We say that «y is a period of $[1]$».

REMARK 4.2.6.2. The conclusion of the proposition need not hold if (4.2.1) is not satisfied. Thus if $m = 3$, $n = 1$, $A = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $a = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, then $(1 + t\sigma)[1] = 0$ while $y(a, t) = \sum_{s=0}^{\infty} (t^s / s + 1)$ (cf. Proposition 7.4).

4.3. *Adjoint of* $\tilde{R}$. Let

$$\tilde{R}^* = \left\{ \sum_{u \in \tilde{H}_0} B_u \frac{1}{X^u} \,\middle|\, B_u \in \Omega(t)\, \forall u \in \tilde{H}_0 \right\}.$$

The pairing of R'^* with R' restricts to a pairing of $\tilde{R}^*$ with $\tilde{R}$ by which $\tilde{R}^*$ may be identified with $\mathrm{Hom}(\tilde{R}, \Omega(t))$. The injection $\tilde{R} \hookrightarrow R'$ has an adjoint mapping, $\tilde{\gamma}_-$ of R'^* onto $\tilde{R}^*$, a projection

$$\tilde{\gamma}_- \frac{1}{X^u} = \begin{cases} \dfrac{1}{X^u} & \text{if } u \in \tilde{H}_0\,, \\ 0 & \text{if } u \neq \tilde{H}_0\,. \end{cases}$$

The adjoint of $D_{a,i,t}$ is now

$$\tilde{D}^*_{a,i,t} = \tilde{\gamma}_- \circ D^*_{a,i,t}\,, \qquad 1 \leq i \leq m$$

and the connection on $\tilde{R}^*$ is given by

$$\tilde{\sigma}^*_j = \tilde{\gamma}_- \circ \sigma^*_j\,, \qquad 1 \leq j \leq n\,.$$

We again have the relation

$$\text{(4.3.1)} \qquad \frac{\partial}{\partial t_j} \langle \xi^*, \xi \rangle = \langle \tilde{\sigma}^*_j \xi^*, \xi \rangle + \langle \xi^*, \sigma_j \xi \rangle \qquad \text{for } (\xi^*, \xi) \in \tilde{R}^* \times \tilde{R}\,.$$

We define $\tilde{\mathcal{K}}_{a,t}$ to be the annihilator of $\sum D_{a,i,t} \tilde{R}$ in $\tilde{R}^*$, i.e.

$$\tilde{\mathcal{K}}_{a,t} = \{ \xi^* \in \tilde{R}^* \mid D^*_{a,i,t} \xi^* = 0,\ 1 \leq i \leq m \}.$$

We have a connection on $\tilde{\mathcal{K}}_{a,t}$ induced by $\{\tilde{\sigma}^*_j\}_{1 \leq j \leq m}$.

4.4. We describe the Ω-space $\tilde{\mathcal{K}}_{a,0}$. It is of dimension 1. If a satisfies the condition

$$\text{if } a_i \in \mathbb{N}^{\times} \text{ then } i \in \mathfrak{S}_2\,, \tag{4.4.1}$$

then the basis element of $\tilde{\mathcal{K}}_{a,0}$ may be chosen to be

$$\tilde{\xi}^*_{a,0} = \sum_{u \in \tilde{H}_0} \frac{1}{X^u} \prod_{i=1}^{m} (a_i)_{u_i}\,. \tag{4.4.2}$$

(The formula is the same as in (4.2.2) but the sum is over a smaller set). By (1.4) this series is well defined. If on the contrary $\mathfrak{S}_1 = \mathcal{F} \cup \mathcal{F}'$ where $a_i \notin \mathbb{N}^{\times}$ for all $i \in \mathcal{F}$ and $a_i \in \mathbb{N}^{\times}$ for all $i \in \mathcal{F}'$, then the basis element may by chosen to be

$$\tilde{\xi}^*_{a,0} = \prod_{i \in \mathfrak{S}_2} \sum_{u_i \in \mathbb{N}} \frac{(a_i)_{u_i}}{X_i^{u_i}} \cdot \prod_{i \in \mathcal{F}} \sum_{u_i \in \mathbb{Z}} \frac{(a_i)_{u_i}}{X_i^{u_i}} \cdot \prod_{i \in \mathcal{F}} X_i^{a_i} \cdot \exp\left(- \sum_{i \in \mathcal{F}} X_i \right). \tag{4.4.3}$$

We now put

$$\tilde{\xi}^*_{a,t} = \tilde{\gamma} - \xi^*_{a,0} \exp\left(g(t, X) - g(0, X)\right).$$

We conclude that

$$\tilde{\xi}^*_{a,t} \in \sum_{u \in \tilde{H}_0} \frac{1}{X^u} \Omega[\![t]\!]\,, \tag{4.4.4.1}$$

$$\tilde{D}^*_{a,i,t}\, \tilde{\xi}^*_{a,t} = 0\,, \qquad 1 \leqslant i \leqslant m\,, \tag{4.4.4.2}$$

$$\tilde{\sigma}^*_j\, \tilde{\xi}^*_{a,t} = 0\,, \qquad 1 \leqslant j \leqslant n\,. \tag{4.4.4.3}$$

PROPOSITION 4.4.5. *If a satisfies both* (2.1) *and* (4.4.1) *then*

$$y(a, t) = \langle \tilde{\xi}^*_{a,t}, 1 \rangle .$$

If a fails to satisfy (4.4.1) *but does satisfy* (2.1) *then*

$$0 = \langle \tilde{\xi}^*_{a,t}, 1 \rangle .$$

PROOF. The proof is the same as that of Proposition 4.2.5, except that for the first assertion we must use (3.2.1)

PROPOSITION 4.4.6. *If a satisfies both* (2.1) *and* (4.4.1) *then*

$$\tilde{\mathfrak{A}}_1(a) \subset \mathfrak{A}(a),$$

i.e. $y(a, t)$ *is a period of* $\tilde{1}$, *the class of* 1 *in* $\tilde{\mathfrak{W}}_{a,t}$.

PROOF. The proof is the same as that of Proposition 4.2.6.

REMARK 4.4.7. Trivially $\tilde{\mathfrak{A}}_1(a) \subset \mathfrak{A}_1(a)$.

5. Differential relations.

The symbols $A, a, \delta, \mathcal{R}$ are as in § 2.

NOTATION 5.0. For $\mathbb{N}^m$, let

$$h_u(a, \delta) = \prod_{i=1}^{m} (a_i + l_i(\delta))_{u_i} \in \Omega[\delta].$$

For $1 \leqslant j \leqslant n$ we define m-tuples in $\mathbb{N}^m$, $v^{(j)}$, $u^{(j)}$, by

$$v_i^{(j)} = \sup(0, A_{i,j}), \qquad u_i^{(j)} = \sup(0, -A_{i,j}).$$

Thus

$$A^{(j)} = v^{(j)} - u^{(j)}. \tag{5.0.1}$$

We define

$$L_j(a, t, \delta) = \delta_j \circ h_{u^{(j)}}(a, \delta) + t_j h_{v^{(j)}}(a, \delta).$$

For $x \in \mathbb{R}$ let $\bar{x} = \sup(0, -x)$, $\hat{x} = \sup(0, x)$.

PROPOSITION 5.1. *If a satisfies* (2.1) *then* $L_j(a, t, \delta) \in \mathfrak{A}(a)$.

PROOF. It is enough to check that for $s \in \mathbb{N}$, $s_j \geqslant 1$

$$\delta_j \frac{(-t_j)^{s_j}}{s_j!} + t_j \frac{(-t_j)^{s_j - 1}}{(s_j - 1)!} = 0, \tag{5.1.1}$$

$$(a_i + l_i(s))_{u_i^{(j)}} (a_i)_{l_i(s)} = (a_i + l_i(s - \varepsilon_j))_{v_i^{(j)}} (a_i)_{l_i(s - \varepsilon_j)}, \tag{5.1.2}$$

where ε_j is the unit vector in the j-th direction in n-space. The second relation follows from $v_i^{(j)} - l_i(\varepsilon_j) = u_i^{(j)}$.

PROPOSITION 5.2. *For $w \in \mathbb{N}^n$, subject to* 2.1, *we have*

$$h_w(a - w, \delta)y(a - w, t) = y(a, t)\prod_{i=0}^{m}(a_i - w_i)_{w_i}. \tag{5.2.1}$$

PROOF. If a satisfies (2.1) then so does $a - w$ and hence both $y(a, t)$ and $y(a - w, t)$ are well defined. The assertion follows from (1.8), from which we deduce $(a_i - w_i + l_i(s))_{w_i}(a_i - w_i)_{l_i(s)} = (a_i)_{l_i(s)}(a_i - w_i)_{w_i}$ for all $s \in \mathbb{N}^n$.

REMARK 5.2.2. If, say, $a_1 \in -\mathbb{N}^\times$, $w_1 \geq 1 + |a_1|$, then the right hand side in 5.2.1 is zero.

PROPOSITION 5.3. *If $u, v \in \mathbb{N}^m$, $a \in \mathbb{C}^m$ then*

$$h_v(a + u)h_u(a) = h_{v+u}(a).$$

PROOF. If is enough to check that $(a_i + u_i + l_i(\delta))_{v_i}(a_i + l_i(\delta))_{u_i} = (a_i + l_i(\delta)))_{u_i + v_i}$.

PROPOSITION 5.4. *If $a \in \mathbb{C}^m$ then*

$$L_j(a + \varepsilon_i) \circ (a_i + l_i(\delta)) = (a_i + l_i(\delta) + \overline{l_i(\varepsilon_j)}) \circ L_j(a).$$

PROOF. The assertion is equivalent to the two identities in the commutative ring $\Omega[\delta]$

$$\delta_j h_{u^{(j)}}(a + \varepsilon_i, \delta)(a_i + l_i(\delta)) = (a_i + l_i(\delta) + \overline{l_i(\varepsilon_j)})\delta_j h_{u_{(j)}}(a, \delta), \tag{5.4.1}$$

$$h_{v^{(j)}}(a + \varepsilon_i, \delta)(a_i + l_i(\delta)) = (a_i + l_i(\delta + \varepsilon_j) + \overline{l_i(\varepsilon_j)})h_{v_{(j)}}(a, \delta), \tag{5.4.2}$$

Discarding the obviously identical factors on the two side of these assertions, we reduce, using $\overline{l_i(\varepsilon_j)} = \overline{A}_{i,j} = u_i^{(j)}$, to the assertions

$$(a_i + 1 + l_i(\delta))_{u_i^{(j)}}(a_i + l_i(\delta)) = (a_i + l_i(\delta) + u_i^{(j)})(a_i + l_i(\delta))_{u_i^{(j)}}, \tag{5.4.1$'$}$$

$$(a_i + 1 + l_i(\delta))_{v_i^{(j)}}(a_i + l_i(\delta)) = (a_i + l_i(\delta) + v_i^{(j)})(a_i + l_i(\delta))_{v_i^{(j)}}. \tag{5.4.2$'$}$$

These assertion are implied by the identity $(x + 1)_b x = (x + b)(x)_b$ for $b \in \mathbb{N}$.

PROPOSITION 5.5. *For* $a \in \mathbb{C}^m$, $v \in \mathbb{N}^m$

$$L_j(a+v, \delta)\, h_v(a, \delta) = h_v(a + \overline{l(\varepsilon_j)}, \delta)\, L_j(a, \delta)$$

where $\overline{l(\varepsilon_j)}$ *is the m-tuple* $(\overline{l_1(\varepsilon_j)}, \ldots, \overline{l_m(\varepsilon_j)})$.

PROOF. We use induction on $\sum_{i=1}^{m} v_i = \text{weight}(v)$. The assertion is trivial for weight $(v) = 0$ and the case of weight $(v) = 1$ is given by Proposition 5.4. By that proposition (with a replaced by $a + v$)

$$L_j(a + v + \varepsilon_i, \delta)(a_i + v_i + l_i(\delta)) = (a_i + v_i + l_i(\delta) + \overline{l_i(\varepsilon_j)})\, L_j(a + v, \delta).$$

Multiplying on the right by $h_v(a, \delta)$, the left side becomes $L_j(a + v + \varepsilon_i, \delta)\, h_{v+\varepsilon_i}(a, \delta)$ while the right side becomes $(a_i + v_i + l_i(\delta) + \overline{l_i(\varepsilon_j)})\, L_j(a + v, \delta)\, h_v(a, \delta)$, which by the induction hypothesis is $(a_i + v_i + l_i(\delta) + \overline{l_i(\varepsilon_j)})\, h_v(a + \overline{l_i(\varepsilon_j)}, \delta)\, L_j(a, \delta)$ which coincides with $h_{v+\varepsilon_i}(a + \overline{l(\varepsilon_j)}, \delta)\, L_j(a, \delta)$.

DEFINITION 5.5.1. *For* $\theta \in \Omega[t, \delta]$ *viewed as a polynomial ring in* $t = (t_1, \ldots, t_n)$ *with coefficients in* $\Omega[\delta]$, *let*

$$\text{rank}\, \theta = \sum_{j=1}^{n} \sup(0, \deg_{t_j} \theta).$$

For $a \in \mathbb{C}^m$ *let* $\mathscr{B}(a) = \sum \Omega[t, \delta]\, L_j(a, \delta)$, *a left ideal in* $\Omega[t, \delta]$.

PROPOSITION 5.6. *Let* $a \in \mathbb{C}^m$. *For* $\theta \in \Omega[t, \delta]$, *let*

$$w = \sum_{j=1}^{n} \sup(0, \deg_{t_j} \theta)\, v^{(j)}$$

an element of $\mathbb{N}^m$. *There exists* $P \in \Omega[\delta]$ *such that*

$$\theta \circ h_w(a - w) \in P + \mathscr{B}(a - w).$$

The assertion remains valid if w *is replaced by* $w + u$ *for any* $u \in \mathbb{N}^m$.

PROOF. The assertion is trivial if $\text{rank}\, \theta = 0$. We use induction on the rank of θ. We may assume $\deg_{t_1} \theta \geqslant 1$. We write

$$\theta = P_1 t_1 + P_2 \tag{5.6.1}$$

where $P_1, P_2 \in \Omega[t, \delta]$, with $\deg_{t_1} P_2 = 0$ and

$$\text{(5.6.2)} \qquad \operatorname{Sup}(\deg_{t_j} P_1, \deg_{t_j} P_2) \leqslant \deg_{t_j} \theta , \qquad 2 \leqslant j \leqslant n .$$

Multiplyng on the right we obtain

$$\text{(5.6.3)} \qquad \theta \circ h_{v^{(1)}}(a - v^{(1)}, \delta) = P_1 t_1 h_{v^{(1)}}(a - v^{(1)}, \delta) + P_3 = \\ = P_1 L_1 (a - v^{(1)}, \delta) + P_4$$

$$\text{(5.6.4)} \quad P_3 = P_2 \circ h_{v^{(1)}}(a - v^{(1)}, \delta) , \quad P_4 = -P_1 \delta_1 h_{u^{(1)}}(a - v^{(1)}, \delta) + P_3 .$$

It follows from these formulae that

$$\text{(5.6.5)} \qquad \deg_{t_j} P_4 \leqslant \deg_{t_j} \theta - \delta_{1,j} , \qquad 1 \leqslant j \leqslant n$$

and hence $P_4 < \operatorname{rank} \theta$. Letting $w' = w - v^{(1)}$ and applying the induction hypothesis to $a - v^{(1)}$,

$$\text{(5.6.6)} \qquad P_4 \circ h_{w'}(a - v^{(1)} - w', \delta) \in P + \mathcal{B}(a - w)$$

where $P \in \Omega[\delta]$. Multiplying (5.6.3) on the right by $h_{w'}(a - v^{(1)} - w', \delta)$ and applying Proposition 5.3 with (a, u, v) replaced by $(a - v^{(1)} - w', w', v^{(1)})$ we obtain

$$\text{(5.6.3)} \qquad \theta \circ h_w(a - w) = P_1 L_1 (a - v^{(1)}, \delta) \circ h_{w'}(a - v^{(1)} - w', \delta) + \\ + P_4 \circ h_{w'}(a - v^{(1)} - w', \delta).$$

Applying Proposition 5.5 with (a, v, j) replaced by $(a - w, w', 1)$ we see that the first term on the right side of (5.6.7) lies in $\mathcal{B}(a - w)$. By (5.6.6) the second term lies in $P + \mathcal{B}(a - w)$. This completes the proof of the proposition.

PROPOSITION 5.7. *If a satisfies* (2.1) *and if $\theta \in \mathfrak{A}(a) \cap \Omega[t, \delta]$ then the operator P of Proposition* 5.6 *lies in* $\mathfrak{A}(a - w) \cap \Omega[\delta]$.

PROOF. It follows from Proposition 5.1 that $\mathcal{B}(a - w) \subset \mathfrak{A}(a - w)$. It follows from Proposition 5.2 that $\theta \circ h_w(a - w) \in \mathfrak{A}(a - w)$. The assertion is now clear.

6. Differential relations for σ.

We consider $h_u(a, t\sigma)$ and $L_j(a, t, t\sigma)$ elements of $\mathcal{R}_1$ defined as in 5.0 but with $\partial/\partial t_j$ replaced by σ_j for $1 \leqslant j \leqslant n$.

PROPOSITION 6.1. *For $u \in \mathbb{N}^n$, $a \in \mathbb{C}^m$*

$$h_u(a, t\sigma)[t] = [X^u] \text{ in } \mathcal{W}'_{a,t} \tag{6.1.1}$$

$$h_u(a, t\sigma)\tilde{1} = \tilde{X}^u, \quad \text{the class of } X^u \text{ in } \tilde{\mathcal{W}}_{a,t}. \tag{6.1.2}$$

PROOF. We observe that

$$a_i + l_i(t\sigma) = a_i + \sum A_{i,j}\left(t_j \frac{\partial}{\partial t_j} + t_j \frac{\partial g}{\partial t_j}\right) =$$

$$= l_i(\delta) + a_i - \sum_{j=1}^{n} A_{i,j} t_j X^{A^{(j)}} = l_i(\delta) + a_i + g_i + X_i .$$

Thus

$$a_i + l_i(t\sigma) = l_i(\delta) + X_i - E_i + D_{a,i,t}. \tag{6.1.3}$$

Thus for $v \in \mathbb{Z}^m$

$$(a_i + l_i(t\sigma))[X^v] = [(X_i - E_i)X^v] \tag{6.1.4}$$

and so (6.1.1) is a consequence of the calculation for $u \in \mathbb{N}^m$

$$\prod_{i=1}^{m}(X_i - E_i)_{u_i} 1 = X^u . \tag{6.1.5}$$

The proof of (6.1.2) is precisely the same except that (6.1.4) now takes the form

$$(a_i + l_i(t\sigma))\tilde{X}^v = \text{the class in } \tilde{\mathcal{W}}_{a,t} \text{ of } (X - E_i)X^v \tag{6.1.6}$$

for all $v \in \tilde{H}_0$ and in particular for all $v \in \mathbb{N}^m$.

PROPOSITION 6.2.

$$L_j(a, t\sigma)[1] = 0 \quad \text{in } \mathcal{W}'_{a,t}, \tag{6.2.1}$$

$$L_j(a, t\sigma)\tilde{1} = 0 \quad \text{in } \tilde{\mathcal{W}}_{a,t}. \tag{6.2.2}$$

PROOF. It follows from the definition and Proposition 6.1 that we need only show the vanishing of the class of $t_j\sigma_j X^{u^{(j)}} + t_j X^{v^{(j)}}$. This is trivial since $\sigma_j X^{u^{(j)}} = (\partial g/\partial t_j)\cdot X^{u^{(j)}} = -X^{A^{(j)}+u^{(j)}} = -X^{v^{(j)}}$.

REMARK 6.2.3. Proposition 6.2 together with (4.4.5) gives a second proof of Proposition 5.1. (We may assume (2.1) and then choose $\mathfrak{S}_2$ so that (3.2.1) and (4.4.1) are satisfied).

PROPOSITION 6.3. *If* $\theta \in \mathfrak{A}(a) \cap \Omega[t, \delta]$ *then subject to the conditions*

(4.2.1) $$a_i \notin \mathbb{N}^\times \quad \text{for any } i \in \{1, \ldots m\},$$

(6.3.1) $$\textit{if } a_i \in -\mathbb{N} \textit{ then } A_{i,j} \in -\mathbb{N} \quad (1 \leq j \leq n).$$

we have

$$\theta \circ h_w(a - w) \in \mathscr{B}(a - w)$$

where w *is defined in Proposition* 5.6.

PROOF. It follows from (1.5) and the hypotheses that $y(a - w, t) = \sum_{s \in \mathbb{N}^n} C(s) t^s$ where $C(s) \in \mathbb{C}^\times$ for all $s \in \mathbb{N}^n$. The point is that by (4.2.1), $a_i - w_i \in -\mathbb{N} \Rightarrow a_i \in -\mathbb{N} \Rightarrow l_i(s) \in -\mathbb{N}$ by (6.3.1).

Since the operator P of Proposition 5.6 lies in $\Omega[\delta]$ and by Proposition 5.7 must annihilate $y(a - w, t)$, we conclude that $0 = P(s)C(s)$ for all $s \in \mathbb{N}^n$ and so $P = 0$, which completes the proof.

PROPOSITION 6.4. *If* a *satisfies* (4.2.1) *and* (6.3.1) *then*

$$\mathfrak{A}_1(a) \supset \mathfrak{A}(a).$$

PROOF. Let $\theta \in \mathfrak{A}(a)$. Without loss in generality we may assume $\theta \in \Omega[t, \delta]$. Hence by Proposition 6.3, letting w be as in Proposition 5.6; $\theta \circ h_w(a - w, \delta) \in \mathscr{B}(a - w)$. Thus by (6.2.1), replacing $\partial/\partial t_j$ by σ_j $(1 \leq j \leq n)$,

(6.4.1) $$\theta(t, \sigma) h_w(a - w, t\sigma)[1] = 0 \quad \text{in } \mathscr{W}'_{a-w,t}.$$

Thus by (6.1.1)

(6.4.2) $$\theta(t, \sigma)[X^w] = 0 \quad \text{in } \mathscr{W}'_{a-w,t}.$$

Now multiplication by X^{-w} commutes with σ and this multiplication in R' induces a mapping of $\mathscr{W}'_{a-w,t}$ into $\mathscr{W}'_{a,t}$. We conclude that

(6.4.3) $$\theta(t, \sigma)[1] = 0 \quad \text{in } \mathscr{W}'_{a,t}.$$

This completes the proof.

COROLLARY 6.5. *Subject to* (4.2.1) *and* (6.3.1), $\mathfrak{A}_1(a) = \mathfrak{A}(a)$.

PROPOSITION 6.6. *Subject to* (2.1), (4.4.1) *and* (6.3.1), *a period* $z = \sum_{s \in \mathbb{N}^n} C(s) t^s$ *of* $\tilde{1}$ *in* $\Omega[\![t]\!]$, *is uniquely determined by* $C(0)$.

PROOF. It follows from (6.2.2) that $L_j(a, t, \delta) z = 0$ and hence

$$(6.6.1) \qquad (1 + s_j) h_{u^{(j)}}(a, s + \varepsilon_j) C(s + \varepsilon_j) + h_{v^{(j)}}(a, s) C(s) = 0$$

for each $s \in \mathbb{N}^n$. By (2.1) $(a_i + l_i(s))_{v_i^{(j)}}$ and $(a_i + l_i(s + \varepsilon_j))_{u_i^{(j)}}$ lie in $\mathbb{C}^\times$ for all $s \in \mathbb{N}^n$ if $a_i \in \mathbb{N}^\times$. If $a_i \in -\mathbb{N}$ then by (6.3.1) $A_{i,j} \in -\mathbb{N}$ so $v_i^{(j)} = 0$ while $l_i(s + \varepsilon_j) \leqslant l_i(\varepsilon_j) = A_{i,j} = -u_i^{(j)}$ and so by (1.7), $(a_i + l_i(s + \varepsilon_j))_{u_i^{(j)}} \neq 0$ and $(a_i + l_i(s))_{v_i^{(j)}} \neq 0$. We conclude that for all $s \in \mathbb{N}^n$, $C(s + \varepsilon_j)$ is fixed by $C(s)$. This completes the proof.

COROLLARY 6.7.1. *Subject to* (2.1), (4.4.1), (6.3.1), $y(a, t)$ *is up to a constant factor the unique period in* $\Omega[\![t]\!]$ *of* $\tilde{1}$, *the class of* 1 *in* $\tilde{\mathfrak{W}}_{a,t}$.

REMARK 6.7.2. We know under the hypotheses of the corollary that $\mathfrak{A}(a) \supset \tilde{\mathfrak{A}}_1(a) \supset \mathscr{B}(a)$. We believe but have not shown equality of $\mathfrak{A}$ and $\tilde{\mathfrak{A}}_1$.

7. Examples.

We give some examples involving $a \in \mathbb{Z}^3$. In particular we give an example in which $[1] = 0$.

Let

$$-g = X_1 + X_2 + X_3 + t\frac{X_1 X_2}{X_3}, \qquad a = (a_1, a_2, a_3).$$

Let C be the cone in $\mathbb{Q}^3$ generated by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $A(= (1, 1, -1))$. This cone is identical with the cone defined by the inequalities $f_i(u) \geqslant 0$ $(i = 1, 2, 3, 4)$ where

$$(7.1) \qquad \begin{cases} f_1(u) = u_1, \\ f_2(u) = u_2, \\ f_3(u) = u_1 + u_3, \\ f_4(u) = u_2 + u_3. \end{cases}$$

Let $\hat{H}_0$ be the intersection $C \cap \mathbb{Z}^3$. It coincides with the monoid generated by $\varepsilon_1, \varepsilon_2, \varepsilon_3, A$. Let $\hat{R}$ be the $\mathbb{Q}(t)$ span of $\{X^u\}_{u \in \hat{H}_0}$ and let $\hat{\mathfrak{W}}_{a,t} = \hat{R} / \sum_{I=1}^{3} D_{a,i} \hat{R}$. The mapping of $\hat{\mathfrak{W}}_{a,t}$ into $\mathfrak{W}'_{a,t}$ induced by the injection

$\hat{R} \hookrightarrow R'$ is known to be an isomorphism subject to condition [D, § 6.4.1]

$$f_i(a) \notin \mathbb{N}^\times, \qquad i = 1, 2, 3, 4. \tag{7.2}$$

Furthermore we know that (g_1, g_2, g_3) is a regular sequence in $\hat{R}$ (for $t \neq 0, 1, \infty$) and that any set of representatives in $\hat{R}$ of a basis of $\hat{R}/\sum g_i \hat{R}$ also represents a basis of $\hat{\mathcal{W}}_{a,t}$ and hence represents a basis of $\mathcal{W}'_{a,t}$ subject to (7.2). In particular $\{1, (X_1X_2/X_3)\}$ and $\{1, X_1\}$ represent bases of $\mathcal{W}'_{a,t}$ subject to (7.2).

PROPOSITION 7.3. *$\mathfrak{A}_1(0)$ is the left ideal generated by $(t\sigma)^2$, $\mathfrak{A}(0)$ is generated by δ and so $\mathfrak{A}(0) \supsetneq \mathfrak{A}_1(0)$.*

PROOF. The operator $L_1 = \delta(a_3 - \delta) + t(a_1 + \delta)(a_2 + \delta)$ takes the form $L_1 = (t-1)\delta_2$. Hence by Proposition 6.2

$$(t\sigma)^2[1] = 0.$$

Since $t\sigma 1 \equiv X_1$ in $\mathcal{W}'_{0,t}$ (Proposition 6.1) and since $1, X_1$ represent a basis of $\mathcal{W}'_{0,t}$, it follows that $[1]$ cannot be annihilated by any operator of degree 1 in σ. Hence $\mathfrak{A}_1(0)$ is generated by $(t\sigma)^2$. The assertion for $\mathfrak{A}(0)$ follows from $y(0, t) = 1$.

PROPOSITION 7.4.

$$\mathfrak{A}_1((1, 1, -1)) \quad \textit{is generated by } 1 + t\sigma,$$

$$\mathfrak{A}((1, 1, -1)) \quad \textit{is generated by } \delta \circ (1 - t) \circ (1 + \delta).$$

Therefore

$$\mathfrak{A}_1(1, 1, -1)) \supsetneq \mathfrak{A}((1, 1, -1)).$$

PROOF. Let $b = (1, 1, -1)$. By Proposition 6.1 and Proposition 7.3

$$X_1X_2 \equiv (t\sigma)(t\sigma)1 \equiv 0 \quad \text{in } \mathcal{W}'_{0,t}.$$

Multiplication by X_3/X_1X_2 in R' induces an isomorphism of $\mathcal{W}'_{0,t}$ onto $\mathcal{W}'_{b,t}$. Hence

$$X_3 \equiv 0 \quad \text{in } \mathcal{W}'_{b,t}.$$

But by Proposition 6.1

$$X_3 \equiv (-1 - t\sigma)1 \quad \text{in } \mathcal{W}'_{b,t}.$$

Since X_1X_2/X_3 is a basis element (and hence non zero) in $\mathcal{W}'_{0,t}$ the isomorphism shows $1 \not\equiv 0$ in $\mathcal{W}'_{b,t}$. Thus $\mathfrak{A}_1(b)$ is generated by $1 + t\sigma$. The assertion for $\mathfrak{A}(b)$ follows form $y(b, t) = \sum t^s/(1+s)$ which cannot be annihilated by a first order operator but is annihilated by L_1.

PROPOSITION 7.5.

$$[1] = 0 \quad \textit{in } \mathcal{W}'_{(1,2,-1),t}.$$

PROOF. By Proposition 6.1, $X_2 \equiv (1+t\sigma)1 \equiv 0$ in $\mathcal{W}'_{b,t}$ where $b = (1, 1, -1)$ as in Proposition 7.4. Multiplication by $1/X_2$ induce an isomorphism of $\mathcal{W}'_{b,t}$ onto $\mathcal{W}'_{(1,2,-1),t}$ thus $[1] = 0$ as asserted.

8. Delsarte sums.

The object of this section is to show that very general exponential modules have hypergeometric series as periods. We fill in some lacunes in the corresponding treatment in [D-L2].

Let $\omega^{(1)}, \ldots, \omega^{(m)}$ be a set of elements of $\mathbb{Z}^m$ which are linearly independent over $\mathbb{Q}$. Let Λ be the lattice $\sum_{i=1}^{m} \mathbb{Z}\omega^{(i)}$. Let

$$-h = \sum_{i=1}^{m} X^{\omega^{(i)}}$$

viewed as element of $\Omega[X_1, \ldots X_m, X_1^{-1}, \ldots, X_m^{-1}] = R'$. Let $L_1, \ldots, L_m$ be $\mathbb{Q}$-linear forms in m variables.

$$L_i(\omega^{(k)}) = \delta_{i,k} \qquad 1 \leq i, k \leq m.$$

Let $\tilde{R}$ be the Ω span of $\{X^v\}_{v\in\Lambda}$. Let $\mathfrak{a}$ be a set of representatives of $\mathbb{Z}^m/\Lambda$. Then $R' \bigoplus_{u\in\mathfrak{a}} X^u\tilde{R}$ as Ω-spaces. For $i = 1, 2, \ldots, m$ let $D_{a,i} = E_i + a_i + h_i$, $E_i = X_i(\partial/\partial X_i)$, $h_i = E_i h$.

PROPOSITION 8.1. *$D_{a,i}$ is stable on $\tilde{R}$ and on cosets $X^u\tilde{R}$.*

PROOF. The assertion in easily verified. For later use we state some formalities. We observe that

$$-\boldsymbol{h} = -\begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = (\omega^{(1)}, \ldots, \omega^{(m)}) \begin{pmatrix} X^{\omega^{(1)}} \\ \vdots \\ X^{\omega^{(m)}} \end{pmatrix}$$

and hence

$$X^{\omega^{(i)}} = -L_i(\boldsymbol{h})$$

where if $L_i(u) = \sum C_{i,k} u_k$ then $L_i(\boldsymbol{h})$ denotes $\sum C_{i,k} h_k$. We write $L_i(D) = \sum C_{i,k} D_{a,k}$ and observe that

(8.1.1) $\quad L_i(D) = L_i(a) + L_i(E) + L_i(\boldsymbol{h}) = L_i(a) + L_i(E) - X^{\omega^{(i)}}$.

PROPOSITION 8.2. *Let* $\mathfrak{W}'_a = R'/\sum D_{a,i} R'$, $\tilde{\mathfrak{W}}'_a = \tilde{R}/\sum D_{a,i}\tilde{R}$. *Then*

$$\mathfrak{W}'_a \simeq \bigoplus_{u \in \mathfrak{a}} \tilde{\mathfrak{W}}_{a+u}.$$

PROOF. The assertion follows from the direct sum decomposition of R' into cosets $X^u \tilde{R}$ and from the commutativity relation

$$D_{a,i} \circ X^u = X^u D_{a+u,i}.$$

PROPOSITION 8.3.

$$\dim \tilde{\mathfrak{W}}'_a = 1.$$

PROOF. Let $Y_i = X^{\omega^{(i)}}$ $(i = 1, \ldots, m)$. Then for $v \in \mathbb{Z}^m$ we compute

$$L_i(D)\, Y^v = \left(L_i \left(a + \sum_{k=1}^{m} v_k \omega^{(k)} \right) - Y_i \right) Y^u = (L_i(a) + v_i - Y_i)\, Y^v.$$

Thus putting $b \in \Omega^m$, $b_i = L_i(a)$ we reduce to the case in which $-h(X) = X_1 + \ldots + X_m$ and we must show that $\mathfrak{W}'_b$ is of dimension 1. Summarizing a well known method, we first let $\hat{R} = \Omega[X_1, \ldots, X_m]$. We know that $\sum \hat{R} h_i$ is a maximal ideal of $\hat{R}$ and that

(8.3.1) $$\hat{R} = \Omega \cdot 1 \oplus \left(\sum \hat{R} h_i \right).$$

Furthermore $(h_1, \ldots, h_m)$ is a regular sequence in $\hat{R}$ and hence any relation $0 = \sum_{i=1}^{m} P_i h_i$ where P_i is homogeneous of degree l must be trivial in the sense that

$$P_i = \sum_{i=1}^{m} h_k q_{ik}$$

where each q_{ik} is a homogeneous element of $\hat{R}$ of degree $l-$ and $q_{ik} = -q_{ki}$. Thus if ξ lies in $\hat{R}$, $\deg \xi = l \geqslant 1$, then its homogeneous part of maximal degree may be written in the form $\sum_{i=1}^{m} X_i Z_i$ where Z_i is homogeneous of degree $l-1$ and so $\sum X_i Z_i = \sum D_{a,i}(-Z_i) + \sum (b_i + E_i) Z_i$ which reduces the degree of ξ modulo $\sum D_{a,i} \mathcal{R}$. This shows that

$$\hat{\mathcal{R}} = \Omega \cdot 1 + \sum D_{a,i} \hat{\mathcal{R}} .$$

To show that $1 \notin \sum D_{a,i} \hat{\mathcal{R}}$, suppose otherwise. So $1 = \sum D_{a,i} \xi_i$ and let $l = \sup \deg \xi_i$. Thus $\xi_i = P_i + Z_i$ where P_i is homogeneous of degree l and $\deg Z_i \leqslant l-1$. We show that l cannot be minimal. Clearly $0 = \sum P_i X_i$ and hence $\exists \{q_{i,j}\}$ homogeneous of degree $l-1$, $q_{i,j} = -q_{j,i}$ $(1 \leqslant i, j \leqslant m)$ such that

$$P_i = \sum_{j=1}^{m} q_{i,j} X_j = \sum_{j=1}^{m} D_{a,j} q_{i,j} + \rho_i$$

where $\deg \rho_i \leqslant l-1$. Let $\xi_i' = \rho_i + Z_i$. Then

$$1 = \sum D_{a,i} \xi_i = \sum_{i=1}^{m} D_{a,i} \left(\xi_i' + \sum_{i=1}^{m} D_{a,j} q_{i,j} \right) = \sum_{i=1}^{m} D_{a,i} \xi_i' \text{ and } \deg \xi_i' \leqslant l-1.$$

This then shows that $\dim \hat{\mathcal{W}}_a = 1$ without condition on a. To continue our demonstration we introduce the hypothesis.

$$a_i \notin \mathbb{N}^{\times} . \tag{8.3.2}$$

This will not be needed for the final proof. Subject to 8.3.2 we show that for $u \in \mathbb{N}^m$

$$\hat{R} = X^u \hat{R} + \sum D_{a-u,i} \hat{R} . \tag{8.3.3}$$

Let $v \in \mathbb{N}^m$. If $v - u \in \mathbb{N}^m$ the $X^v \in X^u \hat{R}$ there is nothing to prove. Suppose that $v_1 - u_1 < 0$. Then $D_{a-u,1} X^v = (a_1 - (u_1 - v_1)) X^v - X_1 X^v$ and so we may replace X^v by $X_1 X^v \bmod \sum D_{a-u,i} \hat{R}$ provided $a_1 \neq u_1 - v_1 \in \mathbb{N}^{\times}$. By iteration, assertion 8.3.3 is clear.

This shows that subject to (8.3.2) the mapping of $\hat{\mathcal{W}}_a$ into $\hat{\mathcal{W}}_{a-u}$ induced by multiplication by X^u is surjective. The dimensions are equal and hence the mapping is an isomorphism. Hence subject to (8.3.2) we have

$$X^u \hat{R} \cap \sum D_{a-u,i} \hat{R} = X^u \sum D_{a,i} \hat{R} . \tag{8.3.4}$$

Dividing by X^u we deduce for all $u \in \mathbb{N}^m$.

(8.3.3′) $$\frac{1}{X^u}\hat{R} = \hat{R} + \sum D_{a,i}\frac{1}{X^u}\hat{R}$$

(8.3.4′) $$\sum D_{a,i}\hat{R} = \hat{R} \cap \sum_{i=1}^{m} D_{a,i}\frac{1}{X_u}\hat{R}\,.$$

Since $R' = \bigcap_{u \in \mathbb{N}^m} (1/X^u)\hat{R}$ we deduce

(8.3.3″) $$R' = \hat{R} + \sum D_{a,i}R'\,,$$

(8.3.4″) $$\sum D_{a,i}\hat{R} = \hat{R} \cap \sum D_{a,i}R'\,.$$

Thus subject to (8.3.2) the natural mapping of $\hat{R}$ into R' induces an isomorphism of $\hat{\wp}_a$ with $\wp'_a$. This proves the assertion subject to (8.3.2).

But given $a \in \mathbb{C}^m$ there exists $u \in \mathbb{N}^m$ such that $a-u$ satisfies (8.3.2). Hence $\wp'_{a-u}$ is of dimension one.

But multiplication in R' by $1/X^u$ induces an isomorphism of $\wp'_{a-u}$ with $\wp'_a$. This completes the proof.

PROPOSITION 8.4.

(*a*) *dimension of* $\wp'_a$ = *index of* Λ *in* $\mathbb{Z}^m$,

(*b*) *for* $v \in \mathbb{Z}^m$, $X^{v+w^{(i)}} \equiv L_i(a+v)X^v$ *in* $\wp'_a$,

(*c*) $X^{v+\sum_{i=1}^{m} r_i w^{(i)}} \equiv \prod_{i=1}^{m} (L_i(a+v))_{r_i} X^v$ *in* $\wp'_a$ *for* $(r_1, \ldots, r_m) \in \in \mathbb{N}^m$.

PROOF. Part (*a*) follows from the proceding proposition. As noted before $L_i(D) = L_i(a) + L_i(E) - X^{w^{(i)}}$. Assertion (*b*) follows by computing $L_i(D)X^v$. Assertion (*c*) follows from (*b*) by induction on $\sum_{i=1}^{m} r_i$.

We now introduce the hypothesis (for a particular $v \in \mathbb{Z}^m$)

(8.5) $$L_i(a+v) \notin \mathbb{Z} \quad (1 \leqslant i \leqslant m).$$

COROLLARY 8.6. *Subject to* (8.5)

$$X^{v+\sum r_i w^{(i)}} \equiv \prod_{i=1}^{m} (L_i(a+v))_{r_i} X^v \text{ in } \wp'_a \text{ for all } (r_1, \ldots, r_m) \in \mathbb{Z}^m$$

PROOF. It follows from 8.4(b) replacing v by $v - w^{(i)}$ that

$$X^{v-w^{(i)}} \equiv (L_i(a+v))_{-1} X^v \quad \text{in } \mathcal{W}'_a .$$

The assertion now follows by induction on $\sum |r_i|$.

8.7. We now assume that $L_i(a+v) \notin \mathbb{Z}$ for any $v \in \mathbb{Z}^m$ and any $1 \leq \leq i \leq m$. This is equivalent to the hypotheses that $L_i(a+u) \notin \mathbb{Z}$ for any $u \in \mathfrak{a}$. Let

$$\mathcal{R}'^* = \left\{ \sum_{v \in \mathbb{Z}^m} B_v \frac{1}{X^u} \,|\, B_v \in \Omega \right\}.$$

We define

$$\mathcal{K}'_a = \{ \xi^* \in \mathcal{R}'^* \,|\, D^*_{a,i} \zeta^* = 0,\ 1 \leq i \leq m \}$$

where $D^*_{a,i} = -E_i + a_i + h_i$. Then $\mathcal{K}'_a$ is dual to $\mathcal{W}'_a$ and a dual basis indexed by $u \in \mathfrak{a}$ is given by

$$\xi^*_{a,u} = \frac{1}{X^u} \sum_{(r_1, \ldots, r_m) \in \mathbb{Z}^m} \prod_{i=1}^m (L_i(a+u))_{r_i} \frac{1}{X^{\sum r_i w^{(i)}}} \cdot$$

Now let $g(t, X) = -h(X) + \sum_{j=1}^n t_j X^{\mu^{(j)}}$. Multiplication by $\left(\sum t_j X^{\mu^{(j)}}\right)$ gives an injection of $\mathcal{K}'_a$ into $\mathcal{K}'_{a,t} \otimes_{\Omega(t)} \Omega((t))$ and the image consists of horizontal elements, the connection being given by

$$\sigma^*_j = \frac{\partial}{\partial t_j} - \frac{\partial g}{\partial t_j} \qquad (1 \leq j \leq n).$$

Let

$$\xi^*_{u,a,t} = \xi^*_{u,a} \exp\left(\sum t_j X^{\mu^{(j)}}\right).$$

Then for $v \in \mathfrak{a}$, $[X^v]$ is an element of $\mathcal{W}'_{a,t} = \mathcal{R}' / \sum D_{a,i,t} \mathcal{R}'$ (here $\mathcal{R}' = = \Omega(t)[X, X_1^{-1}, \ldots, X_m^{-1}]$) with periods $C_{u,v}$ (for each $u \in \mathfrak{a}$),

$$C_{u,v} = \langle \xi^*_{u,a,t}, X^v \rangle .$$

We find

$$C_{u,v} = \sum_{s \in \mathbb{N}^n,\, r \in \mathbb{Z}^m} \frac{t^s}{s_1! \ldots s_n!} \prod_{i=1}^m (L_i(a+u))_{r_i}$$

where

$$u + r_1 \omega^{(1)} + \ldots + r_m \omega^{(m)} = v + s_1 \mu^{(1)} + \ldots + s_n \mu^{(n)} .$$

This condition means that

$$r_i = L_i(v - u + s_1 \mu^{(1)} + \ldots + s_n \mu^{(n)}), \qquad 1 \leqslant i \leqslant m .$$

This means that we must restrict $s \in \mathbb{N}^n$ so that

$$L_i\left(v - u + \sum_{j=1}^{n} s_j \mu^{(j)}\right) \in \mathbb{Z}, \qquad 1 \leqslant i \leqslant m .$$

We choose $N \in \mathbb{N}$ such that $l_i = NL_i$ is a $\mathbb{Z}$-linear form. Having defined l_i, the condition on s is that $s \in \mathbb{N}^n$ and

$$l_i\left(\sum_{j=1}^{n} s_j \mu^{(j)}\right) \equiv l_i(u - v) \bmod N\mathbb{Z}, \qquad 1 \leqslant i \leqslant m .$$

Fixing u, v this has a finite set of solutions for s mod N. Let $\bar{S}$ be the set of representatives of these solutions in the box $0 \leqslant s_j < N$, $1 \leqslant j \leqslant n$. If s is any solution of the congruence in $\mathbb{N}^n$ then there exists a unique representation

$$s = \bar{s} + N(\lambda_1, \ldots, \lambda_n) \quad \text{where } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n .$$

Thus

$$C_{u,v} = \sum_{\bar{s} \in \bar{S}} C_{u,v,\bar{s}} .$$

Here

$$C_{u,v,\bar{s}} = t^{\bar{s}} \sum_{\lambda \in \mathbb{N}^n} \frac{t^{N\lambda}}{\prod_{j=1}^{n} (\bar{s}_j + N\lambda_j)!} \prod_{i=1}^{m} L_i(a + u)_{r_i}$$

where

$$r_i = L_i\left(v - u + \sum_{j=1}^{n} \bar{s}_j \mu^{(j)} + N \sum_{j=1}^{n} \lambda_j \mu^{(j)}\right) = \beta_i + l_i\left(\sum \lambda_j \mu^{(j)}\right) = \beta_i + f_i(\lambda)$$

and

$$\beta_i = L_i\left(v - u + \sum_{j=1}^{n} \bar{s}_j \mu^{(j)}\right) \in \mathbb{Z},$$

$$f_i(\lambda) = l_i\left(\sum \lambda_j \mu^{(j)}\right), \qquad \text{a } \mathbb{Z}\text{-linear form in } \lambda .$$

Now

$$(L_i(a+u))_{\beta_i + f_i(\lambda)} = (\beta_i + L_i(a+u))_{f_i(\lambda)} (L_i(a+u))_{\beta_i}$$

i.e.

$$C_{u,v,\bar{s}} = \prod_{i=1}^{m} (L_i(a+u))_{\beta_i} \cdot t^{\bar{s}} \sum_{\lambda \in \mathbb{N}^n} \frac{t^{N\lambda}}{\prod_{i=1}^{n} (\bar{s}_j + N\lambda_j)!} \prod_{i=1}^{m} (\gamma_i)_{f_i(\lambda)}$$

where $\gamma_i = \beta_i + L_i(a+u)$. To complete the description we use the Gauss multiplication formula to compute $(s+N\lambda)!/s!$ for $s, \lambda \in \mathbb{N}$, $n = 1$. We have

$$(s+N\lambda)! = \Gamma(1+s+N\lambda) = \Gamma\left(N\left(\lambda + \frac{1+s}{N}\right)\right) =$$

$$= N^{N\left(\frac{1+s}{N} + \lambda\right)} \frac{\prod_{i=0}^{N-1} \Gamma\left(\frac{1+s}{N} + \lambda + \frac{i}{N}\right)}{(2\pi)^{\frac{N-1}{2}} \sqrt{N}},$$

and so dividing by the same formula with $\lambda = 0$,

$$\frac{(s+N\lambda)}{s!} = N^{N\lambda} \prod_{i=0}^{N-1} \left(\frac{1+s}{N} + \frac{i}{N}\right)_{\lambda}.$$

Thus

$$C_{u,v,\bar{s}} = \prod_{i=1}^{n} (L_i(a+u))_{\beta_i} \cdot \frac{t_1^{\bar{s}_1} \dots t_n^{\bar{s}_n}}{\bar{s}_1! \dots \bar{s}_n!} \cdot y$$

where

$$y = \sum_{\lambda \in \mathbb{N}^n} \frac{(t/N)^{\lambda N}}{\lambda_1! \dots \lambda_n!} H(\lambda),$$

$$H(\lambda) = \prod_{i=1}^{m} (\gamma_i)_{f_i(\lambda)} \Big/ \prod_{j=1}^{m} \prod_{\substack{k_j=0 \\ k_j \neq N-1-\bar{s}_j}}^{N-1} \left(\frac{1+\bar{s}_j}{N} + \frac{k_j}{N} \right)_{\lambda_j}.$$

9. More exponential modules.

We mention two more exponential modules, each more natural than $\tilde{\mathcal{W}}_{a,t}$ of § 3.2 which may be useful in the case in which condition (4.2.1) is not satisfied.

Let $\hat{\hat{\mathcal{H}}}_0$ be the monoid generated by $\varepsilon_1, \dots \varepsilon_m, A^{(1)}, \dots, A^{(n)}$. Let $\hat{\mathcal{H}} = \mathbb{Z}^m \cap C$, where C is the cone in $\mathbb{R}^m$ generated by $\varepsilon_1, \dots \varepsilon_m, A^{(1)}, \dots, A^{(n)}$. We construct $\hat{\hat{R}}$ (resp: $\hat{R}$), the $\Omega(t)$ span of all X^u for $u \in \hat{\hat{\mathcal{H}}}_0$ (resp: $\hat{\mathcal{H}}_0$). The operators $D_{a,t,i}$, σ_j operate on these spaces and the definition of the modules $\hat{\hat{\mathcal{W}}}_{a,t}$ (resp: $\hat{\mathcal{H}}_{a,t}$) is clear. The adjoint spaces $\hat{\hat{R}}^*$ and $\hat{R}^*$ are defined as § 3.2 and likewise for $\hat{\hat{\mathcal{K}}}_{a,t}$ and $\hat{\mathcal{K}}_{a,t}$, the construction of the projections $\hat{\hat{\gamma}}_-$, $\hat{\gamma}_-$ being obvious.

If a satisfies the condition

(9.1) if $a_i \in \mathbb{N}^{\times}$ then C lies in the region $u_i \geq 0$,

then the basis element of $\hat{\mathcal{K}}_{a,0}$ (resp: $\hat{\hat{\mathcal{K}}}_{a,0}$) may be taken to be

$$\hat{\xi}^*_{a,t} = \sum_{u \in \hat{H}_0} \frac{1}{X^u} \prod_{i=1}^{m} (a_i)_{u_i}$$

with a similar formula for $\hat{\hat{\xi}}^*_{a,0}$. We note that condition (9.1) is implied by 2.1.

If condition 9.1 is not satisfied then a basis may be constructed using the condition that $\sum_{u \in \hat{H}_0} B_u \cdot 1/X^u$ lies in $\hat{\mathcal{K}}_{a,0}$ if and only if

(9.2) $$(a_i + u_i) B_u = B_{u+\varepsilon_i} \quad \forall u \in \hat{\mathcal{H}}_0, \ 1 \leq i \leq m.$$

The condition for $\hat{\hat{\mathcal{K}}}_{a,0}$ is similar. We put

$$\hat{\zeta}^*_{a,t} = \hat{\gamma}^*_- \hat{\xi}^*_{a,0} \exp(g(t,X) - g(0,X)).$$

The formula for $\hat{\hat{\xi}}^*_{a,t}$ is similar. Subject to (2.1) we have

$$y(a, t) = \langle \hat{\xi}^*_{a,t}, 1 \rangle = \langle \hat{\hat{\xi}}^*_{a,t}, 1 \rangle .$$

Letting $\hat{1}$ (resp: $\hat{\hat{1}}$) denote the class of 1 in $\hat{\mathcal{W}}_{a,t}$ (resp: $\hat{\hat{\mathcal{W}}}_{a,t}$) we conclude that y is a period of $\hat{1}$ (resp: $\hat{\hat{1}}$).

Let $\hat{\mathfrak{A}}_1(a)$ (resp: $\hat{\hat{\mathfrak{A}}}_1(a)$) denote the annihilator of $\hat{1}$ (resp: $\hat{\hat{1}}$) in $\mathcal{R}_1$. The inclusion $\hat{\hat{R}} \subset \hat{R} \subset R'$ implies

$$\hat{\hat{\mathfrak{A}}}_1 \subset \hat{\mathfrak{A}}_1 \subset \mathfrak{A}_1$$

and subject to (2.1) we have $\hat{\mathfrak{A}}_1 \subset \mathfrak{A}_1$. The advantage of the present section is that for all $a \in \mathbb{C}^m$, $\hat{\hat{1}}$ is a cyclic element of $\hat{\hat{\mathcal{W}}}_{a,t}$.

We do not know if $\{g_1, \ldots, g_m\}$ is a regular sequence in $\hat{R}$ (more precisely if $\{\bar{g}_1, \ldots, \bar{g}_m\}$ is a regular sequence in the graded ring associated with $\hat{R}$ by means of the grading given by the polyhedron of g) but we do know that the dimension of $\hat{\mathcal{W}}_{a,t}$ is bounded by the volume of this polyhedron.

If however $\hat{R}$ and $\hat{\hat{R}}$ coincide, then (as explained to us by A. Adolphson) the regular sequence property does follow from the work of Kouchnirenko. In particular this holds if

$$\sum_{j=1}^{n} \sup(0, -A_{i,j}) \leq 1 \qquad \text{for } 1 \leq i \leq m .$$

An example has been brought our attention by Kita [K1, 2] who has studied the hypergeometric function that we would associate with

$$-g = \sum_{i=1}^{m} X_i + \sum_{i=1}^{n} \sum_{j=1}^{m-n-1} t_{i,j} X_i X_{n+j} / X_m$$

where $m > n \geq 1$.

If $\hat{R} = \hat{\hat{R}}$ then we may conclude that $\tilde{\mathcal{W}}_{a,t}$ is a differential module generated by $\hat{\hat{1}}$ and has dimension given by the volume of the polyhedron of g. The dimension of $\mathcal{W}'_{a,t}$ would be the same but we know [1] to be a generator (as $\mathcal{R}_1$-module) only subject to the conditions of [D, equation 6.13].

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Manoscritto pervenuto in redazione il 3 dicembre 1992.