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On the Algebraic and Arithmetical Structure of Generalized Polynomial Algebras.

FRANZ HALTER-KOCH (*)

ABSTRACT - We introduce a new kind of polynomial rings in infinitely many indeterminates (called large polynomial rings). The large polynomial ring over a factorial or a Krull domain is itself factorial or a Krull domain. The algebra of polynomial functions on an abelian group turns out to be essentially a large polynomial ring.

Introduction.

The classical notion of a polynomial function permits far-reaching generalizations, see [3], Ch. IV, [7], [13] and only recently [12]. In this paper we deal with polynomial functions defined on a module over a commutative ring $R$ with values in an $R$-algebra. These polynomial functions form a commutative ring, whose algebraic structure is determined by means of a new kind of formal polynomial rings (called large polynomial rings). These large polynomial rings have nice arithmetical properties: They are factorial resp. Krull domains if the base ring is a factorial resp. a Krull domain.

1. Large polynomials and power series.

Throughout this paper, let $I \neq \emptyset$ be a set, denote by $\mathcal{S}(I)$ the set of all finite subsets of $I$, and let $\leq$ be a total order on $I$. For $n \in \mathbb{N}_0$, we set

$$I^n = \{(i_1, \ldots, i_n) \in I^n \mid i_1 \leq i_2 \leq \ldots \leq i_n\};$$

in particular, \( I_0 \) is the singleton consisting of the empty sequence.

Let \( R \) be a commutative ring (always with \( 1 \neq 0 \)) and \( X = (X_i)_{i \in I} \) a family of (algebraically independent) indeterminates over \( R \). For a subset \( J \subset I \), we set \( X_J = (X_i)_{i \in J} \). Let

\[
R[X] = R[(X_i)_{i \in I}] = R[\mathcal{F}(X)]
\]

be the total algebra of the free abelian monoid \( \mathcal{F}(X) \) with basis \( X \); see [3], ch. III, § 2, no. 11, 12. We call \( R[X] \) the large power series ring in \( X \) over \( R \); it coincides with the ring \( A_1 \) investigated in [2] and with the ring \( R[(X_i)_{i \in I}] \), investigated in [9].

**Proposition 1.** Let \( R \) be a domain.

i) \( R[X] \) is a domain.

ii) Suppose that all power series rings \( R[X_1, \ldots, X_m] \) in finitely many indeterminates over \( R \) are factorial; then \( R[X] \) is also factorial. In particular, if \( R \) is a regular factorial ring, then \( R[X] \) is factorial.

iii) If \( R \) is a Krull domain, then \( R[X] \) is a Krull domain.

**Proof.** i) [3], ch. IV, § 4, no. 8 or [6] or [15].

ii) [2], [6] or [15].

iii) [9].

In [15] a more general class of rings is dealt with.

Every \( f \in R[X] \) has a unique representation in the form

\[
f = \sum_{\mathcal{F}(X)} \lambda_P P = \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in J^n_x} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n}
\]

with coefficients \( \lambda_P, \lambda_{i_1, \ldots, i_n} \in R \); addition and multiplication in \( R[X] \) are defined in the usual way. For \( f \) as above and \( J \subset I \), we set

\[
f_J = \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in J^n_x} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n} \in R[X_J],
\]

and we define \( \pi_J : R[X] \to R[X_J] \) by \( \pi_J(f) = f_J \). For \( J' \subset J \subset I \) we define \( \pi_{J, J'} : R[X_J] \to R[X_{J'}] \) by

\[
\pi_{J, J'} \left( \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in J^n_x} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n} \right) =
\]

\[
= \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in J'^n_x} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n}.
\]
\( \pi_J \) and \( \pi_{J, J'} \) are ring epimorphisms satisfying \( \pi_{J, J'} \circ \pi_J = \pi_{J'} \) and \( \pi_{J, J'} \circ \pi_J = \pi_{J, J'} \) whenever \( J'' \subseteq J' \subseteq J \subseteq I \). With the mappings \( \pi_{J, J'} \), the system \( (R[X_J])_{J \in \mathcal{J}(I)} \) becomes a projective system of \( R \)-algebras, and

\[
\pi = \lim_{J \in \mathcal{J}(I)} \pi_J : R[X] \rightarrow \lim_{J \in \mathcal{J}(I)} R[X_J]
\]
is an isomorphism of \( R \)-algebras; if \( f \in R[X] \), then \( \pi(f) = (f_J)_{J \in \mathcal{J}(I)} \).

If \( \mathcal{J} \subseteq \mathcal{J}(I) \) is cofinal, we identify

\[
\lim_{J \in \mathcal{J}(I)} R[X_J] = \lim_{J \in \mathcal{J}} R[X_J],
\]
and we shall in the sequel simply write \( \lim \) to denote the inverse limit over \( \mathcal{J}(I) \) or some cofinal subset.

The constructions performed so far suggest to endow \( R[X] \) with a topology as follows; for the topological concepts used in the sequel we refer to [4].

For every \( J \in \mathcal{J}(I) \), we give \( R[X_J] \) the discrete topology. We endow \( \lim_{J \in \mathcal{J}(I)} R[X_J] \) with the topology of the projective limit and shift this topology to \( R[X] \) by means of \( \pi \). This topology on \( R[X] \) (which makes \( \pi \) into a homeomorphism) will be called the limit topology; it is obviously different from the usual topology on power series rings, and it is discrete if \( I \) is finite.

The limit topology makes \( R[X] \) into a separated complete topological \( R \)-algebra. For \( f \in R[X] \) and \( J \in \mathcal{J}(I) \), we set

\[
\mathcal{U}_J(f) = \{ g \in R[X] \mid g_J = f_J \};
\]
then \( \{ \mathcal{U}_J(f) \mid J \in \mathcal{J}(I) \} \) is a fundamental system of neighbourhoods of \( f \), and the family \( (f_J)_{J \in \mathcal{J}(I)} \) converges to \( f \) in the limit topology.

As an \( R \)-module, \( R[X] \) is of the form

\[
R[X] = \prod_{d \geq 0} R[X]_d,
\]
where \( R[X]_d = \prod_{(i_1, \ldots, i_d) \in I_d^+} RX_{i_1} \cdots X_{i_d} \);

in particular, \( R[X]_0 = R \), and the elements of \( R[X]_d \) are of the form

\[
\sum_{(i_1, \ldots, i_d) \in I_d^+} \lambda_{i_1, \ldots, i_d} X_{i_1} \cdots X_{i_d} \quad \text{where} \quad \lambda_{i_1, \ldots, i_d} \in R;
\]
they are called large forms of degree \( d \); if \( f \in R[X]_d \) and \( g \in R[X]_e \), then \( fg \in R[X]_{d+e} \).
The ring $R[X]$ contains the usual polynomial ring $R[X]$, consisting of all elements

$$
\sum_{P \in \mathcal{P}(X)} \lambda_P p = \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in I_n^\mathbb{N}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n} \in R[X]
$$

where only finitely many of the coefficients $\lambda_P$ resp. $\lambda_{i_1, \ldots, i_n}$ are different from zero.

The main purpose of this paper is to investigate the subring

$$
R\langle X \rangle = R[\langle X_i \rangle_{i \in I}] = \bigsqcup_{d \geq 0} R[X]_d \subset R[X]
$$

consisting of all $f \in R[X]$ of the form

$$
f = \sum_{n=0}^{N} \sum_{(i_1, \ldots, i_n) \in I_n^\mathbb{N}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n}
$$

for some $N \in \mathbb{N}_0$; we call $R\langle X \rangle$ the large polynomial ring and its elements large polynomials (in $X$ over $R$). Any $f \in R\langle X \rangle$ has a unique representation in the form

$$
f = \sum_{d \geq 0} f_d ,
$$

where $f_d \in R[X]_d$ are large forms of degree $d$, and $f_d = 0$ for all but finitely many $d \geq 0$. As in the classical case, we call

$$
deg (f) = \sup \{ d \geq 0 \mid f_d \neq 0 \} \in \mathbb{N}_0 \cup \{-\infty\}
$$

the degree of $f$.

Clearly, an element $f \in R[X]$ belongs to $R\langle X \rangle$ if and only if $f_J \in R[X_J]$ for all $J \in \mathcal{E}(I)$ and $\sup \{ \deg (f_J) \mid J \in \mathcal{E}(I) \} < \infty$; in this case, $\deg (f) = \max \{ \deg (f_J) \mid J \in \mathcal{E}(I) \}$.

We introduce a more general class of polynomial rings, containing $R[X]$ and $R\langle X \rangle$ as special cases, as follows. Let $\aleph$ be an infinite cardinal, and let $R\langle X \rangle_{\aleph}$ be the set of all large polynomials

$$
f = \sum_{(i_1, \ldots, i_n) \in I_n^\aleph} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdots X_{i_n}
$$

for which

$$\text{card} \{(i_1, \ldots, i_n) \in I_n^\aleph \mid \lambda_{i_1, \ldots, i_n} \neq 0\} < \aleph ;
$$

large polynomials with this property will be called large $\aleph$-polynomials. Clearly, $R\langle X \rangle_{\aleph}$ is a subring of $R\langle X \rangle$, $R\langle X \rangle_{\aleph} \subseteq R[X]$, and $\aleph \geq \max \{ \aleph_0, \text{card}(I) \}$ implies $R\langle X \rangle_{\aleph} = R\langle X \rangle$.

We say that an indeterminate $X_J$ occurs in a large polynomial

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PROPOSITION 2. Let $R$ be a domain.

PROOF. Since it is a domain by Proposition 1; i) and iii) are proved as in the classical case, see [3], Ch. IV, § 9, no. 5. Since $R$ is a domain, we have $fg = fUg$ for all $f, g \in R \{0\}$, which implies ii).

We endow $R\langle X \rangle$ with the subspace topology induced from the limit topology on $R[X]$. If $I$ is infinite, $R\langle X \rangle$ is not closed in $R[X]$ and hence it is not complete. Its closure $\overline{R\langle X \rangle}$ consists of all $f \in R[X]$ such that $f_J \in R[X]_J$ for all $J \in \mathcal{I}(I)$. The ring $R\langle X \rangle$ coincides with the ring $A_2$ investigated in [2]; it was proved there, that this ring does not even satisfy the ascending chain condition for principal ideals (if $I$ is infinite).

The ring $R\langle X \rangle$ has the following universal mapping property.

PROPOSITION 3. Let $\varphi: R \to S$ be a homomorphism of commutative rings, $(x_i)_{i \in I} \in S^{(I)}$, and give $S$ the discrete topology. Then there exists a unique continuous ring homomorphism $\hat{\varphi}: R\langle X \rangle \to S$ satisfying $\hat{\varphi}|R = \varphi$ and $\hat{\varphi}(x_i) = x_i$ for all $i \in I$.

PROOF. Clearly there exists exactly one ring homomorphism $\varphi^*: R[X] \to S$ satisfying $\varphi^*|R = \varphi$ and $\varphi^*(x_i) = x_i$ for all $i \in I$. For any $f \in R[X]$, we have $(\varphi^*)^{-1}(\varphi^*f) = \cup_J(f)$, where $J = \{i \in I \mid x_i \neq 0\}$; hence $\varphi^*$ is continuous and has a unique extension to a continuous homomorphism $\hat{\varphi}$ as asserted.

2. Arithmetical properties of the large polynomial ring.

In this section we shall prove that the large polynomial ring $R\langle X \rangle$ is a factorial domain resp. a Krull domain if $R$ is so (Theorems 1 and 2).
First we recall from [1] the notation of a finite factorization domain (FFD). An integral domain \( R \) is called an FFD, if every \( a \in R \setminus (R^\times \cup \{0\}) \) is a product of irreducible elements of \( R \) and possesses (up to associates) only finitely many divisors in \( R \). If \( R \) is an FFD, then every polynomial ring \( R[X_1, \ldots, X_m] \) is an FFD by [1], Prop. 5.3; every Krull domain is an FFD by [10], Theorem 5.

**Proposition 4.** Let \( R \) be an FFD and \( f \in R[(X)] \setminus R \). Then \( f \) is irreducible in \( R[(X)] \) if and only if there exists some \( J_0 \in \mathcal{E}(I) \) such that \( f_J \) is irreducible in \( R[X_J] \) for all \( J \in \mathcal{E}(I) \) satisfying \( J \supset J_0 \).

**Proof.** If \( f \) is reducible in \( R[(X)] \), then \( f = gh \), where \( g, h \in R[(X)] \setminus R^\times \). This implies \( f_J = g_J h_J \), and if \( J \) is sufficiently large; then \( g_J, h_J \notin R^\times \). Therefore \( f_J \) is irreducible in \( R[X_J] \) for all sufficiently large \( J \in \mathcal{E}(I) \).

For the converse, suppose that, for any \( J_0 \in \mathcal{E}(I) \), there exists some \( J \in \mathcal{E}(I) \) such that \( J \supset J_0 \) and \( f_J \) is reducible in \( R[X_J] \). We set \( n = \deg(f) \in \mathbb{N} \), and we shall prove that \( f \) is reducible in \( R[(X)] \). By assumption, the set

\[ \mathcal{J} = \{ J \in \mathcal{E}(I) : \deg(f_J) = n, f_J \text{ is reducible in } R[X_J] \} \]

is cofinal in \( \mathcal{E}(I) \), and for \( J \in \mathcal{J} \) the set

\[ E_J = \{ \varphi \in R[X_J] \setminus R^\times : \deg(\varphi) < n, \varphi \mid f_J \} \]

is not empty. If \( J, J' \in \mathcal{J}, J' \subset J \) and \( \varphi \in E_J \), then there exists some \( \psi \in R[X_J] \setminus R \) such that \( f_J = \varphi \psi \), and consequently \( f_{J'} = \pi_{J,J'}(f_J) = \pi_{J,J'}(\varphi) \pi_{J,J'}(\psi) \). This implies \( \pi_{J,J'}(\varphi) \mid f_{J'} \) and \( n = \deg(\pi_{J,J'}(f_J)) = \deg(\pi_{J,J'}(\varphi)) + \deg(\pi_{J,J'}(\psi)) \leq \deg(\varphi) + \deg(\psi) = \deg(f_J) = n \), whence \( \deg(\pi_{J,J'}(\varphi)) = \deg(\varphi) < n \). If \( \pi_{J,J'}(\varphi) \in R \), then \( 0 = \deg(\pi_{J,J'}(\varphi)) = \deg(\varphi) \), which implies \( \varphi \in R \) and thus \( \pi_{J,J'}(\varphi) = \varphi \notin R^\times \). In any case, we obtain \( \pi_{J,J'}(\varphi) \notin R^\times \) and therefore \( \pi_{J,J'}(\varphi) \in E_{J'} \).

Now we consider the projective system \((E_J)_{J \in \mathcal{J}}, (\pi_{J,J'} | E_J : E_J \to E_{J'})_{J,J' \in \mathcal{J}}\), and we assert that \( \text{lim } E_J \neq 0 \).

Since \( R[X_J] \) is an FFD, the set

\[ E_J = \{ \varphi R^\times : \varphi \in E_J \} \]

of classes of associates in \( E_J \) is finite. For \( J, J' \in \mathcal{J}, J' \subset J \), we define \( \pi_{J,J'} : E_J \\to E_{J'} \), by \( \pi_{J,J'}(\varphi R^\times) = \varphi_{J,J'}(\varphi) R^\times \); then we have \( \text{lim } E_J \neq 0 \).

By [5], (Ch. III, § 7, no. 4, Ex. II). Let \( (\varphi_J)_{J \in \mathcal{J}} \) be a family of polynomials \( \varphi_J \in R[X_J] \) such that \( (\varphi_J R^\times)_{J \in \mathcal{J}} \) is \( E_J \), and fix some \( J_0 \in \mathcal{J} \). If \( J \in \mathcal{J}, J \supset J_0 \), then \( \pi_{J,J_0}(\varphi_J) = u_J \varphi_{J_0} \) for some \( u_J \in R^\times \), and we set \( \varphi_J = \cdots \)
If and then $g_J = \varphi_J \in E_J$ for all $J \supset J_0$. If $J, J' \in \Xi$, $J \supset J' \supset J_0$, then $\pi_J J_0 (\varphi_J) = \nu \varphi_J$ for some $\nu \in R^\times$, and $\varphi_J = \pi_J J_0 (\varphi_J) = \pi_{J'} J_0 (\nu \varphi_J) = \nu \varphi_J$ implies $\nu = 1$; therefore $(\varphi_J)_{J \in \Xi, J \supset J_0} \in \lim E_J$.

If $\varphi = (\varphi^J)_{J \in \Xi} \in \lim E_J \subset \lim R[X_J]$ and $g = \varphi^{-1}(\varphi) \in R[X_J]$, then $g_J = \varphi^J \in R[X_J]$ for all $J \in \Xi$ and $\deg(g_J) = \deg(\varphi^J) < n$, which implies $g \in R[(x)]$ and $\deg(g) < n$. If $g \in R$, then $\varphi^J = g_J = g$ for all $J \in \Xi$, and consequently $g \notin R^\times$.

For any $J \in \Xi$, we have $f_J = \varphi^J \varphi^J$ for some polynomial $\varphi^J \in R[X_J]$; this implies $\psi = (\psi^J)_{J \in \Xi} \in \lim R[X_J]$ and (as above) $h = \pi^{-1}(\psi) \in R[(x)]$. Since $f_J = g_J h_J = (gh)_J$ for all $J \in \Xi$, we obtain $f = gh$. Since $g \notin R^\times$ and $\deg(g) < n$, $f$ is reducible in $R[(x)]$.

**Proposition 5.** Let $R$ be a domain, $f \in R[(x)]$ and suppose that there exists some $J_0 \in \Xi(I)$ such that, for any $J \in \Xi(I)$ satisfying $J \supset J_0$, $f_J$ is a prime element of $R[X_J]$. Then $f$ is a prime element of $R[(x)]$.

**Proof.** Suppose that $f \mid gh$ for some $g, h \in R[(x)]$. For any $J \in \Xi(I)$, this implies $f_J \mid g_J h_J$, and if $J \supset J_0$, then either $f_J \mid g_J$ or $f_J \mid h_J$. We set

$$\Xi' = \{J \in \Xi(I) \mid f_J \mid g_J\}, \quad \Xi'' = \{J \in \Xi(I) \mid f_J \mid h_J\},$$

and we obtain

$$\{J \in \Xi(I) \mid J \supset J_0\} \subset \Xi' \cup \Xi''$$

which implies that either $\Xi'$ or $\Xi''$ is cofinal in $\Xi(I)$. Without restriction, let $\Xi'$ be cofinal in $\Xi(I)$. For $J \in \Xi'$, there exists some polynomial $\varphi^J \in R[X_J]$ such that $g_J = f_J \varphi^J$. If $J, J' \in \Xi'$ and $J \supset J'$, then $g_J = \pi_{J, J'} (g_J) = \pi_{J, J'} (f_J \pi_{J, J'} (\varphi^J)) = f_J \pi_{J, J'} (\varphi^J) = f_J \varphi^J$ implies $\pi_{J, J'} (\varphi^J) = \varphi^J$ and hence $\varphi = (\varphi^J)_{J \in \Xi} \in \lim R[X_J]$. If $q = \pi^{-1}(\varphi)$, then $q_J = \varphi^J \in R[X_J]$ and $\deg(q_J) = \deg(\varphi^J) \leq \deg(g_J) \leq \deg(g)$ for all $J \in \Xi'$; this implies $q \in R[(x)]$. Since $g_J = f_J q_J = (fq)_J$ for all $J \in \Xi'$, we obtain $f = q g$, whence $f \mid g$ in $R[(x)]$.

Next we adopt Gauss' Lemma for large polynomials. An element $f \in R[(x)]$ is called primitive, if $f = \lambda f^*$ where $\lambda \in R$ and $f^* \in R[(x)]$ implies $\lambda \in R^\times$ (i.e., $1$ is a g.c.d. of all coefficients of $f$ in $R$). Hence an element of $R$ is primitive if and only if it lies in $R^\times$. 

On the algebraic and arithmetical structure etc. 127
PROPOSITION 6. Let \( R \) be an FFD and \( f \in R[\langle X \rangle] \setminus R \). Then the following assertions are equivalent:

a) \( f \) is primitive.

b) \( f_J \) is primitive for some \( J \in \mathcal{F}(I) \).

c) There exists some \( J_0 \in \mathcal{F}(I) \) such that \( f_J \) is primitive for all \( J \supset J_0 \).

PROOF. Obviously, c) \( \Rightarrow \) b) \( \Rightarrow \) a). Now set

\[
f = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[\langle X \rangle]
\]

and suppose that \( f \) is primitive, i.e., 1 is a g.c.d. of \( \{\lambda_P \mid P \in \mathcal{F}(X)\} \). Since \( R \) is an FFD, there exists a finite subset \( \mathcal{P} \subset \mathcal{F}(X) \) such that 1 is a g.c.d. of \( \{\lambda_P \mid P \in \mathcal{P}\} \). If \( J_0 \in \mathcal{F}(I) \) is such that \( \mathcal{P} \subset \mathcal{F}(X_{J_0}) \), then \( \mathcal{P} \subset \mathcal{F}(X_J) \) for all \( J \in \mathcal{F}(I) \) satisfying \( J \supset J_0 \); therefore 1 is a g.c.d. of \( \{\lambda_P \mid P \in \mathcal{F}(X_J)\} \) for any such \( J \), which means that

\[
f_J = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[X_J]
\]

is primitive. ■

PROPOSITION 7 (Gauss' lemma). Let \( R \) be a factorial domain and \( K \) a quotient field of \( R \).

i) If \( f, g \in R[\langle X \rangle] \) are primitive, then \( fg \) is also primitive.

ii) If \( f \in R[\langle X \rangle] \) is primitive, \( g \in K[\langle X \rangle] \) and \( fg \in R[\langle X \rangle] \), then already \( g \in R[\langle X \rangle] \).

PROOF. For classical polynomials \( f \in K[X] \), we use the notation of the content \( c(f) \) as in [8], § 8. Then we have \( c(fg) = c(f)c(g) \) for all \( f, g \in K[X] \); \( f \in R[X] \) if and only if \( c(f) \) is integral; \( f \in R[X] \) is primitive if and only if \( c(f) = 1 \).

i) If \( f, g \in R[\langle X \rangle] \) are primitive, then \( f_J, g_J \in R[X_J] \) are primitive for some \( J \in \mathcal{F}(I) \) by Proposition 6. Then \( (fg)_J = f_J g_J \) is also primitive, and again Proposition 6 implies that \( fg \) is primitive.

ii) By Proposition 6, there exists some \( J_0 \in \mathcal{F}(I) \) such that \( f_J \) is primitive for all \( J \in \mathcal{F}(I) \) satisfying \( J \supset J_0 \). For such \( J \), \( c(f_J g_J) = c(g_J) \) is integral, since \( f_J g_J = (fg)_J \in R[X_J] \); this implies \( g_J \in R[X_J] \), and consequently \( g \in R[\langle X \rangle] \). ■

PROPOSITION 8 Let \( R \) be a factorial domain, \( K \) a quotient field of \( R \) and \( f \in R[\langle X \rangle] \setminus R \). Then the following assertions are equivalent:
a) $f$ is a prime element of $\mathcal{R}\langle X\rangle_{K}$.

b) $f$ is irreducible in $\mathcal{R}\langle X\rangle_{K}$.

c) $f$ is primitive and irreducible in $K\langle X\rangle$.

**Proof.** For finite $I$, this is classical; see [14], Ch. V, § 6.

a) $\Rightarrow$ b) is obvious.

b) $\Rightarrow$ c) If $f$ is irreducible in $\mathcal{R}\langle X\rangle_{K}$, then $f$ is irreducible in $\mathcal{R}\langle X\rangle$ by Proposition 2, ii). By Proposition 4, there exists some $J_{0} \in \mathcal{E}(I)$ such that $f_{j}$ is irreducible in $\mathcal{R}\langle X_{j}\rangle$ and hence in $K\langle X_{j}\rangle$ for all $J \in \mathcal{E}(I)$ satisfying $J \supsetneq J_{0}$. Again by Proposition 4 it follows that $f$ is irreducible in $K\langle X\rangle$. Being irreducible in $\mathcal{R}\langle X\rangle$, $f$ is primitive by definition.

c) $\Rightarrow$ a) By Propositions 4 and 6, there exists some $J_{0} \in \mathcal{E}(I)$ such that $f_{j}$ is primitive and irreducible in $K\langle X_{j}\rangle$ for all $J \in \mathcal{E}(I)$ satisfying $J \supsetneq J_{0}$. Hence $f_{j}$ is a prime element in $\mathcal{R}\langle X_{j}\rangle$ for all such $J$, and Proposition 5 implies that $f$ is a primitive element of $\mathcal{R}\langle X\rangle$. By Proposition 2, ii),

$$f\mathcal{R}\langle X\rangle \cap \mathcal{R}\langle X\rangle_{K} = f\mathcal{R}\langle X\rangle_{K},$$

and hence $f$ is also a prime element of $\mathcal{R}\langle X\rangle_{K}$. □

**Theorem 1.** Let $R$ be a factorial domain and $K$ a quotient field of $R$. Then $\mathcal{R}\langle X\rangle_{K}$ is a factorial domain; the prime elements of $\mathcal{R}\langle X\rangle_{K}$ are the primes of $R$ and the primitive polynomials $f \in \mathcal{R}\langle X\rangle_{K} \setminus R$ which are irreducible in $K\langle X\rangle$.

**Proof.** If $f \in \mathcal{R}\langle X\rangle_{K} \setminus R$ is primitive and irreducible in $K\langle X\rangle$, then $f$ is a prime element of $\mathcal{R}\langle X\rangle_{K}$ by Proposition 8. If $p \in R$ is a prime element of $R$, then $\mathcal{R}\langle X\rangle_{K} / p\mathcal{R}\langle X\rangle_{K}$ is a domain, and thus $p$ is a prime element of $\mathcal{R}\langle X\rangle_{K}$.

We must prove that every $f \in \mathcal{R}\langle X\rangle_{K} \setminus (R^{\times} \cup \{0\})$ has a factorization $f = p_{1} \cdots p_{r} f_{1} \cdots f_{s}$, where $p_{i} \in R$ are prime elements and $f_{j} \in \mathcal{R}\langle X\rangle_{K} \setminus R$ are irreducible. For $f \in R$, this is obvious. Thus we may suppose that $f \in \mathcal{R}\langle X\rangle_{K} \setminus R$ and that the assertion is proved for all large polynomials of smaller degree. Clearly, $f = p_{1} \cdots p_{r} f^{*}$, where $p_{i} \in R$ are primes of $R$ and $f^{*} \in \mathcal{R}\langle X\rangle_{K} \setminus R$ is primitive. If $f^{*}$ is irreducible, we are done; otherwise $f^{*} = f_{1}^{*} f_{2}^{*}$ where $f_{i}^{*} \in \mathcal{R}\langle X\rangle_{K} \setminus R$ and hence $\deg (f_{i}^{*}) < \deg (f^{*}) = \deg (f) (i = 1, 2)$. Applying the induction hypothesis for $f_{i}^{*}$, the assertion follows. □

For the next result, we need a Lemma.
LEMMA 1. Let \((R_a)_{a \in A}\) be a family of Krull domains contained in a field \(K\), and set
\[ R = \bigcap_{a \in A} R_a. \]
Suppose that for every \(0 \neq x \in R\) the set \(\{a \in A \mid x \not\in R_a^\times\}\) is finite. Then \(R\) is a Krull domain.

PROOF. \([9], \text{Lemma 1.2.}\)

THEOREM 2. If \(R\) is a Krull domain, then \(R[\langle X \rangle]_K\) is also a Krull domain.

PROOF. Let \(K\) be a quotient field of \(R\) and \((V_a)_{a \in A}\) a family of discrete valuation rings of \(K\) such that \(R = \bigcap_{a \in A} V_a\), and, for each \(0 \neq x \in R\), the set \(\{a \in A \mid x \not\in V_a^\times\}\) is finite. For \(a \in A\), set
\[ N_a = \{f \in V_a[\langle X \rangle] \mid f \text{ is primitive}\}. \]
By Proposition 7, \(N_a\) is a multiplicatively closed subset of \(V_a[\langle X \rangle]\). By Theorem 1, the domains \(K[\langle X \rangle]_K\) and \(V_a[\langle X \rangle]\) are factorial and hence the localisations \(V_a[\langle X \rangle]_{N_a}\) are also factorial. If \(0 \neq f \in R[\langle X \rangle]\) then the set \(\{a \in A \mid x \not\in N_a^\times\}\) is finite; this implies \(f \in (V_a[\langle X \rangle]_{N_a})^\times\) for all but finitely many \(a \in A\). By Lemma 1 it is sufficient to prove that
\[ R[\langle X \rangle]_K = K[\langle X \rangle]_K \cap \bigcap_{a \in A} V_a[\langle X \rangle]_{N_a}. \]
Obviously \(R[\langle X \rangle]_K\) is contained in \(K[\langle X \rangle]_K\) and in each \(V_a[\langle X \rangle]_{N_a}\). If \(a \in A\) and \(f \in K[\langle X \rangle] \cap V_a[\langle X \rangle]_{N_a}\), then there exists some \(g_a \in N_a\) such that \(fg_a \in V_a[\langle X \rangle]\). Since \(g_a \in V_a[\langle X \rangle]\) is primitive, Proposition 7, ii) implies \(f \in V_a[\langle X \rangle]\). Thus we obtain
\[ K[\langle X \rangle]_K \cap \bigcap_{a \in A} V_a[\langle X \rangle]_{N_a} \subset K[\langle X \rangle]_K \cap \bigcap_{a \in A} V_a[\langle X \rangle] = R[\langle X \rangle]_K. \]

3. Polynomial functions on modules.

Throughout this section, let \(F\) be a commutative ring, \(R\) a commutative \(F\)-algebra and \(V\) an \(F\)-module. A mapping \(p:V \to R\) is called a \textit{homogeneous \(F\)-polynomial function of degree \(d \in \mathbb{N}\)}, if there exists an \(F\)-multilinear mapping \(p^*:V^d \to R\) such that \(p(x) = p^*(x, \ldots, x)\) for all \(x \in V\). We denote by \(\mathcal{P}_F(V, R)_d\) the set of all homogeneous \(F\)-polynomial functions \(p:V \to R\) of degree \(d \in \mathbb{N}\); \(\mathcal{P}_F(V, R)_0\) denotes the set of all con-
stant functions \( p: V \to R \) which we call homogeneous \( F \)-polynomial functions of degree 0. For any \( d \in \mathbb{N}_0 \), \( \mathcal{P}_F(V, R)_d \) is an \( R \)-module under pointwise addition and scalar multiplication. If \( p \in \mathcal{P}_F(V, R)_d \), \( x \in V \) and \( t \in F \), then \( p(tx) = t^d p(x) \).

It is usual to define polynomial functions with values in \( F \)-modules, see e.g. [3; Ch. IV, §5, no. 9]. In this paper however, we are mainly interested in the polynomial algebra (with a pointwise multiplication), and therefore we restrict ourselves to polynomial functions taking values in an \( F \)-algebra.

**PROPOSITION 9.** Let \( d, e \in \mathbb{N}_0 \), \( p \in \mathcal{P}_F(V, R)_d \) and \( q \in \mathcal{P}_F(V, R)_e \) be given. If \( pq: V \to R \) is defined pointwise, i.e. \( (pq)(x) = p(x)q(x) \), then \( pq \in \mathcal{P}_F(V, R)_{d+e} \).

**PROOF.** Let \( p^*: V^d \to R \) and \( q^*: V^e \to R \) be \( F \)-multilinear mappings such that \( p(x) = p^*(x, \ldots, x) \) and \( q(x) = q^*(x, \ldots, x) \) for all \( x \in V \). If \( r: (V^d \times V^e) \to R \) is defined by \( r(x_1, \ldots, x_d, y_1, \ldots, y_e) = p(x_1, \ldots, x_d)q(y_1, \ldots, y_e) \), then \( r \) is \( F \)-multilinear, and \( (pq)(x) = r(x, \ldots, x) \) for all \( x \in V \).

A mapping \( p: V \to R \) is called an \( F \)-polynomial function, if there exists some \( d \in \mathbb{N}_0 \) and homogeneous \( F \)-polynomial functions \( p_0, \ldots, p_d: V \to R \) such that \( p(x) = p_0(x) + \ldots + p_d(x) \) for all \( x \in V \); \( p \) is called a local \( F \)-polynomial function, if \( p|_M: M \to R \) is an \( F \)-polynomial function for every finitely generated \( R \)-submodule \( M \) of \( V \). We denote by \( \mathcal{P}_F(V, R) \) the set of all \( F \)-polynomial functions and by \( \mathcal{P}_F(V, R) \) the set of all local \( F \)-polynomial functions \( p: V \to R \). Obviously,

\[
\mathcal{P}_F(V, R) \subset \mathcal{P}_F(V, R) \subset R^V
\]

are \( R \)-subalgebras if \( R^V \) is viewed as the \( R \)-algebra of all functions \( f: V \to R \) under pointwise addition, multiplication and scalar multiplication.

On the algebra \( R^V \) we introduce a topology as follows. Denote by \( \mathcal{E}(V) \) the set of all finitely generated \( F \)-submodules of \( V \). For \( M \in \mathcal{E}(V) \), define \( \pi_M: R^V \to R^M \) by \( \pi_M(f) = f|_M \), and for \( M, M' \in \mathcal{E}(V), M \supset M' \), define \( \pi_{M, M'}: R^M \to R^{M'} \) by \( \pi_{M, M'}(g) = g|_{M'} \). With the mappings \( \pi_{M, M'} \), the system \( (R^M)_{M \in \mathcal{E}(V)} \) becomes a projective system of \( R \)-algebras, and

\[
\pi = \lim_{M \in \mathcal{E}(V)} \pi_M: R^V \to \lim_{M \in \mathcal{E}(V)} R^M
\]

is an \( R \)-algebra isomorphism. If \( f \in R^V \), then \( \pi(f) = (f|_M)_{M \in \mathcal{E}(V)} \). For
every $M \in \mathcal{E}(V)$, we give $R^M$ the discrete topology, and we shift the topology of the projective limit to $R^V$ be means of the isomorphism $\pi$. This topology on $R^V$ (obviously different from the product topology) will be called the limit topology.

With the limit topology, $R^V$ is a separated complete topological $R$-algebra. For $f \in R^V$ and $M \in \mathcal{E}(V)$, we set

$$\mathcal{U}_M(f) = \{ g \in R^V \mid g|_M = f|_M \}. $$

Then $\{\mathcal{U}_M(f) \mid M \in \mathcal{E}(V)\}$ is a fundamental system of neighbourhoods of $f$, and therefore the limit topology on $R^V$ coincides with the topology of $\mathcal{E}(V)$-convergence; see [4], Ch. X, § 1.

For the next result, let $\mathcal{E}^+(V)$ be the set of all finitely generated $F$-submodules of $V$ which are $F$-direct summands.

**Proposition 10.** i) $\varphi_F(V, R)$ is closed in $R^V$, and

$$\pi(\varphi_F(V, R)) = \lim_{M \in \mathcal{E}(V)} \varphi_F(M, R) \subset \lim_{M \in \mathcal{E}(V)} R^M. $$

ii) Let $M \in \mathcal{E}(V)$ be given and suppose that either $M \in \mathcal{E}^+(V)$ or $R$ is an injective $F$-module. Then the restriction map

$$\varphi : \begin{cases} \varphi_F(V, R) & \rightarrow \varphi_F(M, R), \\ f & \mapsto f|_M, \end{cases} $$

is surjective.

iii) Suppose that either $\mathcal{E}^+(V)$ is cofinal in $\mathcal{E}(V)$ or $R$ is an injective $F$-module. Then

$$\varphi_F(V, R) = \varphi_F(V, R) \subset R^V. $$

**Proof.** i) A function $f \in R^V$ lies in $\varphi_F(V, R)$ if and only if $f|_M \in \varphi_F(M, R)$ for all $M \in \mathcal{E}(V)$, i.e.,

$$\pi(f) = (f|_M)_{M \in \mathcal{E}(V)} \in \lim_{M \in \mathcal{E}(V)} \varphi_F(M, R).$$

This implies $\pi(\varphi_F(V, R)) = \lim \varphi_F(M, R)$. If $f \in \varphi_F(V, R)$, then

$$\pi(f) \in \varphi_F(V, R) = \lim_{M \in \mathcal{E}(V)} \varphi_F(V, R) \subset \lim_{M \in \mathcal{E}(V)} \varphi_F(M, R),$$

and consequently $f \in \varphi_F(V, R)$. Hence $\varphi_F(V, R)$ is closed in $R^V$. 


ii) It is sufficient to prove that every homogeneous \( F \)-polynomial function \( q: M \to R \) of degree \( d \geq 1 \) can be extended to an \( F \)-polynomial function \( \tilde{q}: V \to R \). Let \( q^*: M^d \to R \) be \( F \)-multilinear such that \( q(x) = q^*(x, \ldots, x) \) for all \( x \in M \). If either \( M \in \mathfrak{E}^+(V) \) or \( R \) is \( F \)-injective, then there exists an \( F \)-multilinear mapping \( \tilde{q}^*: V^d \to R \) such that \( \tilde{q}^*|M^d = q^* \), and \( \tilde{q}: V \to R \), defined by \( \tilde{q}(x) = \tilde{q}^*(x, \ldots, x) \), is an \( F \)-polynomial function extending \( q \).

iii) If \( M \in \mathfrak{E}^+(V) \) or \( R \) is \( F \)-injective, ii) implies

\[
\mathcal{P}_F(M, R) = \pi_M \mathcal{P}_F(V, R) \subset \pi_M \overline{\mathcal{P}_F(V, R)} \subset \mathcal{P}_F(M, R),
\]

whence equality holds. This implies

\[
\overline{\pi \mathcal{P}_F(V, R)} = \lim_{M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R) = \pi \overline{\mathcal{P}_F(V, R)},
\]

and consequently \( \mathcal{P}_F(V, R) = \overline{\mathcal{P}_F(V, R)} \). □

Next we investigate the connection between \( F \)-polynomial functions and large polynomials; we start with the case of polynomials in a finite number of indeterminates.

We say that \( F \) has no zero divisors on \( R \) if

\[
t \in F, x \in R, tx = 0 \text{ implies } t = 0 \text{ or } x = 0;
\]

notice that this condition implies that \( F \) itself is a domain.

A polynomial \( f \in R[X_1, \ldots, X_n] \) (in \( n \in \mathbb{N} \) indeterminates) is called \( q \)-reduced (for some \( q \in \mathbb{N} \)), if \( \deg_{X_j}(f) < q \) for all \( j \in \{1, \ldots, n\} \).

**Lemma 2.** Suppose that \( F \) has no zero divisors on \( R \), and let \( f \in R[X_1, \ldots, X_n] \) be a polynomial

i) If \( F \) is infinite and \( f(x_1, \ldots, x_n) = 0 \) for all \( (x_1, \ldots, x_n) \in F^n \), then \( f = 0 \).

ii) If \( \# F = q \in \mathbb{N} \), then there exists a unique \( q \)-reduced polynomial \( f_0 \in R[X_1, \ldots, X_n] \) such that \( f(x_1, \ldots, x_n) = f_0(x_1, \ldots, x_n) \) for all \( (x_1, \ldots, x_n) \in F^n \).

**Proof.** Exactly as in the classical case; cf. [14], Ch. V, § 4. □

Now let again \( X = (X_i)_{i \in I} \) be a family of indeterminates, and adopt all notations of section 1.
THEOREM 3. For

\[ f = \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in I^n} \lambda_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n} \in \overline{R[X]} \]

we define \( f^F : F^{(I)} \to R \) by

\[ f^F((x_i)_{i \in I}) = \sum_{n \geq 0} \sum_{(i_1, \ldots, i_n) \in I^n} \lambda_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n}. \]

i) If \( f \in \overline{R[\langle X \rangle]} \) implies \( f^F \in \overline{\mathcal{P}_F}(F^{(I)}, R) \), and \( f \in R[\langle X \rangle] \) implies \( f^F \in \mathcal{P}_F(F^{(I)}, R) \).

ii) The mapping

\[ \phi^F : \begin{cases} \overline{R[\langle X \rangle]} & \to \overline{\mathcal{P}_F}(F^{(I)}, R), \\ f & \mapsto f^F, \end{cases} \]

is a homomorphism of \( R \)-algebras satisfying

\[ \phi^F(\overline{R[\langle X \rangle]}) = \mathcal{P}_F(F^{(I)}, R). \]

iii) Suppose that \( F \) has no zero divisors on \( R \); then

\[ \phi^F(\overline{R[\langle X \rangle]}) = \overline{\mathcal{P}_F}(F^{(I)}, R). \]

If moreover \( F \) is infinite, then \( \phi^F \) is a topological isomorphism.

PROOF. We set \( V = F^{(I)} \), and we denote by \((e_i)_{i \in I}\) the canonical basis of \( V \), i.e.,

\[ x = (x_i)_{i \in I} = \sum_{i \in I} x_i e_i \quad \text{for all} \quad x \in V. \]

For \( J \subset I \), we set

\[ V_J = \bigoplus_{i \in J} R e_i \subset V. \]

If \( J \in 8(I) \), then \( V_J \in \mathcal{C}^+(V) \), and the system \( \{V_J \mid J \in 8(I)\} \) is cofinal in \( \mathcal{C}(V) \). Identifying \( V_J \) with \( F^{(J)} \) we obtain, for any \( f \in \overline{R[\langle X \rangle]} \),

\[ f^F|V_J = (f_J)^F. \]

i) We show first that \( f \in R[\langle X \rangle] \) implies \( f^F \in \mathcal{P}_F(V, R) \) and it suffices to do this for large forms \( f \in R[X]_n \), where \( n \in \mathbb{N} \). Suppose that

\[ f = \sum_{(i_1, \ldots, i_n) \in I^n} \lambda_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n} \in R[X]_n, \]
and let $p^* : V^n \to R$ be the unique $F$-multilinear mapping satisfying

$$p^*(e_{i_1}, \ldots, e_{i_n}) = \begin{cases} \lambda_{i_1, \ldots, i_n} & \text{if } (i_1, \ldots, i_n) \in I^n, \\ 0 & \text{otherwise}. \end{cases}$$

If $p \in \mathcal{P}_F(V, R)$ is defined by $p(x) = p^*(x, \ldots, x)$, then

$$p((x_i)_{i \in I}) = p^* \left( \sum_{i \in I} x_i e_i, \ldots, \sum_{i \in I} x_i e_i \right) = \sum_{(i_1, \ldots, i_n) \in I^n} x_{i_1} \cdot \ldots \cdot x_{i_n} \lambda_{i_1, \ldots, i_n} = f^F((x_i)_{i \in I}),$$

whence $f^F = p \in \mathcal{P}_F(V, R)$.

If $f \in R[(X)]$, then $f_J \in R[X_J]$ for all $J \in \mathcal{E}(I)$ and consequently $(f_J)^F = f^F | V_J \in \mathcal{P}_F(V_J, R)$, which implies $f^F \in \mathcal{P}_F(V, R)$.

ii) Clearly, $\phi^F$ is a homomorphism of $R$-algebras. In order to prove the equality $\phi^F(R[(X)]) = \mathcal{P}_F(V, R)$, it is sufficient to show that every homogeneous $F$-polynomial function $p : V \to R$ of degree $n \geq 1$ is of the form $p = f^F$ for some $f \in R[X]_n$.

Let $p : V \to R$ be a homogeneous $F$-polynomial function, and let $p^* : V^n \to R$ be $F$-multilinear such that $p(x) = p^*(x, \ldots, x)$ for all $x \in V$. For $(i_1, \ldots, i_n) \in I^n$, we set

$$[i_1, \ldots, i_n] = \{(i_{\sigma(1)}, \ldots, i_{\sigma(n)}) \mid \sigma \in S_n\},$$

and we define $f \in R[X]_n$ by

$$f = \sum_{(i_1, \ldots, i_n) \in I^n} \left( \sum_{e_{j_1}, \ldots, e_{j_n} \in [i_1, \ldots, i_n]} p^*(e_{j_1}, \ldots, e_{j_n}) \right) X_{i_1} \cdot \ldots \cdot X_{i_n}.$$

Then we obtain

$$f^F((x_i)_{i \in I}) = \sum_{(i_1, \ldots, i_n) \in I^n} p^*(e_{i_1}, \ldots, e_{i_n}) x_{i_1} \cdot \ldots \cdot x_{i_n} =$$

$$= p^* \left( \sum_{i=1}^n x_i e_i, \ldots, \sum_{i=1}^n x_i e_i \right) = p((x_i)_{i \in I}),$$

whence $p = f^F$.

iii) For a large polynomial $f \in R[(X)]$, we have $f^F = 0$ if and only if $0 = f^F | V_J = (f_J)^F$ for all $J \in \mathcal{E}(I)$. If $F$ is infinite, this implies $f_J = 0$ for all $J \in \mathcal{E}(I)$ (by Lemma 2) and hence $f = 0$; therefore

$$\phi^F : R[(X)] \to \mathcal{P}_F(V, R)$$
is an isomorphism and

$$(\phi^F | R[X_J]: R[X_J] \overset{\sim}{\to} \mathcal{O}_F(V_J, R))_{J \in \mathcal{E}(I)}$$

is a family of isomorphisms, compatible with the mappings $\pi_{J, J'}$ of the projective systems on either side. Taking projective limits and observing the commutative diagram

$$
\begin{array}{ccc}
R[X_J] & \xrightarrow{\phi^F} & \mathcal{O}_F(V_J, R) \\
\pi & & \pi \\
\lim R[X_J] & \xrightarrow{\sim} & \lim \mathcal{O}_F(V_J, R)
\end{array}
$$

it follows from Proposition 9, iii) that $\phi^F$ is an isomorphism if $F$ is infinite.

Now consider the case $\# F = q \in \mathbb{N}$. If $g \in \mathcal{O}_F(V, R)$ and $J \in \mathcal{E}(I)$, then $g|V_J \in \mathcal{O}_F(V_J, R)$, and by ii), there exists a polynomial $f^{(J)} \in R[X_J]$ such that $(f^{(J)})^F = g|V_J$. By Lemma 2, there exists a unique $q$-reduced polynomial $f_0^{(J)} \in R[X_J]$ such that $(f_0^{(J)})^F = g|V_J$. If $J, J' \in \mathcal{E}(I)$ and $J \supseteq J'$, then $\pi_{J, J'}(f_0^{(J)})$ is $q$-reduced, and $\pi_{J, J'}(f_0^{(J)})^F = (f_0^{(J)})^F|V_{J'} = g|V_{J'}$, whence $\pi_{J, J'}(f_0^{(J)}) = \pi_{J, J'}$. This implies $(f_0^{(J)})_{J \in \mathcal{E}(I)} \in \lim R[X_J]$, $f = \pi^{-1}((f_0^{(J)})_{J \in \mathcal{E}(I)}) \in R[\langle X \rangle]$ and $f^F|V_J = (f_J)^F = (f_0^{(J)})^F = g|V_J$, whence $f^F = g$. 

4. Polynomial functions on groups.

In this section we study $(\mathbb{Z}-)$ polynomial functions and local $(\mathbb{Z}-)$ polynomial functions $q: G \to R$, where $G$ is an abelian group and $R$ is a commutative ring containing a prime field $F$.

Let $G$ be an (additively written) abelian group, $F$ a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for some prime number $p$) and $R$ a commutative $F$-algebra. We shall be concerned with the $R$-algebras $\mathcal{O}(G, R) = \mathcal{O}_Z(G, R)$ and $\mathcal{O}(G, R) = \mathcal{O}_Z(G, R)$; we always write $\otimes$ instead of $\otimes_Z$.

$F \otimes G$ is a vector space over $F$, and $F \otimes G = \{ \lambda \otimes g \mid \lambda \in F, g \in G \}$. Let $\omega: G \to F \otimes G$ be the group homomorphism defined by $\omega(g) = 1 \otimes g$.

Let $\mathcal{E}^n(G, R)$ resp. $\mathcal{E}^n_F(F \otimes G, R)$ be the $R$-module of all multiadditive functions $G^n \to R$ resp. $F$-multilinear functions $(F \otimes G)^n \to R$. For $p^* \in$
Then we obtain the following Lemma.

**Lemma 3.** The mapping $\omega^n: \mathcal{L}_F^n(F \otimes G, R) \rightarrow \mathcal{L}_F^n(G, R)$ is an isomorphism of $R$-modules.

**Proof.** $\omega^n$ is $R$-linear by definition. Now we consider the canonical isomorphism

$$\mathcal{L}_F^n(G, R) \rightarrow \text{Hom}(G \otimes \ldots \otimes G, R),$$

$$\mathcal{L}_F^n(F \otimes G, R) \rightarrow \text{Hom}_F((F \otimes G) \otimes \ldots \otimes (F \otimes G), R)$$

and

$$(F \otimes G) \otimes \ldots \otimes (F \otimes G) \rightarrow F \otimes (G \otimes \ldots \otimes G);$$

they induce a commutative diagram

$$\mathcal{L}_F^n(F \otimes G, R) \xrightarrow{\omega^n} \text{Hom}_F(F \otimes G \otimes^n, R) \xrightarrow{\omega^*} \text{Hom}(G \otimes^n, R)$$

where $\omega^*(\varphi)(g_1 \otimes \ldots \otimes g_n) = \varphi(1 \otimes g_1 \otimes \ldots \otimes g_n)$. By [11], Lemma 2, $\omega^*$ is an isomorphism (there $R$ is assumed to be a field, but this is immaterial). Hence $\omega^*$ is also an isomorphism. $lacksquare$

**Theorem 4.** The mapping

$$\omega^*: \{\mathcal{P}_F(F \otimes G, R) \rightarrow \mathcal{P}(G, R), p \mapsto p \circ \omega, \}

is an isomorphism of $K$-algebras satisfying

$$\omega^*(\mathcal{P}_F(F \otimes G, R)) = \mathcal{P}(G, R).$$

**Proof.** We prove first that $p \in \mathcal{P}_F(F \otimes G, R)$ implies $\omega^*(p) \in \mathcal{P}(G, R)$, and it is sufficient to do this in the case where $p: F \otimes G \rightarrow R$ is a homogeneous $F$-polynomial function of degree $n \geq 1$. In this case, let
$p^* \in \mathcal{L}_R^n(F \otimes G, R)$ be such that $p(z) = p^*(z, \ldots, z)$ for all $z \in F \otimes G$. Then we obtain, for $g \in G$,

$$\omega^*(p)(g) = p(1 \otimes g) = p^*(1 \otimes g, \ldots, 1 \otimes g) = (\omega^n p)(g, \ldots, g),$$

which implies $\omega^*(p) \in \mathcal{P}(G, R)$. Now we set

$$\tilde{\omega} = \omega^* | \mathcal{P}_F(F \otimes G, R) : \mathcal{P}_F(F \otimes G, R) \to \mathcal{P}(G, R),$$

and we prove that $\tilde{\omega}$ is an isomorphism of $R$-algebras.

In order to prove that $\tilde{\omega}$ is surjective it suffices to show that every homogeneous polynomial function $q : G \to R$ of degree $n \geq 1$ lies in the image of $\tilde{\omega}$. Let $q : G \to R$ be a homogeneous polynomial function of degree $n \geq 1$, and let $q^* \in \mathcal{L}_n^n(G, R)$ be such that $q(g) = q^*(g, \ldots, g)$ for all $g \in G$. By Lemma 3, $q^* = \omega^n(p^*)$ for some $p^* \in \mathcal{L}_R^n(F \otimes G, R)$. If $p : F \otimes G \to R$ is defined by $p(z) = p^*(z, \ldots, z)$ then

$$\omega^*(p)(g) = p^*(1 \otimes g, \ldots, 1 \otimes g) = q^*(g, \ldots, g) = q(g)$$

for all $g \in G$, whence $\omega^*(p) = q$.

In order to prove that $\tilde{\omega}$ is injective, let $p \in \mathcal{P}_F(F \otimes G, R)$ be in the kernel of $\tilde{\omega}$, i.e., $p(1 \otimes g) = 0$ for all $g \in G$.

**CASE 1.** $\text{char}(R) = p > 0$, $F = \mathbb{F}_p$. In this case, all elements of $F \otimes G$ are of the form $z = \bar{m} \otimes g = 1 \otimes mg$ for some $m \in \mathbb{Z}$, which implies $p = 0$.

**CASE 2.** $\text{char}(R) = 0$, $F = \mathbb{Q}$. We write $p$ in the form $p = p_1 + \ldots + p_d$, where $p_i : F \otimes G \to R$ is a homogeneous $F$-polynomial function of degree $i$. For $t \in \mathbb{Q}$ and $g \in G$, we obtain

$$p(t \otimes g) = \sum_{i=0}^d t^i p_i(1 \otimes g) \in R,$$

and if $t \in \mathbb{Z}$, then $p(t \otimes g) = p(1 \otimes tg) = 0$. Hence the polynomial

$$\sum_{i=0}^d p_i(1 \otimes g) T^i \in R[T]$$

vanishes on $\mathbb{Z}$ which, by Lemma 2, implies $p_i(1 \otimes g) = 0$ for all $i \in \{0, \ldots, d\}$ and $g \in G$. Therefore we obtain $p(t \otimes g) = 0$ for all $t \in \mathbb{Q}$ and $g \in G$, i.e., $p = 0$.

Now we consider local polynomial functions. Let $\mathcal{E}(G)$ be the set of all finitely generated subgroups of $G$ and $\mathcal{E}(F \otimes G)$ the set of all finitely generated $F$-submodules of $F \otimes G$. Obviously, the set

$$\mathcal{E}_0(F \otimes G) = \{ F \otimes C | C \in \mathcal{E}(G) \}$$
is cofinal in $\mathcal{S}(F \otimes G)$ and therefore a function $p: F \otimes G \rightarrow R$ lies in $\mathcal{P}_F(F \otimes G, R)$ if and only if $p|F \otimes C \in \mathcal{P}_F(F \otimes G, R)$ for all $C \in \mathcal{G}(G)$. If $p \in \mathcal{P}_F(F \otimes G, R)$, then $(p \circ \omega)|C = (p|F \otimes C) \circ (\omega|C) \in \mathcal{P}(C, R)$ for all $C \in \mathcal{G}(G)$ which implies $\omega^*(p) = p \circ \omega \in \mathcal{P}(G, R)$. For $C \in \mathcal{G}(G)$, we have established an isomorphism

$$\tilde{\omega}_C: \mathcal{P}_F(F \otimes C, R) \cong \mathcal{P}(C, R)$$

satisfying $\tilde{\omega}_C(p) = p \circ \omega$; the family $(\tilde{\omega}_C)_{C \in \mathcal{G}(G)}$ is compatible with the morphisms of the projective system, and therefore we get a commutative diagram.

$$
\begin{array}{ccc}
\mathcal{P}_F(F \otimes G, R) & \xrightarrow{\omega^*} & \mathcal{P}(G, R) \\
\pi \downarrow & & \downarrow \pi \\
\lim_{C \in \mathcal{G}(G)} \mathcal{P}_F(F \otimes C, R) & \xrightarrow{\sim} & \lim_{C \in \mathcal{G}(G)} \mathcal{P}(C, R).
\end{array}
$$

The left vertical arrow is an isomorphism by Proposition 9. Hence the right vertical arrow is surjective, and since it clearly is injective it is also an isomorphism. Therefore $\omega^*$ is an isomorphism. 

**Corollary.** Let $K$ be a field of characteristic zero. Then $\mathcal{P}(G, K)$ is a factorial domain.

**Proof.** By Theorem 4, $\mathcal{P}(G, K) = \mathcal{P}_F(F \otimes G, K)$, where $F = \mathbb{Q}$ is the prime field of $K$. If $F \otimes G = \{0\}$ then $\mathcal{P}_F(F \otimes G, K) = K$; thus we suppose that $F \otimes G = F(I)$ for some set $I \neq \emptyset$. Then we obtain $\mathcal{P}_F(F \otimes G, K) = \mathcal{P}_F(F(I), K) = K[\{x\}]$ by Theorem 3, and the latter ring is a factorial domain by Theorem 1.

**Remark and acknowledgments.** Without using to formalism of large polynomials it was proved in [12] that $\mathcal{P}(G, K)$ is a domain if char$(K) = 0$. I am indebted to Professor Jens Schwaiger for interesting me in generalized polynomial functions and for valuable discussions.

**REFERENCES**
