

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

Z. Y. DUAN

Extensions of abelian by hyper-(cyclic or finite) groups (II)

Rendiconti del Seminario Matematico della Università di Padova,
tome 89 (1993), p. 113-126

[<http://www.numdam.org/item?id=RSMUP_1993__89__113_0>](http://www.numdam.org/item?id=RSMUP_1993__89__113_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1993, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

Extensions of Abelian by Hyper-(Cyclic or Finite) Groups (II).

Z. Y. DUAN(*)

ABSTRACT - If G is a hypercyclic (or hyperfinite and locally soluble) group and A a noetherian $\mathbb{Z}G$ -module with no nonzero cyclic (or finite) $\mathbb{Z}G$ -factors then Zaïcev proved that any extension E of A by G splits conjugately over A . For G being a hyper-(cyclic or finite) locally soluble group, if A is a periodic artinian $\mathbb{Z}G$ -module with no nonzero finite $\mathbb{Z}G$ -factors, then we have shown that any extension E of A by G splits conjugately over A , too. Here we consider the noetherian case and prove the splitting theorem which generalizes that of Zaïcev for G being a hyperfinite and locally soluble group.

In [1], we have proved: if G is a hyper-(cyclic or finite) locally soluble group and if A is a periodic artinian $\mathbb{Z}G$ -module with no nonzero finite $\mathbb{Z}G$ -factors, then any extension E of A by G splits conjugately over A . Now we continue the work and are going to prove the same result for A being noetherian.

The following lemma generalizes the corresponding one in Zaïcev's paper [6] and is very important in our later proof.

LEMMA 1. Let H be a normal hyper-(cyclically or finitely) embedded subgroup of a group G , and let A be a nonzero noetherian $\mathbb{Z}G$ -module. If $C_A(H) = 0$, then there is a subgroup K of H and a nonzero $\mathbb{Z}G$ -submodule B of A such that K is normal in G , $C_B(K) = 0$, and K induces in B a cyclic or finite group of automorphisms.

PROOF. Suppose the lemma is false. Using the noetherian condition we may assume that the lemma is true in all proper $\mathbb{Z}G$ -module A . We may also assume that G acts faithfully on A .

(*) Indirizzo dell'A.: Department of Mathematics, Southwest Teachers University, Beibei, ChongQing, 630715, P. R. China.

There is a cyclic or finite subgroup $F \leq H$ with F being normal in G . If $C_A(F) = 0$ then the lemma is true taking F, A for K, B .

Consider the second possibility $C_A(F) \neq 0$. We let A_1 be the $\mathbb{Z}G$ -submodule $C_A(F)$ and let $H_1 = C_H(F)$. Then H_1 is normal in G and $|H/H_1| < \infty$.

(1) Suppose that the centralizer $A_2/A_1 = C_{A/A_1}(H)$ is nonzero, i.e., $A_2 \neq A_1$. Consider the $\mathbb{Z}H_1$ -isomorphism $A_2/C_{A_2}(f) \cong_{\mathbb{Z}H_1} A_2(f-1)$, where $f \in F$. Since $A_1 \leq C_{A_2}(f)$ and A_2/A_1 is H_1 -trivial, we have that $A_2(f-1)$ is H_1 -trivial for any $f \in F$. It follows that

$$[A_2, F] = \sum_{f \in F} A_2(f-1)$$

is H_1 -trivial and so H induces a finite group of automorphisms on $[A_2, F]$. Since $A_2 \neq A_1$ the $\mathbb{Z}G$ -submodule $[A_2, F] \neq 0$ and $C_{[A_2, F]}(H) = 0$ since $C_A(H) = 0$. Therefore the lemma is true with $K = H$, $B = [A_2, F]$.

(2) Suppose now that $A_2 = A_1$, i.e., $C_{A/A_1}(H_1) = 0$. Then the $\mathbb{Z}G$ -module A/A_1 and the normal subgroup H_1 satisfy the hypotheses of the lemma and so there is a subgroup K_1 of H_1 and nonzero $\mathbb{Z}G$ -submodule B_1/A_1 of A/A_1 such that K_1 is normal in G , $C_{B_1/A_1}(K_1) = 0$, and K_1 induces in B_1/A_1 a cyclic or finite group of automorphisms.

Put $G_1 = C_G(F)$; clearly $H_1 = H \cap G_1$, $|G/G_1| < \infty$.

(a) We consider firstly the case that $K_1/C_{K_1}(B_1/A_1)$ is cyclic. Let $B_2 = [B_1, F]$ and let $K_0 = C_{K_1}(B_1/A_1)$. Since $A_1 = C_A(F)$, so

$$[K_0, B_1, F] = [[K_0, B_1], F] \leq [A_1, F] = 0;$$

also by $K_0 \leq K_1 \leq H_1 = C_H(F)$, we have

$$[F, K_0, B_1] = [[F, K_0], B_1] = [1, B_1] = 0.$$

Thus by the three subgroup lemma,

$$[B_2, K_0] = [[B_1, F], K_0] = [B_1, F, K_0] = 0.$$

Therefore $B_2 \leq C_A(K_0)$ and we then can view the noetherian $\mathbb{Z}G$ -module B_2 as a noetherian $\mathbb{Z}(G/K_0)$ -module. Applying Lemma 3 in [5] to the cyclic normal subgroup K_1/K_0 of G/K_0 , there is an integer m such that

$$B_2(k-1)^m \cap C_{B_2}(k) = 0,$$

where k is an element such that $K_1 = K_0 \langle k \rangle$.

If $B_2(k-1)^m = 0$, then

$$\begin{aligned} 0 = B_2(k-1)^m &= \left(\sum_{f \in F} B_1(f-1) \right) (k-1)^m = \sum_{f \in F} B_1((f-1)(k-1)^m) = \\ &= \sum_{f \in F} B_1((k-1)^m(f-1)) = \sum_{f \in F} (B_1(k-1)^m)(f-1). \end{aligned}$$

That is, $B_1(k-1)^m \leq C_A(F) = A_1$. But this is contrary to

$$C_{B_1/A_1}(k) = C_{B_1/A_1}(K) = 0.$$

So we have $B_2(k-1)^m$ and then the lemma is true by taking $B = B_2(k-1)^m$ and $K = K_1$.

(b) Secondly, we consider the case that $K_1/C_{K_1}(B_1/A_1)$ is finite.

Choose in F a least set of elements $\{x_1, \dots, x_n\}$ satisfying

$$A_1 = C_{B_1}(F) = C_{B_1}(x_1) \cap \dots \cap C_{B_1}(x_n)$$

and put $B_2 = C_{B_1}(x_1) \cap \dots \cap C_{B_1}(x_{n-1})$ if $n > 1$ and $B_2 = B_1$ if $n = 1$. Then

$$(1) \quad B_2 \neq A_1$$

and $C_{B_2}(x_n) = C_{B_1}(x_1) \cap \dots \cap C_{B_1}(x_n) = A_1$. Consider the $\mathbb{Z}G_1$ -isomorphism

$$(2) \quad B_2/A_1 = B_2/C_{B_2}(x_n) \cong_{\mathbb{Z}G_1} B_2(x_n - 1).$$

Since $K_1 \leq G_1$, $B_2 \leq B_1$, and K_1 induces a finite group of automorphisms on B_1/A_1 , so K_1 induces a finite group of automorphism on B_2/A_1 and hence on $B_2(x_n - 1)$. Since $C_{B_1/A_1}(K_1) = 0$ we also have $C_{B_2(x_n-1)}(K_1) = 0$.

Let $D = B_2(x_n - 1)$. Then D is a $\mathbb{Z}G_1$ -submodule of B_1 , $C_D(K_1) = 0$, and $|K_1/C_{K_1}(D)| < \infty$. Let $\overline{D}$ be the $\mathbb{Z}G$ -module generated by D , then $\overline{D} = \sum_{g \in T} Dg$ is a finite sum of $\mathbb{Z}G_1$ -submodules Dg , where T is a transversal to G_1 in G .

Note that since K_1 is normal in G , $C_{Dg}(K_1) = C_D(K_1)g = 0$, and $C_{K_1}(Dg) = g^{-1}C_{K_1}(D)g$. It follows that $|K_1 / \bigcap_{g \in T} C_{K_1}(Dg)| < \infty$ and so K_1 induces a finite group of automorphisms in $\overline{D}$.

Now consider two cases.

(A) D contains an element of finite order.

Then D contains a maximal elementary abelian p -subgroup $D_1 (\neq 0)$

and we let $\bar{D}_1 = \sum_{g \in T} D_1 g$. Let S be the K_1 -socle of the $\mathbb{Z}G_1$ -submodule D_1 , i.e., the sum of all irreducible $\mathbb{Z}G_1$ -submodules (these irreducible $\mathbb{Z}G_1$ -submodules are all finite since K_1 induces a finite group of automorphisms in D). Since D_1 is a $\mathbb{Z}G_1$ -submodule and K_1 is normal in G so S is a $\mathbb{Z}G_1$ -submodule and $\bar{S} = \sum_{g \in T} Sg$ is a $\mathbb{Z}G$ -submodule. Now Sg is a sum of irreducible $\mathbb{Z}K_1$ -submodules and so $\bar{S}$ is a sum of irreducible $\mathbb{Z}K_1$ -submodules each being contained in some Sg . Since $C_{D_g}(K_1) = 0$ it follows that $C_{\bar{S}}(K_1) = 0$. Thus we can take K_1 and $\bar{S}$ satisfying the conclusion of the lemma.

(B) *The group D is torsion-free.*

Let $T(\bar{D})$ be the torsion part of $\bar{D}$. Since $\bar{D}$ is a noetherian $\mathbb{Z}G$ -module, $T(\bar{D})$ has a finite exponent. Therefore $n\bar{D} \cap T(\bar{D}) = 0$ for some n and $n\bar{D}$ is torsion-free.

We put $m = |K_1/C_{K_1}(\bar{D})|$, $C = C_{\bar{D}}(K_1)$ and show that

$$(3) \quad [mn\bar{D}, K_1] \cap C = 0.$$

In fact, if $a \in [mn\bar{D}, K_1] \cap C$, then $a \in [mn\bar{D}, K_1] \cap C$, for some finitely generated K_1 -admissible subgroup $\bar{D}$ of $\bar{D}$. Since $n\bar{D} \cap C = C_{n\bar{D}}(K_1)$, $\bar{D} \leq \bar{D}$, and $n\bar{D}$ is torsion-free, so $n\bar{D}/(n\bar{D} \cap C)$ is torsion-free and then $n\bar{D} = (n\bar{D} \cap C) \oplus V$, where V is a free abelian subgroup. Applying Theorem 4.1 in [2], there is in $n\bar{D}$ a K_1 -admissible subgroup W such that $(n\bar{D} \cap C) \cap W = 0$ and the factor group $n\bar{D}/[(n\bar{D} \cap C) \oplus W]$ has a finite exponent, dividing m . Thus $mn\bar{D} \leq (n\bar{D} \cap C) \oplus W$. It follows that $[mn\bar{D}, K_1] \leq W$ and so $[mn\bar{D}, K_1] \cap C = 0$. Hence $a = 0$ and (3) is proved.

Note now that $[mn\bar{D}, K_1] \neq 0$. In fact, if $[mn\bar{D}, K_1] = 0$, then $mn\bar{D} \leq C_{\bar{D}}(K_1) = C$. Therefore $mnD \leq C$ and since D is torsion-free, $D \leq C$. This shows that D is a K_1 -trivial $\mathbb{Z}G_1$ -module and since $D = B_2(x_n - 1)$ and is G_1 -isomorphic to B_2/A_1 by (2) we have B_1/A_1 is also K_1 -trivial. But $C_{B_1/A_1}(K_1) = 0$ and so $B_2 = A_1$ contrary to (1). Thus $[mn\bar{D}, K_1] \neq 0$. Since $[mn\bar{D}, K_1]$ is a $\mathbb{Z}G$ -submodule and K_1 induces in it (as in $\bar{D}$) a finite group of automorphisms then it follows from (3) that the conditions of the lemma are satisfied by K_1 and $[mn\bar{D}, K_1]$. The lemma is proved.

As in the hyperfinite case, we need:

LEMMA 2. Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$ -module, and B a $\mathbb{Z}G$ -submodule of A such that B is of finite index in A and B has no nonzero finite $\mathbb{Z}G$ -factors, then

B has a complement in A , i.e., $A = B \oplus C$ for some finite $\mathbb{Z}G$ -submodule C of A .

PROOF. Suppose that B does not have a complement in A . By considering an appropriate factor-module of A we may assume that for every $\mathbb{Z}G$ -submodule D of B with $D \neq 0$, B/D has a complement in A/D .

Put $H = C_G(A/B)$, then, since G/H is finite and the irreducible $\mathbb{Z}G$ -factors of B are all infinite, we have $C_B H = 0$ so we can apply Lemma 1 to the subgroup H and the $\mathbb{Z}G$ -module B . So there is a subgroup K of H and a nonzero $\mathbb{Z}G$ -submodule D of B such that K is normal in G , $C_D(K) = 0$ and K induces on D a cyclic or finite group of automorphisms, i.e., $K/C_K(D)$ is cyclic or finite.

We write A as a sum $A = B + A_1$ with $B \cap A_1 = D$ and we will consider the $\mathbb{Z}G$ -submodule A_1 as a faithful $\mathbb{Z}G_0$ -module, where $G_0 = G/C_G(A_1)$. It is clear that D is a $\mathbb{Z}G_0$ -submodule of A_1 such that D is of finite index in A_1 and D has no nonzero finite $\mathbb{Z}G_0$ -factors. Also D has no complements in A_1 for otherwise if $A_1 = D \oplus C_1$ for some $\mathbb{Z}G_0$ -submodule C_1 of A_1 then C_1 can be viewed as a $\mathbb{Z}G$ -submodule of A by $G_0 = G/C_G(A_1)$ and then $A = B + A_1 = B \oplus C_1$, a contradiction.

Since $C_D(K) = 0$ and $D \leq A_1$, so K is not contained in $C_G(A_1)$. Let $K_0 = (KC_G(A_1))/C_G(A_1)$, then $K_0 \neq 1$. Also, it is clear that $C_D(K_0) = 0$ and K_0 induces on the $\mathbb{Z}G_0$ -submodule D of A_1 a cyclic or finite group of automorphisms. We prove that $C_{K_0}(D) = 1$. For suppose $C_{K_0}(D) \neq 1$ and let F_0 be a nontrivial cyclic or finite normal subgroup of G_0 contained in $C_{K_0}(D)$. If $x \in F_0$, then $D \leq C_{A_1}(x)$. Since $|A_1/D| = |A/B| < \infty$ and, as groups, $A_1/C_{A_1}(x) \cong A_1(x-1)$, we see that $A_1(x-1)$ is finite. Thus the $\mathbb{Z}G_0$ -submodule $[A_1, F_0]$ is finite. Also

$$\begin{aligned} F_0 \leq C_{K_0}(D) \leq K_0 &= (KC_G(A_1))/C_G(A_1) \leq (HC_G(A_1))/C_G(A_1) = \\ &= (C_G(A/B)C_G(A_1))/C_G(A_1), \end{aligned}$$

thus $[A_1, F_0] \leq B$, and then $[A_1, F_0] \leq D$. By D having no nonzero finite $\mathbb{Z}G_0$ -factors, we have $[A_1, F_0] = 0$ contrary to G_0 acting faithfully on A_1 . So $C_{K_0}(D) = 1$ and hence K_0 is cyclic or finite.

Now put

$$\begin{aligned} G_1 &= C_{G_0}(K_0), \quad K_0 = \langle x_1 = 1, x_2, \dots, x_m \rangle, \quad C_n = C_{A_1}(\langle x_1, \dots, x_n \rangle), \\ n &= 1, 2, \dots, m. \end{aligned}$$

We prove that $A_1 = D + C_n$, $n = 1, 2, \dots, m$.

It is clear that $A_1 = D + C_1$. Suppose $A_1 = D + C_n$ we prove $A_1 =$

$= D + C_{n+1}$. Consider the isomorphism of $\mathbb{Z}G_1$ -modules

$$C_n/C_{n+1} = C_n/C_{C_n}(x_{n+1}) \cong_{\mathbb{Z}G_1} C_n(x_{n+1} - 1),$$

where $C_n(x_{n+1} - 1)$ may not be contained in C_n if K_0 is nonabelian. Since

$$\begin{aligned} x_{n+1} \in K_0 &= (KC_G(A_1))/C_G(A_1) \leq (HC_G(A_1))/C_G(A_1) \leq \\ &\leq (C_G(A/B)C_G(A_1))/C_G(A_1), \end{aligned}$$

the $\mathbb{Z}G_1$ -module $C_n(x_{n+1} - 1)$ of A_1 is contained in B and then in D . Since $|G_0/G_1| < \infty$ it follows from Proposition 2 in [4] that the irreducible $\mathbb{Z}G_1$ -factors of D are all infinite, hence so are the factors of C_n/C_{n+1} . But

$$C_n/(C_{n+1} + (D \cap C_n)) \cong_{\mathbb{Z}G_1} (C_n + D)/(C_{n+1} + D),$$

a factor module of the finite module A_1/D . Hence $C_n + D = C_{n+1} + D$ and so $A_1 = C_{n+1} + D$. Thus $A_1 = C_n + D$ for all $n = 1, 2, \dots, m$. In particular, put $n = m$, $C_m = C_{A_1}(K_0)$ and $A_1 = D + C_{A_1}(K_0)$. Since $C_{A_1}(K_0)$ is clearly a $\mathbb{Z}G_0$ -submodule of A_1 and since $D \cap C_{A_1}(K) = C_D(K) = 0$ we have $A_1 = D \oplus C_{A_1}(K)$, contrary to D having no complements in A_1 . The proof is completed.

From the proof of Lemma 2, we have:

LEMMA 3. Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$ -module, and B a $\mathbb{Z}G$ -submodule of A such that, as group, A/B is a finite $\mathfrak{p}$ -group for some prime $\mathfrak{p}$ and the $\mathbb{Z}G$ -submodule B contains no nonzero $\mathbb{Z}G$ -factors being finite $\mathfrak{p}$ -groups. Then B has a complement in A , i.e., $A = B \oplus C$ for some $\mathbb{Z}G$ -submodule C of A .

Dual to Lemma 2, we have:

LEMMA 4. Let G be a hyper-(cyclic or finite) group, A a $\mathbb{Z}G$ -module, and B a finite $\mathbb{Z}G$ -submodule of A such that all irreducible $\mathbb{Z}G$ -factors of A/B are infinite. Then B has a complement on A , i.e., $A = B \oplus C$ for some $\mathbb{Z}G$ -submodule C of A .

PROOF. By Zorn's Lemma, A has a $\mathbb{Z}G$ -submodule D maximal with respect to $B \cap D = 0$. We show that $A = B \oplus D$. Suppose not, then by replacing A by A/D we may assume that for any nonzero $\mathbb{Z}G$ -submodule C of A , $B \cap C \neq 0$. We also assume that G acts faithfully on A .

Put $H = C_G(B)$, $|G/H| < \infty$ so there is a normal subgroup K of G contained in H such that K is either cyclic or finite. Put $H_1 = C_H(K)$.

Since H_1 is normal in G and $|G/H_1| < \infty$ it follows from Proposition 2 in [4] that the irreducible $\mathbb{Z}H_1$ -factors of A/B are infinite. If $x \in K$, then $B \leq C_A(x)$ and so the irreducible $\mathbb{Z}H_1$ -factors of $A/C_A(x)$ and hence $A(x-1)$ are infinite.

We prove that $[A, K] \cap B = 0$. If not, then there is a minimal set of elements $x_1, \dots, x_n$ such that $B_1 = B \cap \sum_{i=1}^n A(x_i - 1) \neq 0$. Then

$$\begin{aligned} B_1 &\cong_{\mathbb{Z}H_1} \left(b_1 \oplus \sum_{i=1}^{n-1} A(x_i - 1) \right) \left/ \left(\sum_{i=1}^{n-1} A(x_i - 1) \right) \right. = \\ &= \left(\sum_{i=1}^n A(x_i - 1) \right) \left/ \left(\sum_{i=1}^{n-1} A(x_i - 1) \right) \right. \cong_{\mathbb{Z}H_1} \\ &\cong_{\mathbb{Z}H_1} A(x_n - 1) \left/ \left(A(x_n - 1) \cap \sum_{i=1}^{n-1} A(x_i - 1) \right) \right. . \end{aligned}$$

This shows that $A(x_n - 1)$ has a nonzero finite $\mathbb{Z}H_1$ -factor contrary to the fact that the irreducible $\mathbb{Z}H_1$ -factors of $A(x-1)$ are all infinite. Thus $[A, K] \cap B = 0$ and hence $[A, K] = 0$, contrary to G acting faithfully on A . So the result is true.

An immediate consequence of Lemma 4 is:

COROLLARY 5. Let G be a hyper-(cyclic or finite) group, and A a noetherian $\mathbb{Z}G$ -module. Then A has a nonzero finite $\mathbb{Z}G$ -factor if and only if A has a nonzero finite $\mathbb{Z}G$ -image.

PROOF. We only need to suppose that A has a finite $\mathbb{Z}G$ -factor B/C , then using the noetherian condition we may assume that every irreducible $\mathbb{Z}G$ -factor of A/B is infinite. Then applying Lemma 4 to A/C with the finite $\mathbb{Z}G$ -submodule B/C we obtain a finite $\mathbb{Z}G$ -image of A .

As before, we have:

LEMMA 6. Let G be a hyper-(cyclic or finite) group, A a $\mathbb{Z}G$ -module and B a $\mathbb{Z}G$ -submodule of A . If as a group B is a finite p -group for some prime p , and if the factor module A/B contains no nonzero finite $\mathbb{Z}G$ -factors being p -groups, then B has a complement in A , i.e., $A = B \oplus C$ for some $\mathbb{Z}G$ -submodule C of A .

COROLLARY 7. Let G be a hyper-(cyclic or finite) group, and A a noetherian $\mathbb{Z}G$ -module. Then A has a nonzero $\mathbb{Z}G$ -image being

a finite $\mathfrak{p}$ -group for some prime $\mathfrak{p}$ if and only if A has such a nonzero $\mathbb{Z}G$ -factor.

Before we prove the main splitting theorem, we need to prove the following three results.

LEMMA 8. Let G be a hyper-(cyclic or finite) group, B a $\mathbb{Z}G$ -module, and A a noetherian $\mathbb{Z}G$ -submodule of B such that all irreducible $\mathbb{Z}G$ -factors of A are infinite. If B/A is torsion-free and G -trivial, then $B = A \oplus B_1$ for some $\mathbb{Z}G$ -submodule B_1 of B .

PROOF. Suppose that A has no complements in B . Since A is noetherian, we may assume that for each nonzero $\mathbb{Z}G$ -submodule C of A , A/C has a complement in B/C .

In B , we choose a $\mathbb{Z}G$ -submodule M maximal with respect to $A \cap M = 0$. We show that if S is any $\mathbb{Z}G$ -submodule such that $B = A + S$ then $M \leq S$.

Since $B/A \geq (A \oplus M)/A \cong_{\mathbb{Z}G} M$, we have M is a G -trivial $\mathbb{Z}G$ -module and hence all of its irreducible $\mathbb{Z}G$ -factors are finite. Also

$$A/(A \cap S) \cong_{\mathbb{Z}G} (A + S)/S = B/S \geq (M + S)/S \cong_{\mathbb{Z}G} M/(M \cap S).$$

Since A is noetherian and having no nonzero finite $\mathbb{Z}G$ -factors, we must have $M = M \cap S$, i.e., $M \leq S$.

Consider the factor-module B/M . Every nonzero $\mathbb{Z}G$ -submodule of B/M has nonzero intersection with $(A \oplus M)/M$. In particular, $(A \oplus M)/M$ has no complements in B/M . If V/M is a nonzero $\mathbb{Z}G$ -submodule of $(A \oplus M)/M$ then $V = C \oplus M$, where $C = A \cap V$ is nonzero and so $B/C = A/C \oplus S_1/C$ for some $\mathbb{Z}G$ -submodule S_1 of B . As above, $M \leq S_1$ and so $(A \oplus M) \cap S_1 = (A \cap S_1) \oplus M = C \oplus M = V$. Thus S_1/V is a complement to $(A \oplus M)/V$ in B/V .

By passing to the factor-module B/M we may assume that $M = 1$ so that: (a) A has no complements in B but for any nonzero $\mathbb{Z}G$ -submodule C of A , A/C has a complement in B/C ; (b) if N is a nonzero $\mathbb{Z}G$ -submodule of B then $A \cap N \neq 0$.

We may assume that A is torsion-free. For otherwise, we may let $A[\mathfrak{p}]$ be the nonzero $\mathbb{Z}G$ -submodule generated by all the elements of order $\mathfrak{p}$, where $\mathfrak{p}$ is a prime. By (a), $B/A[\mathfrak{p}] = A/A[\mathfrak{p}] \oplus B_1/A[\mathfrak{p}]$. Since $B_1/A[\mathfrak{p}] (\cong_{\mathbb{Z}G} B/A)$ is torsion-free, $\mathfrak{p}B_1 \neq 0$, then, by (b), $0 \neq A \cap \mathfrak{p}B_1 \leq A[\mathfrak{p}] \cap B_1$. That is, B_1 has elements of order $\mathfrak{p}^2$, contrary to $B_1/A[\mathfrak{p}]$ being torsion-free. So A is torsion-free and then B is torsion-free. Since A has no nonzero finite $\mathbb{Z}G$ -factors, we have $C_A(G) = 0$. By Lemma 1, G has a normal subgroup K and A has a nonzero $\mathbb{Z}G$ -submodule A_1 such that $C_{A_1}(K) = 0$ and $K/C_K(A_1)$ is cyclic or finite. By (a),

$B/A_1 = A/A_1 \oplus B_1/A_1$. Consider the $\mathbb{Z}G$ -module B_1 and we prove that $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$ -submodule B_2 (and hence we get $B = A \oplus B_2$ as required).

Suppose $B_1 \neq A_1 \oplus B_2$ for any $\mathbb{Z}G$ -submodule B_2 and suppose that G acts faithfully on B_1 , i.e., $C_G(B_1) = 1$. It is clear that we still have that K is normal in G , $C_{A_1}(K) = 0$, and $K/C_K(A_1)$ is cyclic or finite. If $C_K(A_1) \neq 1$, then, since $C_K(A_1) = K \cap C_G(A_1)$ is a normal subgroup of G , $C_K(A_1)$ contains a nontrivial cyclic or finite subgroup F being normal in G . Let $F = \langle f_i, \dots, f_n \rangle$ and let $G_1 = C_G(F)$, then $|G/G_1| < \infty$. By Proposition 2 in [4], the irreducible $\mathbb{Z}G_1$ -factors of A_1 are infinite. Since B_1/A_1 is G -trivial, it is also G_1 -trivial. By $B_1/C_{B_1}(f_i) \cong_{\mathbb{Z}G_1} B_1(f_i - 1) \leq A_1$ and $A_1 \leq C_{B_1}(f_i)$, we must have $B_1(f_i - 1) = 0$, for all i . That is, $1 \neq F \leq C_G(B_1)$, contrary to G acting faithfully on B_1 . So $C_K(A_1) = 1$ and so K is a nontrivial cyclic or finite normal subgroup of G . Let $K = \langle k_1, \dots, k_t \rangle$. Being similar with the above, we have $B_1/C_{B_1}(k_i) \cong_{\mathbb{Z}G_2} B_1(k_i - 1) \leq A_1$ for all i , where $G_2 = C_G(K)$. Thus $B_1/(A_1 + C_{B_1}(k_i))$ must be zero for all i . That is, $B_1 = A_1 + C_{B_1}(k_i)$ for any i . Let $C_m = C_{B_1}(\langle k_1, \dots, k_m \rangle)$, $m = 1, \dots, t$. Then we have $B_1 = A_1 + C_1$. Suppose that $B_1 = A_1 + C_m$; we prove that $B_1 = A_1 + C_{m+1}$.

Consider the $\mathbb{Z}G_2$ -modules

$$C_m/C_{m+1} = C_m/C_{C_m}(k_{m+1}) \cong_{\mathbb{Z}G_2} C_m(k_{m+1} - 1).$$

Since B_1/A_1 is G -trivial, $C_m(k_{m+1} - 1) \leq A_1$ and so $C_m(k_{m+1} - 1)$ has no nonzero finite $\mathbb{Z}G_2$ -factors; hence the irreducible $\mathbb{Z}G_2$ -factors of C_m/C_{m+1} are all infinite. But

$$C_m/(C_{m+1} + (A_1 \cap C_m)) \cong_{\mathbb{Z}G_2} (C_m + A_1)/(C_{m+1} + A_1),$$

a factor module of the G_2 -trivial $\mathbb{Z}G_2$ -module B_1/A_1 . Hence $A_1 + C_m = A_1 + C_{m+1}$. That is, $B_1 = A_1 + C_{m+1}$. Therefore $B_1 = A_1 + C_m$ for all m . Put $m = n$, then $C_n = C_{B_1}(K)$ and $B_1 = A_1 + C_{B_1}(K)$, which implies that $C_{B_1}(K) \neq 0$. Hence, by (b) and $B/A_1 = A/A_1 \oplus B_1/A_1$, we have $C_{A_1}(K) = A_1 \cap C_{B_1}(K) = A \cap C_{B_1}(K) = 0$, a contradiction. So $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$ -submodule B_2 and hence the lemma is proved.

COROLLARY 9. Let G be a hyper-(cyclic or finite) group, B a $\mathbb{Z}G$ -module, and A a noetherian $\mathbb{Z}G$ -submodule of B such that all irreducible $\mathbb{Z}G$ -factors of A are infinite. If B/A is an infinite cyclic group, the $B = A \oplus B_1$ for some $\mathbb{Z}G$ -submodule B_1 of B .

PROOF. Let $G_1 = C_G(B/A)$, then $|G/G_1| \leq 2$ and B/A is torsion-free and G_1 -trivial. By Lemma 8, $B = A \oplus B_1$ for some G_1 -trivial $\mathbb{Z}G_1$ -

submodule B_1 of B . For $g \in G$, if $B_1 g \neq B_1$, then $B_1 g$ is G_1 -trivial and

$$0 \neq B_1 g / (B_1 \cap B_1 g) \cong_{\mathbb{Z}G_1} (B_1 + B_1 g) / B_1 \leq B / B_1 \cong_{\mathbb{Z}G_1} A.$$

That is, A has a nonzero G_1 -trivial $\mathbb{Z}G_1$ -factor and then a nonzero finite irreducible $\mathbb{Z}G_1$ -factor, which will imply that A has a nonzero finite irreducible $\mathbb{Z}G$ -factor, a contradiction. So $B_1 g = B_1$ for all $g \in G$. That is, B_1 is a $\mathbb{Z}G$ -submodule of B . The result is proved.

LEMMA 10. Let E be an extension of the abelian group A by a hyper-(cyclic or finite) group G such that A is a noetherian $\mathbb{Z}G$ -module and all irreducible $\mathbb{Z}G$ -factors of A are infinite. Then if C/A is a normal subgroup of E/A and $C \leq C_E(A)$, then $C = A \times N$, where N is a normal subgroup of E and is contained in every supplement to A in E .

PROOF. Let N be a normal subgroup of E contained in C and maximal subject to $N \cap A = 1$. By considering the factor group E/N we may suppose that $N = 1$. Then E satisfies the following condition: if S is normal in E , $S \leq C$, and $S \neq 1$, then $S \cap A \neq 1$. We show that this implies that $A = C$.

Suppose that $A \neq C$. Since E/A is hyper-(cyclic or finite), there is a nontrivial finite subgroup $K/A \leq C/A$ such that K is normal in E or an infinite cyclic subgroup $L/A \leq C/A$ such that L is normal in E .

For K , by the hypothesis of the lemma, $K \leq C_E(A)$ and so K is a finite extension of its central subgroup A . Hence K' is finite (Theorem 10.1.4 in [3]). It follows that $A \cap K'$ is finite and so $A \cap K' = 1$ by A having no nonzero finite $\mathbb{Z}G$ -factors. By the condition above, we have $K' = 1$ and so K is abelian. Apply Lemma 2 to the $\mathbb{Z}(E/K)$ -module K and its submodule A , then $A = A \times K_1$ for some normal subgroup K_1 of E , contrary to the condition above.

For L , by the hypothesis of the lemma, $L \leq C_E(A)$ and so L is a cyclic extension of its central subgroup A . Thus L is abelian. By Corollary 9, $L = A \times L_1$ for some normal subgroup L_1 of E , contrary to the condition above.

Thus we have proved that $C = A \times N$, where N is normal in E .

Now let E_1 be a supplement to A in E so that $E = AE_1$, $C = A(C \cap E_1)$ and $C \cap E_1$ is normal in AE_1 . We have

$$N(C \cap E_1) / (C \cap E_1) \leq C / (C \cap E_1) = A(C \cap E_1) / (C \cap E_1).$$

Since N is hyper-(cyclically or finitely) embedded in E and the irreducible $\mathbb{Z}G$ -factors of A are all finite, we must have $N(C \cap E_1) / (C \cap E_1) = 1$, i.e., $N(C \cap E_1) = C = E_1$. Hence $N \leq E_1$ as required.

Now we prove the main result of this paper.

THEOREM. Let G be a hyper-(cyclic or finite) locally soluble group and A a noetherian $\mathbb{Z}G$ -module. If A has no nonzero finite $\mathbb{Z}G$ -images, then the extension E of A by G splits conjugately over A and A has no nonzero finite $\mathbb{Z}G$ -factors.

PROOF. By Corollary 5, A has no nonzero finite $\mathbb{Z}G$ -factors.

Suppose the theorem is false, then using the fact that A is a noetherian $\mathbb{Z}G$ -module we may assume that: A has conjugate complements in E modulo any nontrivial E -invariant subgroup of A .

Since A has no nonzero finite $\mathbb{Z}G$ -factors, $C_A(E) = 1$. By Lemma 1, E/A has a normal subgroup K/A and A has a nontrivial E -invariant subgroup A_0 such that $C_{A_0}(K) = 1$ and $K/C_K(A_0)$ is cyclic or finite.

(1) If $K/C_K(A_0)$ is finite, then we may choose K and A_0 such that $K/C_K(A_0)$ is minimal and so $K/C_K(A_0)$ is a chief factor of E . (For if L is normal in E and $C_K(A_0) < L < K$ then if $C_{A_0}(L) = 1$ we have L, A_0 contrary to minimality of $|K/C_K(A_0)|$ and if $C_{A_0}(L) \neq 1$ then $K, C_{A_0}(L)$ is contrary to minimality of $|K/C_K(A_0)|$.) Hence $K/C_K(A_0)$ has order p^k for some prime p and integer $k \geq 1$. From $C_{A_0}(K) = 1$ it follows that $A_0[p] = 1$ and so $A_0^{p^k} \neq 1$.

By the assumption on A , we have E splits conjugately over A modulo $A_0^{p^k} \neq 1$.

Let E_1 be a complement to A in E modulo $A_0^{p^k} \neq 1$: $E = AE_1$, $A \cap E_1 = A_0^{p^k} \neq 1$; put $E_0 = A_0 E_1$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 10, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in E_1 . Consider the factor group $\bar{E}_0 = E_0/N$ and the subgroups $\bar{K}_0, \bar{A}_0$. Since

$$\bar{K}_0/\bar{A}_0 = \bar{K}_0/\bar{C}_0 \cong K_0/C_0 \cong K/C_K(A_0),$$

we have $|\bar{K}_0/\bar{A}_0| = p^k$. Corresponding to $C_{A_0}(K) = 1$ we have $C_{\bar{A}_0}(\bar{K}_0) = 1$ and also $\bar{A}_0 \cap \bar{E}_1 = \bar{A}_0^{p^k}$. It follows, by applying Lemma 6 in [6] to $\bar{E}_0$ and its subgroups $\bar{K}_0, \bar{A}_0$, that $\bar{E}_0$ splits over $\bar{A}_0$: $\bar{E}_0 = \bar{A}_0 \bar{E}_2$, $\bar{A}_0 \cap \bar{E}_2 = 1$. The complete preimage E_2 of $\bar{E}_2$ in E_0 gives $E_0 = A_0 E_2$ and $A_0 \cap E_2 = 1$. So that E_2 is a complement to A in E . Let S_1, S_2 be any two complements to A in E . Then, since E splits conjugately over A modulo $A_0^{p^k}$, we have S_1 and S_2 are conjugate modulo $A_0^{p^k}$ and we may assume that $A_0^{p^k} S_1 = A_0^{p^k} S_2$. Put $E_0 = A_0 S_1 = A_0 S_2$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 10, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in every supplement to A_0 in E_0 ; in particular, $N \leq S_1 \cap S_2$. Consider the factor group $\bar{E}_0 = E_0/N$ and its subgroups $\bar{K}_0, \bar{A}_0$. Since $\bar{K}_0/\bar{A}_0 \cong K/C_K(A_0)$, so $\bar{K}_0/\bar{A}_0$ is a group of order p^k , and also $C_{\bar{A}_0}(\bar{K}_0) =$

$= \bar{1}$ by $C_{A_0}(K) = 1$. From $A_0^{b^k} S_1 = A_0^{b^k} S_2$ it follows that $\bar{S}_1$ and $\bar{S}_2$ are complements to $\bar{A}_0$ in $\bar{E}_0$ which coincide modulo $\bar{A}_0^{b^k}$. Applying Lemma 6 in [6] to the group $\bar{E}_0$ and its subgroups $\bar{K}_0, \bar{A}_0$, we have the conjugacy of the complements: $\bar{S}_1^a = \bar{S}_2, a \in A_0$. Since $\bar{S}_1 = S_1/N, \bar{S}_2 = S_2/N$, and N is normal in E_0 it follows that $S_1^a = S_2$, i.e., E splits conjugately over A , a contradiction.

(2) Now we may suppose that $K/C_K(A_0)$ is cyclic.

In this case, we let $A_1 = [A_0, K] \leq A_0$, then, by $C_{A_0}(K) = 1$, we have $A_1 \neq 1$. Thus E splits conjugately over A modulo A_1 , i.e., $E = AE_1, A \cap E_1 = A_1$. Let $K_1 = K \cap E_1$ and $C_1 = C_{K_1}(A_0)$. It is clear that $A_1 \leq C_1 \leq C_{K_1}(A_1) \leq C_{E_1}(A_1)$. By Lemma 10, $C_1 = A_1 \times N$ for some normal subgroup N of E_1 . Since $K_1/C_1 \cong K/C_K(A_0)$, we have $K_1 = C_1 \langle x \rangle$ for some $x \in K_1$. Let $M = N \langle x \rangle$, then $K_1 = C_1 \langle x \rangle = A_1 M$. Since

$$\begin{aligned} [A_1 \cap M, K] &= [A_1 \cap M, C_K(A_0) \langle x \rangle] = [A_1 \cap M, x] = \\ &= [A_1 \cap M, \langle x \rangle] \leq [A_1, x] \cap [M, x] \leq A_1 \cap N = 1, \end{aligned}$$

we have $A_1 \cap M \leq C_{A_0}(K) = 1$. Thus $K_1 = A_1 M$, i.e., M is a complement to A_1 in K_1 .

Suppose that M_0 is also a complement to A_1 in K_1 with $N \leq M_0$; we show that M and M_0 are conjugate by an element of A_0 . We can write $x = a_1 x_0$ with $a_1 \in A_1$ and $x_0 \in M_0$. Since

$$A_1 = [A_0, K] = [A_0, C_K(A_0) \langle x \rangle] = [A_0, \langle x \rangle] = [A_0, x^{-1}],$$

so $a_1 = [a_0^{-1}, x^{-1}]$ for some $a_0 \in A_0$, and therefore

$$x = a_1 x_0 = [a_0^{-1}, x^{-1}] x_0 = a_0 (a_0^{-1})^{x^{-1}} x_0 = (a_0^{-1})^{x^{-1}} a_0 x_0 = x (x^{-1})^{a_0} x_0,$$

i.e., $x_0 = x^{a_0}$. Since $N \leq M_0$ and $N \leq C_1 = C_{K_1}(A_0)$, we have

$$M^{a_0} (N \langle x \rangle)^{a_0} = N \langle x^{a_0} \rangle = N \langle x_0 \rangle \leq M_0.$$

As $C_K(A_0) = AC_{K_1}(A_0)$ and $K = K_1 C_K(A_0)$, so

$$\begin{aligned} AM_0 &= A(A_1 M_0) = AK_1 = AC_{K_1}(A_0) K_1 = C_K(A_0) K_1 = \\ &= K = K^{a_0} = (AM)^{a_0} = AM^{a_0}, \end{aligned}$$

also $A \cap M_0 = A_1 \cap M_0 = 1$ and $A \cap M = 1$ implies that $A \cap M^{a_0} = 1$. Thus $M_0 = M^{a_0}$.

We now prove that A has conjugate complements in E and that the complements are of the form $L = N_{E_0}(M)$, where $E_0 = A_0 E_1$ and M is, as above, a complement to A_1 in K_1 containing N .

If $g \in E_1$, then since N and K_1 are both normal in E_1 and the sub-

group M^g is a complement to A_1 in K_1 containing N , thus $M^g = M^{a_0}$ for some $a_0 \in A_0$ and so $ga_0^{-1} \in N_{E_0}(M) = L$, hence $E = AE_1 = AL$. We show that L is a complement to A in E . That is, we need to prove that $A \cap L = 1$.

Since $L \leq E_0 = A_0E_1$ and $A \cap E_1 = A_1$, so

$$\begin{aligned} A \cap L &= A \cap (E_0 \cap L) = (A \cap E_0) \cap L = (A \cap A_0E_1) \cap L = \\ &= A_0(A \cap E_1) \cap L = A_0A_1 \cap L = A_0 \cap L; \end{aligned}$$

also A_0 is normal in E and $L = N_{E_0}(M)$, hence $[A_0 \cap L, M] \leq A_0 \cap M$.

Since

$$A_0 \cap M = A_0 \cap (E_1 \cap M) = (A_0 \cap E_1) \cap M = A_1 \cap M = 1,$$

so $A \cap L \leq C_{A_0}(M)$. Therefore, by $K = AM$ and $C_{A_0}(K) = 1$, we have $A \cap L \leq C_{A_0}(M) = C_{A_0}(K) = 1$. That is, $A \cap L = 1$ and so L is a complement to A in E .

Now let S be any complement to A in E . Thus S and L are conjugate modulo A_1 and we may assume that $A_1L = A_1S$. Therefore, we have

$$\begin{aligned} E_0 &= E_0 \cap E = E_0 \cap AL = (E_0 \cap A)L = (A_0E_1 \cap A)L = \\ &= A_0L = A_0A_1L = A_0A_1S = A_0S. \end{aligned}$$

Since $K_1 = A_1M \leq A_1L = A_1S$, so $K_1 = A_1M_1$, $A_1 \cap M_1 = 1$, where $M_1 = K_1 \cap S$; thus M and M_1 are complements to A_1 in K_1 . We show that $N \leq M_1$. By $K_1 \leq A_1S$ and $C_1 = C_{K_1}(A_0) \leq K_1$ we have $C_1 = C_1 \cap A_1S = A_1(C_1 \cap S)$, thus $C_1 = A_1 \times N_1$, where $N_1 = C_1 \cap S \leq M_1$ and N_1 is normal in $A_0S = E_0$ since $C_1 = C_{K_1}(A_0)$ is normal in $A_0E_1 = E_0$. In particular, N_1 is normal in $E_1 \leq E_0$ and, since E_1/A_1 is hyper-(cyclic or finite), N_1 is hyper-(cyclically or finitely) embedded in E_1 . Consider the product NN_1 . If $NN_1 \neq N_1$ then, by $C_1 = A_1 \times N = A_1 \times N_1$, $NN_1 \cap A_1 \neq 1$ and so A_1 contains a nontrivial cyclic or finite subgroup normal in E_1 . By $A_1 \leq A$ and $E_1/A_1 \cong E/A \cong G$, we have A has a nonzero cyclic or finite $\mathbb{Z}G$ -module and hence contains a nonzero finite $\mathbb{Z}G$ -factor, a contradiction. Thus $NN_1 = N_1$, $N \leq N_1$ and so $N \leq M_1$.

This shows that M and M_1 are conjugate by an element $a_0 \in A_0$, i.e., $M^{a_0} = M_1$, and hence $L^{a_0} = N_{E_0}(M)^{a_0} = N_{E_0}(M^{a_0}) = N_{E_0}(M_1)$. From $K_1 = A_1M$ and M is normal in L it follows that K_1 is normal in A_1L . Therefore, by $A_1L = A_1S$, we have K_1 is normal in A_1S , and so $M_1 = K_1 \cap S$ is normal in S and $S \leq N_{E_0}(M_1)$. By $L^{a_0} = N_{E_0}(M_1)$, we have $S \leq L^{a_0}$ and so

$$L^{a_0} = AS \cap L^{a_0} = (A \cap L^{a_0})S = S.$$

That is, S and L are conjugate in E , i.e., E splits conjugately over A , a contradiction again.

Thus, we have finished the proof of the theorem.

Acknowledgements. I am grateful to Dr. M. J. Tomkinson for the guidance of the research. Also I am grateful to the Sino-British Friendship Scholarship Scheme for the financial support.

REFERENCES

- [1] Z. Y. DUAN, *The extension of abelian-by-hyper-(cyclic or finite) groups*, Comm. Alg., 20: 8 (1992), pp. 2305.2321.
- [2] D. S. PASSMAN, *Infinite Crossed Products*, Academic Press (1989).
- [3] D. J. S. ROBINSON, *A Course in the Theory of Groups*, Springer-Verlag, New York (1982).
- [4] J. S. WILSON, *On normal subgroups of $\overline{SI}$ -groups*, Arch. Math., 25 (1974), pp. 574-577.
- [5] D. I. ZAĬCEV, *On estensions of abelian groups*, AN USSR, Inst. Mat., Kiev (1980), pp. 16-40.
- [6] D. I. ZAĬCEV, *Hyperfinit extensions of abelian groups*, AN USSR, Inst. Mat., Kiev (1988), pp. 17-26.

Manoscritto pervenuto in redazione il 12 febbraio 1992.