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A Property Equivalent to Commutativity for Infinite Groups.

FEDERICO MENEGAZZO (*)

According to a well-known result of B. H. Neumann, the class of groups with centre of finite index can be characterized as the class of groups all whose infinite subsets contain a pair of permutable elements [2].

Similar, possibly more general classes have been introduced by J. C. Lennox, A. M. Hassanabadi and J. Wiegold in [1]. If G is a group and n is a positive integer, they define an n -set in G to be a subset of G of cardinality n . The class P_n^* is then defined by

$G \in P_n^*$ if and only if every infinite set of n -sets in G contains a pair X, Y of different members such that $XY = YX$.

In this terminology, P_n^* is the class of centre-by-finite groups. In [1] it is shown that infinite groups in P_n^* are abelian if $n = 2$ or $n = 3$. However, this result holds in general; in fact, we will prove the following

THEOREM. *Suppose G is an infinite non-abelian group. For every integer $n > 1$ there is an infinite set of pairwise non-permutable n -sets in G .*

1. – We will look first at a special case.

PROPOSITION. *If $n > 1$ and $G \in P_n^*$ has an element with finite centralizer, then G is finite.*

PROOF. Assume, by contradiction, that $G \in P_n^*$ is an infinite group, $a \in G$, and $C_G(a)$ is finite. For $g \in G$, we denote $S(a, g)$ the set $\{x \in$

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$\in G|a^x = g\}$: $S(a, g)$ is either empty or a right coset of $C_G(a)$, and in any case it is finite. Let F be a fixed subset of G of cardinality $n - 1$ and such that $a \in F$; the set $B = a^{-1}FF \cup \left(\bigcup_{b \in F} S(a, b)\right)$ is finite. We now define by induction a sequence of elements of G , beginning with an element $y_1 \notin F$: if $y_1, \dots, y_{m-1}$ have already been defined, we choose y_m in the complement of the finite set $B \cup \left(\bigcup_{1 \leq k < m} a^{-1}Fy_k\right) \cup \left(\bigcup_{1 \leq k < m} S(a, y_k)\right)$. Finally, for every integer m we set $X_m = F \cup \{y_m\}$. All these X_m 's are n -sets in G , and $X_s \neq X_t$ if $s \neq t$. Fix now $1 \leq s < t$; certainly $ay_t \in X_s X_t$. On the other hand, $ay_t \notin X_t X_s = FF \cup Fy_s \cup \bigcup y_t F \cup \{y_t y_s\}$, since

$$\begin{aligned} ay_t \in FF &\text{ implies } y_t \in a^{-1}FF, \\ ay_t \in Fy_s &\text{ implies } y_t \in a^{-1}Fy_s \text{ with } s < t, \\ ay_t \in y_t F &\text{ implies } y_t \in S(a, b) \text{ for some } b \in F, \\ ay_t = y_t y_s &\text{ implies } y_t \in S(a, y_s) \text{ with } s < t, \end{aligned}$$

and all of the above possibilities contradict our choice of the y_m 's.

2. — We come now to the proof of our theorem. Let G be any infinite non-abelian group, and n an integer, $n > 1$. By the Proposition, we may assume that G satisfies the following condition: for every infinite subgroup H of G and every $h \in H$, $C_H(h)$ is infinite. It is immediate to check that this condition implies that every element of G belongs to an infinite abelian subgroup of G . At this point, one can apply Theorem C of [1] and conclude. To get a self-contained proof, one can proceed as follows. Suppose $x, y \in G$ and $xy \neq yx$, and let A be an infinite abelian subgroup of G containing y ; of course, $x \notin A$. Choose $a_1, \dots, a_{n-1} \in A$ such that $a_1 = y$ and $a_1, a_1^x, \dots, a_{n-1}$ are different elements of G . Now define a sequence of elements of G by induction: take $c_1 = 1$ and, if $c_1, \dots, c_{m-1}$ have already been defined, choose c_m in the complement in A of the finite set $\bigcup_{1 \leq k < m} a_1^{-1}\{a_1, \dots, a_{n-1}\}c_k$.

For every m , set $Y_m = \{a_1, \dots, a_{n-1}, c_m x\}$; then $|Y_m| = n$ for every m , and $Y_i \neq Y_j$ if $i \neq j$. Now for $i < j$ we have $a_1 c_j x \in Y_i Y_j$ but

$$\begin{aligned} a_1 c_j x \notin Y_j Y_i = \{a_1, \dots, a_{n-1}\}^2 \cup c_j x \{a_1, \dots, a_{n-1}\} \cup \\ \cup \{a_1, \dots, a_{n-1}\} c_i x \cup \{c_j x c_i x\}; \end{aligned}$$

in fact: $a_1 c_j x \in \{a_1, \dots, a_{n-1}\}^2$ implies $x \in A$; $a_1 c_j x \in c_j x \{a_1, \dots, a_{n-1}\}$ implies $a_1^x = a_r$ for some r , $1 \leq r \leq n - 1$; $a_1 c_j x \in \{a_1, \dots, a_{n-1}\} c_i x$ implies $c_j \in a_1^{-1}\{a_1, \dots, a_{n-1}\} c_i$ with $i < j$; and finally $a_1 c_j x = c_j x c_i x$ implies $x \in A$; and all these contradict some earlier assumption.

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