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Existence and Uniqueness of Maps Into Affine Homogeneous Spaces.

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SUMMARY - We extend the usual existence and uniqueness theorem for immersions into spaces of constant curvature to smooth mappings into affine homogeneous spaces. We also obtain a result on reduction of codimension.

1. Statement of the results.

Let S be a smooth manifold with a connection D on its tangent bundle TS with parallel curvature and torsion tensors R and T. If S is simply connected and D is complete, such a space is precisely a reductive homogeneous space $S = G/H$ with its canonical connection (cf. [N], [K]). In this case, G can be chosen to be the group of affine diffeomorphisms; these are diffeomorphisms $g: S \to S$ with $g^* D = D$.

Let M be another smooth manifold and $f: M \to S$ a smooth mapping. Then its differential gives a vectorbundle homomorphism $F = df: TM \to E$ where E is the pull back bundle of TS:

$$E = f^* TS = \{(m, v);\ m \in M,\ v \in T_{f(m)} S\}.$$

The curvature and torsion tensors of S give bumdle homomorphisms $T: \Lambda^2 E \to E$ and $R: \Lambda^2 E \to \operatorname{End}(E)$ (the endomorphisms of E) satisfying the following structure equations (cf. [GKM]):

$$D_V F(W) - D_W F(V) - F([V, W]) = T(F(V), F(W)), \tag{1}$$

$$D_V D_W \xi - D_W D_V \xi - D_{[V, W]} \xi = R(F(V), F(W)) \xi, \tag{2}$$

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for any sections V, W of TM and ξ of E, where D denotes the pull-back connection on E. Abbreviating the left hand side of (1) by $dF(V,W)$ where d is the Cartan derivative of the E-valued 1-form F, and the left hand side of (2) by $R^E(V,W)\xi$ where $R^E: \Lambda^2 TM \to \mathrm{End}(E)$ is the curvature tensor of the connection D on E, we can write these equations shortly as

$$dF = F^* T, \tag{1}$$

$$R^E = F^* R. \tag{2}$$

Since S has parallel torsion and curvature, the tensors R and T are parallel with respect to the connection D on E. More generally, let E be any vector bundle over M, equipped with a connection D. We say that E has *the algebraic structure of* S if there exist parallel bundle homomorphisms $T: \Lambda^2 E \to E$ and $R: \Lambda^2 E \to \mathrm{End}(E)$ and a linear isomorphism $\Phi_0: E_p \to T_o S$ for some fixed $p \in M$, $o \in S$, which preserve R and T. Apparently, $E = f^* TS$ has the algebraic structure of S. We want to prove the following.

THEOREM 1. *Let S be a manifold with complete connection D wuth parallel torsion and curvature tensors. Let M be a simply connected manifold and E a vector bundle with connection D over M having the algebraic structure (R,T) of S. Let F: $TM \to E$ be a vector bundle homomorphism satisfying equations* (1) *and* (2) *above. Then there exists a smooth map f: $M \to S$ and a parallel bundle isomorphism Φ: $E \to$ $\to f^* TS$ preserving T and R such that*

$$df = \Phi \circ F.$$

If S is simply connected, then f is unique up to affine diffeomorphisms of S.

THEOREM 2 (Reduction of codimension). *Let S be as above and f: $M \to S$ a smooth map such that the image of df lies in a parallel subbundle $E' \subset f^* TS$ which is invariant under T and R. Then there is a totally geodesic subspace $S' \subset S$ with $f(M) \subset S'$.*

REMARKS. If $S = \mathbb{R}$, then the conditions (1), (2) are reduced to $dF = 0$. So Theorem 1 holds since $H^1(M) = 0$. If S is a Riemannian space form of constant curvature with its Levi-Civita connection and if F is injective and E is equipped with a parallel metric, then E can be identified with $TM \oplus \perp M$ where $\perp M = F(TM)^{\perp}$, and F is the embedding of the first factor. Now (1) is equivalent to $(dF)^{\perp} = 0$ which means that the second fundamental form is symmetric, and (2) contains precisely the Gauß, Codazzi and Ricci equations. So we receive the usual existence and uniqueness theorems for maps into space forms. In [EGT], a similar theorem for Kähler space forms was proved which is also covered by our result.

After finishing this work we learned that Theorem 1 was already proved in 1978 by a different method ([W], p. 36); unfortunately, this proof was never published in a Journal.

2. Proof of the theorems.

Let M be a manifold, E a vector bundle over M with connection D and F: $TM \to E$ a bundle homomorphism. We need to generalize the Cartan structure equations of the tangent bundle to this situation. Let $b = (b_1, \dots, b_n)$ be a local frame on some open subset $U \subset M$. Then there are 1-forms $\theta = (\theta_i)$, $\omega = ((\omega^i_j))$ on U (where $i, j = 1, \dots, n$) such that

$$F = b \cdot \theta := \sum \theta^i b_i, \qquad Db = b \cdot \omega$$

where the last expression means $Db_j = \sum \omega^i_j b_i$. Then

(3) $$dF = Db \wedge \theta + b \cdot d\theta = b(\omega \wedge \theta + d\theta),$$

(4) $$dDb = Db \wedge \omega + b \cdot d\omega = b(\omega \wedge \omega + d\omega),$$

where $dDb = (dDb_1, \dots, dDb_n) = (R^E b_1, \dots, R^E b_n)$.

Now let there be parallel homomorphisms T: $\Lambda^2 E \to E$ and R: $\Lambda^2 E \to \mathrm{End}(E)$. Using a fixed frame at some point $p \in M$, we identify E_p with $\mathbb{R}^n$ and get linear maps $T_0 \colon \Lambda^2 \mathbb{R}^n \to \mathbb{R}^n$ and $R_0 \colon \Lambda^2 \mathbb{R}^n \to$ $\to \mathrm{End}(\mathbb{R}^n)$. Let $H \subset \mathrm{Gl}(n, \mathbb{R})$ be group of linear automorphisms of $\mathbb{R}^n$ preserving T_0 and R_0. The vector bundle E is associated to a principal H-bundle as follows. For any $m \in M$, a *frame* $(b_1, \dots, b_n)$ of E_m can be considered as a linear isomorphism b: $\mathbb{R}^n \to E_m$ with $b(e_i) = b_i$. Let PE be the bundle of *(T, R)-frames*, i.e. those frames which map T_0 onto T and R_0 onto R. Clearly, PE is a principal H-bundle, where the group H acts from the right on PE. The advantage is that the coefficients of T and R are the same for any $b \in PE$:

$$T(b_i, b_j) = \sum t_{ij}^{\ k} b_k, \qquad R(b_i, b_j)\, b_k = r_{ijk}^{\ \ \ l} b_l$$

where $t_{ij}^{\ k}$ and $r_{ijk}^{\ \ \ l}$ are the coefficients of T_0 and R_0.

Now let us assume equations (1) and (2). Choose a local (T, R)-

frame, i.e. a local section $b\colon U \to PE|U$. Then

$$dF(v, w) = T(F(v), F(w)) = \sum T(\theta^i(v)\, b_i, \theta^j(w)\, b_j) = \sum \theta^1(v)\, \theta^j(w)\, t_{ij}{}^k\, b_k,$$

hence

$$dF = \frac{1}{2} \sum \theta^i \wedge \theta^j t_{ij}{}^k\, b_k.$$

Likewise,

$$dDb_k = \frac{1}{2} \sum \theta^i \wedge \theta^j r_{ijk}{}^l\, b_l.$$

Together with (3) and (4) we get the structure equations of Cartan type

$$\text{(5)} \qquad \begin{aligned} d\theta &= -\,\omega \wedge \theta + \sum t_{ij}\, \theta^i \wedge \theta^j, \\ d\omega &= -\,\omega \wedge \omega + \sum r_{ij}\, \theta^i \wedge \theta^j, \end{aligned}$$

where $t_{ij} = (t_{ij}{}^1, \dots, t_{ij}{}^n)$ and r_{ij} is the matrix $((r_{ijk}{}^l))$, i.e. $r_{ij}(e_k) = \sum r_{ijk}{}^l e_l$.

Now recall that the forms θ and ω on U are just the pull backs by $b\colon U \to PE$ of global forms on PE which we also call θ and ω. Namely, the forms $\theta \in \Omega^1(PE) \otimes \mathbb{R}^n$ and $\omega \in \Omega^1(PE) \otimes \underline{h}$ (where $\underline{h} \subset \mathrm{End}(\mathbb{R}^n)$ is the Lie algebra of H) are defined as follows. If $b \in PE$ and $X \in T_b PE$, then

$$\text{(6)} \qquad b \cdot \theta(X) = F(d\pi_b(X))$$

where $\pi\colon PE \to M$ is the projection, and

$$\text{(7)} \qquad b \cdot \omega(X) = \pi_v(X)$$

where $\pi_v\colon TPE \to VE$ is the vertical projection determined by the connection; here, $VE \subset TPE$ is the vertical distribution $(VE)_b = T_b(bH)$. Clearly, these forms on PE also satisfy (5).

Now let S be as above. Replacing (M, E, F) with (S, TS, Id), we get also forms θ', ω' on PTS satisfying equations (5) which are now the usual Cartan structure equations of TS. We will consider θ, ω, θ', ω' as forms on the product $PE \times PTS$ by pulling back via the projections pr_1, pr_2 onto the two factors. Since both (θ, ω) and (θ', ω') satisfy (5), we get that $d(\theta^i - \theta'^i)$ and $d(\omega^i_j - \omega'^i_j)$ lie in the ideal generated by $\theta^i - \theta'^i$ and $\omega^i_j - \omega'^i_j$; note that in any ring we have the identity

$$ab - a'b' = (a - a')\,b + a'(b - b').$$

Therefore the distribution

$$\underline{D} = \{(X, X') \in T(PE \times PTS);\ \theta(X) = \theta'(X'),\ \omega(X) = \omega'(X')\}$$

on $PE \times PTS$ is integrable.

Let $L \subset PE \times PTS$ be a maximal integral leaf of this distribution. We have $\dim L = \dim PE$ since the number of equations determining L is $n + \dim \underline{h} = \dim PTS$. Moreover, L intersects the second factor $\{b\} \times PTS$ everywhere transversally. Namely, if some vector $(0, X')$ lies in TL, then $\theta'(X') = 0$ and $\omega'(X') = 0$, hence $X' = 0$ since the forms θ'^i, $\omega'^i{}_j$ span T^*PTS. Moreover, L is invariant under H acting diagonally on $PE \times PTS$. Namely, if $(b, b') \in L$ and $\alpha = (\alpha^i{}_j) \in \underline{h}$, then $(b\alpha, b'\alpha) \in T_{(b,b')}L$ because $b\alpha$ and $b'\alpha$ are vertical vectors (so θ and θ' vanish) and $\omega(b\alpha) = \alpha = \omega'(b'\alpha)$. Thus the map $p_L := pr_1 \mid L\colon L \to PE$ is an H-equivariant local diffeomorphism.

Let us assume from now on that S is simply connected (which is no restriction since we may always pass to the universal cover). Then there is a group G which acts transitively on S by affine diffeomorphisms and also transitively on PTS (from the left) via differentials (cf. [K], Thm. I.17). Then also gL is an integral leaf for any $g \in G$, where we let G act only on the second factor of $PE \times PTS$. This is because θ' and ω' are invariant under affine diffeomorphisms of S since their differential preserves the horizontal and vertical distribution on PTS. (In fact, if we identify PTS by the action with G/kernel, then θ' and ω' are the components of the Maurer-Cartan form with respect to the $Ad(H)$-invariant decomposition of the Lie algebra $\underline{g} = \underline{p} \oplus \underline{h}$.)

Now we claim that the mapping $p_L = pr_1 \mid L\colon L \to PE$ is onto. Since it is a local homeomorphism, its image is open. Since M is connected and p_L maps H-orbits diffeomorphically onto H-orbits, it is sufficient to show that the image is closed. So let $(b_k, b_k')_{k \geq 0}$ be a sequence in L such that $b_k \to b$ in PE. We will show that also $b \in pr_1(L)$. Since G acts transitively on PTS, there exists $g_k \in G$ such that $g_k b_k' = b_0'$. Then the maximal integral leaves $g_k L$ contain the points (b_k, b_0'). So they converge to the maximal integral leaf L' through (b, b_0'). Hence $pr_1(L')$ contains a neighborhood of b in PE, and for big enough k, there exists $b' \in PTS$ with $(b_k, b') \in L'$. Therefore $L' = gL$ where $g \in G$ is such that $b' = gb_k'$, and in particular, $b \in pr_1(L)$ since $pr_1(L) = pr_1(gL)$.

It follows that p_L is a covering map. If U is a neighborhood of some $(b, b') \in L$ where $p_L \mid U$ is a diffeomorphism, then $p_L^{-1}(p_L(U))$ is a disjoint union of copies gU of U, where $g \in G$ leaves L invariant. Since M is simply connected, any element of the fundamental group $\pi_1(PE)$ can be represented by a closed curve in some fibre (H-orbit), and since p_L maps any H-orbit in L diffeomorphically onto an H-orbit in PE, it induces a surjective homomorphism of the fundamental groups. Therefore, the covering map p_L is actually a global diffeomorphism which means that L is a graph over PE. So there exists a smooth H-equivariant map $Pf\colon PE \to PTS$ with Graph$(Pf) = L$, and by uniqueness, any other integral leaf is the graph of $g \circ Pf$ for some $g \in G$. The fact that Graph(Pf) is an integral leaf means

$$Pf^*\theta' = \theta, \qquad Pf^*\omega' = \omega. \tag{8}$$

Since Pf maps fibres onto fibres, it is a bundle map, i.e. it determines a smooth mapping of the base spaces $f\colon M \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} PE & \xrightarrow{Pf} & PTS \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & S. \end{array}$$

Moreover, Pf defines a vector bundle isomorphism $\Phi\colon E \to f^*TS$ as follows. If $\xi = \sum x^i b_i = bx \in E_m$ for some $b = (b_1, \ldots, b_n) \in (PE)_m$ and $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, we put

$$\Phi(\xi) = (m, Pf(b)\cdot x) \in (f^*TS)_m = \{m\} \times T_{f(m)}S.$$

The map Φ is well defined, by the H-invariance of Pf, and it is clearly a bundle isomorphism preserving T and R. Moreover, if $\xi(t)$ is a parallel section of E along some curve in M, then $\xi(t) = b(t)x$ for some horizontal curve $b(t)$ in PE, i.e. $\omega\left(\frac{d}{dt}b(t)\right) = 0$. Since $Pf^*\omega' = \omega$, the curve $Pf(b(t))$ in PTS is horizontal again, so $\Phi(\xi(t))$ is also parallel. This shows that Φ is parallel.

Now let $v \in T_mM$ and $V \in T_bPE$ any lift, i.e. $\pi(b) = m$ and $d\pi_b(V) = v$. Then $df(v) = d\pi'(dPf(V))$. Recall that by (5) for any $b' \in PTS$, $V' \in T_{b'}PTS$, $v' = d\pi'(V')$ we have

$$v' = b'\cdot\theta(V').$$

Using the basis $b' = Pf(b)$ of $T_{f(m)}S$ to represent the vector $v' = df(v)$, we get (omitting the base points)

$$df(v) = Pf(b)\cdot\theta'(dPf(V)). \tag{9}$$

On the other hand, $F(v) = F(d\pi(V)) = b \cdot \theta(V)$ hence

$$\Phi(F(v)) = Pf(b) \cdot \theta(V). \tag{10}$$

Since $Pf^*(\theta') = \theta$, we get $df = \Phi \circ f$.

It remains to show the uniqueness of f. So let $f\colon M \to S$ be any smooth map with $df = \Phi \circ f$ for some parallel bundle isomorphism $\Phi\colon E \to f^*TS$ preserving T and R. Then we define a bundle map $Pf\colon PE \to PTS$ covering $f\colon M \to S$ by

$$Pf(b) = \Phi(b)$$

where $\Phi(b) = (\Phi(b_1), \dots, \Phi(b_n))$ for $b = (b_1, \dots, b_n) \in PE$. As above, Pf satisfies (9) and (10), and thus $df = \Phi \circ f$ implies that $Pf^*\theta' = \theta$. Moreover, since Φ is parallel, Pf maps horozontal curves in PE onto horizontal curves in PTS, and therefore $Pf^*\omega' = \omega$. This shows that Graph (Pf) is an integral leaf of the distribution $\underline{D}$. But we have shown that there is only one integral leaf up to the action of G, so f is uniquely determined up to composition with $g \in G$. This finishes the proof of Theorem 1.

Now we prove Theorem 2. Fix $p \in M$ and let $o = f(p)$. Then $V' := E'_p$ is a linear subspace of $V = (f^*TS)_p = T_oS$ which is invariant under R and T. We may assume that S is simply connected, hence an affine homogeneous space G/H. Then there is a totally geodesic homogeneous subspace $S' = G'/H'$ of S through 0 with $T_oS' = E'_p$ (e.g. cf. the Proof of Thm. I.17 in [K]; we put $\underline{h}' = \{A \in \underline{h};\ A(V') \subset V'\}$, $\underline{g}' = \underline{h}' \oplus V'$), and E' has the algebraic structure of S'. By Theorem 1, there exists a smooth map $f'\colon M \to S'$ with $f'(p) = 0$ and $df' = \Phi' \circ df$ for some parallel (R, T) preserving isomorphism $\Phi'\colon E' \to f'^*TS'$. But f'^*TS' is a parallel subbundle of f'^*TS as well as E', and their fibres agree at the point p, so these subbundles are the same, and $\Phi = id$ since Φ is parallel and $\Phi = id$ at the point p. So we see from the unicity part of Theorem 1 that $f' = f$.

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