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On the Extension of Valuations on a Field K to $K(X)$. - I.

NICOLAE POPESCU - CONSTANTIN VRACIU(*)

Let K be a field and $K(X)$ the field of all rational function on an indeterminate X over K . An old and important problem is to give the extension of all valuation on K to $K(X)$ (see [5], par. 10). In the works [1]-[4] and [7] are given a sequence of results which ultimately permit to give a description of the extensions of all valuations on K to $K(X)$. Particularly in [4] are «classified» all valuations on $K(X)$ relative to the valuations on K . The aim of this paper is to study and to complete the results of [4].

This work is divided in two parts. The present first part has four sections. In the first section we give some general result on the extension of valuations on a field K to an extension L of K . Here are defined the so called quasiunramified extensions of valuation and are given examples of quasiunramified extension (Theorems 1.5 and 1.7). In the second section are shown that any r.t. extension of a valuation on K to $K(X)$ (see definition below in par. 2) is also quasiunramified (Theorem 2.4).

In Section 3 some considerations are made on valuation induced on the residue field of a valuation and the link between induced valuation and extension of valuation in some special setting (Theorem 3.3).

Finally in Section 4 is considered an algebraically closed field K a valuation v on K and are defined all extensions w of v to $K(X)$ such that residue field of w is canonically isomorphic to the residue field of v . These extensions are of two kinds. Theorem 4.4 gives a description of all extensions which are of the first kind. In the second part of this work we shall give a description of the extensions of the second kind. Also we shall define and describe

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these extensions in the case when K is not necessarily algebraically closed.

1. General results.

1. – Let K be a field and v a valuation on K . We emphasize sometimes this situation saying that (K, v) is a valued field. Denote by k_v the residue field, by G_v the value group by O_v the valuation ring of v . Also by M_v denote the maximal ideal of O_v and by $p_v: O_v \rightarrow k_v$ the canonical homomorphism. We refer the reader to [5], [6], [8] or [9] for general notions and definitions.

We say that two valuations v, v' on a field K are *equivalent* (see [5]) if they have the same valuation ring, i.e. $O_v = O_{v'}$. Generally denote $V(K)$ the set of all (pair nonequivalent) valuations on K .

Let $v, v' \in V(K)$; we say that v *dominates* v' and we write $v \leq v'$ if $O_v \subseteq O_{v'}$ and $M_v \subseteq M_{v'}$. In this way $V(K)$ becomes naturally an ordered set.

REMARK 1.1. Let $v, v' \in V(K)$. Then $v \leq v'$ if and only if there exists a homomorphism of ordered groups $s: G_v \rightarrow G_{v'}$ such that $v' = sv$. The homomorphism s is unique of its kind and is a onto mapping.

REMARK 1.2. Let $v \in V(K)$, G an ordered group and let $s: G_v \rightarrow G$ be an onto homomorphism of ordered groups. Then $v' = sv$ is a valuation on K such that $G_{v'} = G$ and $v \leq v'$.

In what follows we consider $V(K)$ as an ordered set by the relation \leq of domination.

If $v \in V(K)$ denote $B(v) = \{v' / v' \in V(K) \text{ such that } v \leq v'\}$, and denote $I(G_v)$ the set of all isolated subgroups of G_v (see [5], [9]). The following result is well known (see [5], [9]).

REMARK 1.3. $B(v)$ and $I(G_v)$ are totally ordered sets, and they are naturally isomorphic. Precisely, there exists an isomorphism

$$\varphi_v: B(v) \rightarrow I(G_v)$$

of ordered sets, defined as follows: if $v' \in B(v)$ then there exists $s: G_v \rightarrow G_{v'}$ such that $v' = sv$; then one has: $\varphi_v(v') = \text{Ker } s$.

2. – Let L/K be an arbitrary field extension, v a valuation on K and w a valuation on L . We say that w is an *extension* of v or that v is the *restriction* of w to K , if there exists a one to one morphism of ordered

groups $i_w^v: G_v \rightarrow G_w$ such that $w(x) = i_w^v v(x)$ for every $x \in K$. If $w \in V(L)$, denote $r(w)$ the restriction of w to K defined in the canonical manner. It is clear that the assignements $w \rightsquigarrow r(w)$ define a mapping called *restriction mapping*:

$$(1) \quad r: V(L) \rightarrow V(K).$$

It is well known (see [5], [8]) that r is an onto mapping of ordered sets. Moreover for every $w \in V(L)$ one has: $O_{r(w)} = O_w \cap K$.

The following result is well known (see [5], [6], [8]):

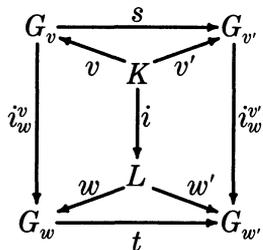
REMARK 1.4. Let $v \in V(K)$, $w \in V(L)$. Then $v = r(w)$ if and only if there exists a unique one to one homomorphism of ordered groups $i_w^v: G_v \rightarrow G_w$ such that for every $x \in K$ one has: $w(x) = i_w^v v(x)$.

Let $w \in V(L)$ and $v = r(w)$. Since the restriction mapping is an order perserving mapping, then it defines in a canonical way a mapping (also denoted by r):

$$(2) \quad r: B(w) \rightarrow B(v).$$

We shall say that w is a *quasiunramified extension* of v to L if the restriction mapping (2) is bijective. Also we say that w is an *unramified extension* of v to L if w is an extension of v and $G_v = G_w$.

3. - Let L/K be an extension of fields and let $w, w' \in V(L)$ be such that $w \leq w'$. Denote $v = r(w)$, $v' = r(w')$. It is clear that also $v \leq v'$ and according to the definition of the relation of domination there exists the morphism of ordered groups $s: G_v \rightarrow G_{v'}$, $t: G_w \rightarrow G_{w'}$ such that $v' = sv$, and $w' = tw$. Then we have the following diagram:



and: $ti_w^v v = twi = w' i = i_w^{v'} v' = i_w^{v'} sv$, i being the natural inclusion. Thus, since v is a onto mapping one has: $ti_w^v = i_w^{v'} s$.

Conversely, let $v, v' \in V(K)$, $v \leq v'$ and let $w \in V(L)$ be such that $v = r(w)$. Let G' be an ordered group, $i': G_{v'} \rightarrow G'$ a one to one homomorphism of ordered groups and $t: G_w \rightarrow G'$ an onto homomorphism of

ordered groups. Then $w = tw$ is a valuation on L such that $G_{w'} = G'$ and $w \leq w'$. If moreover $ti_w^v = i'$ then one has necessarily: $i'v' = i'vs = ti_w^v w = twi = w'i$, and so we may derive that $r(v') = v'$ and $i' = i_{w'}^{v'}$.

4. - At this point we shall consider two general situations of quasiunramification.

THEOREM 1.5. Let K/L be a field extension, and v a valuation on K . Then every unramified extension w of v to L , is quasiunramified.

PROOF. By hypothesis one has $G_v = G_w = G$. Let $\varphi_v: B(w) \rightarrow I(G)$, $\varphi_w: B(v) \rightarrow I(G)$ be respectively the isomorphism of ordered set defined on Remark 1.3. Then $\varphi = \varphi_v^{-1} \varphi_w: B(w) \rightarrow I(v)$ is also an isomorphism of ordered sets. We shall prove that φ coincides with the restriction mapping. For that, let $v' \in B(v)$ and $w' \in B(w)$ be such that $\varphi(w') = v'$. Also let $s: G = G_v \rightarrow G_{v'}$ and $t: G = G_w \rightarrow G_{w'}$ be such that $v' = sv$ and $w' = tw$. According to the definition of φ_v and φ_w one has: $\varphi_v(v') = \text{Ker } s$, and $\varphi_w(w') = \text{Ker } t$. Thus by equality $\varphi(w') = v'$, it follows that $\text{Ker } s = \text{Ker } t$. Now since s and t are onto homomorphism of ordered groups, there exists a homomorphism of ordered groups: $i: G_{v'} \rightarrow G_{w'}$ such that $is = t$. Then according to considerations stated on the point 3, it results that $r(w') = v'$. Hence $\varphi = r$, as claimed.

COROLLARY 1.6. Let $v, v' \in V(K)$ be such that $v \leq v'$, and let $s: G_v \rightarrow G_{v'}$ be a homomorphism of ordered groups such that $v' = sv$. Let w be an unramified extension of v to L . Then one has:

a) There exists only an extension w' of v' to L such that $w \leq w'$.

b) The extension w' , stated in a) is an unramified extension of v' to L and one has: $w' = sw$.

c) The valuation w is a maximal element in the set (ordered by the relation of domination) of all extension of v to L .

PROOF. The assertions a) and c) result obviously from Theorem 1.5. The assertion b) follows from Remarks 1.1 and 1.4.

Now we make some considerations on ordered (abelian) groups, needed in what follows (see also [5]).

We say that an abelian group G is *divisible* if for every $x \in G$ and every positive integer n , there exists $y \in G$ such that $ny = x$. Let G be an abelian group without torsion, and Q the additive group of rational

numbers. Then the tensorial product $G \otimes_{\mathbb{Z}} \mathbb{Q} = \bar{G}$ is a divisible group. Moreover the assignment: $g \mapsto g \otimes 1$ gives a one to one morphism of group $i: G \rightarrow \bar{G}$ such that for every divisible group G' , and every homomorphism of groups $f: G \rightarrow G'$, there exists a unique homomorphism of groups $\bar{f}: \bar{G} \rightarrow G'$ such that $\bar{f}i = f$. The pair (\bar{G}, i) is called the *divisible closure* of G . Particularly if G is an (abelian) ordered group, then \bar{G} the divisible closure of G is, in a natural way, an ordered group, and $i: G \rightarrow \bar{G}$ is a homomorphism of ordered groups.

DEFINITION. Let L/K be a field extension, $w \in V(L)$ and let $v \in V(K)$ be such that $v = r(w)$. We say that w is a *divisible extension* of v , if (G_w, i_w^v) (see 2) is a divisible closure of G_v .

THEOREM 1.7. Let L/K be a field extension and let v be a valuation on K . Then every divisible extension w of v to L is quasiunramified.

PROOF. Let $v' \in B(v)$ and let $s: G_v \rightarrow G_{v'}$ be such that $v' = sv$. Let us denote $(\bar{G}_{v'}, i')$ a divisible closure of $G_{v'}$. Now since (G_w, i_w^v) is by hypothesis a divisible closure of G_v , there exists only a homomorphism of ordered groups $\bar{s}: G_w \rightarrow \bar{G}_{v'}$ such that $\bar{s}i_w^v = i' s$. Since s is an onto mapping, then \bar{s} is also an onto mapping. According to the considerations made at point 3, it follows that $w'_0 = \bar{s}w: L \rightarrow G_{v'}$ is a valuation of L and one has: $G_{w'_0} = \bar{G}_{v'}$, and $w \leq w'_0$. Hence $w'_0 \in B(w)$ and $r(w'_0) = v'$. This show that the canonical mapping:

$$r: B(w) \rightarrow B(v)$$

is onto. Now we shall prove that r is also one to one. For that, let $w' \in B(w)$ be such that $r(w') = v'$. Then, according to Remark 1.4, one has $w'j = i_w^{v'}$, where $j: K \rightarrow L$ is the canonical inclusion. Since $w \leq w'$, there exists a homomorphism of ordered groups: $t: G_w \rightarrow G_{w'}$, such that $tw = w'$. Then as in 3 one has: $i_w^{v'} s' = ti_w^v$. Now, since t is an onto mapping, then G_w is, like $G_{w'}$, a divisible group, and it is easy to see that $(G_w, i_w^{v'})$ is a divisible closure of $G_{v'}$. Then there exists an isomorphism of ordered groups $\rho: G_w \rightarrow G_{v'}$ such that $\rho i_w^{v'} = i'$. Since:

$$\rho t i_w^v = \rho i_w^{v'} s = i' s = \bar{s} i_w^v$$

there results: $\rho t = s$. But one has: $w'_0 = \bar{s}w = \rho t w = \rho w'$, and since ρ is an isomorphism, then by Remark 1.1 it follows $w' = w'_0$. This equality show that the restriction mapping $r: B(w) \rightarrow B(v)$ is one to one, as claimed.

By Theorem 1.7 results:

COROLLARY 1.8. Let $v, v' \in V(K)$ such that $v \leq v'$, and $s: G_v \rightarrow G_{v'}$ be such that $v' = sv$. Let w be a divisible extension of v to L . Then:

a) There exists only an extension w' of v' to L such that $w \leq w'$.

b) The valuation w' stated in a) is a divisible extension of v' to L , and one has: $w' = \bar{s}w$, where $\bar{s}: G_w \rightarrow G_{w'}$ is canonically defined such that $s\bar{i}_w^v = \bar{i}_{w'}^{v'}s$.

c) The valuation w is a maximal element in the set of all extensions of v to L .

2. Residual transcendental extensions.

Generally, let L/K be a field extension, v a valuation on K and w an extension of v to L . We shall say that w is a *residual transcendental extension* if k_w/k_v is a transcendental extension (we shall write briefly: r.t. extension).

If K is a field, $K(X)$ shall denote the field of rational functions of one indeterminate over K . As usual $K[X]$ means the ring of polynomials in the indeterminate X over K .

1. - At this point we shall assume that K is an algebraically closed field. Let $v \in V(K)$ and let $(a, \delta) \in K \times G_v$. Denote $v_{(a, \delta)}$ the valuation of $K(X)$ defined as follows: if $f(X) \in K[X]$ one may write:

$$f(x) = a_0 + a_1(X - a) + \dots + a_n(X - a)^n$$

and thus we define:

$$v_{(a, \delta)}(f) = \inf_i (v(a_i) + i\delta)$$

(see [5], Ch. VI, par. 10). We shall say that $(a, \delta) \in K \times G_v$ is a pair of definition of $w = v_{(a, \delta)} \in V(K(X))$. It is easy to see that $v_{(a, \delta)}$ is in fact a r.t. extension of v to $K(X)$.

According to [1] one knows that every residual transcendental extension w of v to $K(X)$ is of the form $v_{(a, \delta)}$ and for every two pairs (a, δ) and (a', δ') of $K \times G_v$ one has:

$$v_{(a, \delta)} = v_{(a', \delta')} \quad \text{if and only if} \quad \delta = \delta' \quad \text{and} \quad v(a - a') \geq \delta.$$

Moreover if w is a residual transcendental extension of v to $K(X)$, then one has $G_w = G_v$ and so w is an unramified extension. Hence for r.t. extensions of v to $K(X)$ Theorem 1.5 is valid.

THEOREM 2.1. Let $v, v' \in V(K)$ be such that $v \leq v'$ and let $s: G_v \rightarrow G_{v'}$ be the only homomorphism of ordered groups such that $v' = sv$ (Remark 1.1). Then for every pair $(a, \delta) \in K \times G_v$ one has:

$$sv_{(a, \delta)} = v'_{(a, s(\delta))}.$$

PROOF. Let us denote $w = v_{(a, \delta)}$. Let $f \in K[X]$ and let us write:

$$f(x) = a_0 + a_1(X - a) + \dots + a_n(X - a)^n.$$

Then one has:

$$\begin{aligned} (sw)(f) &= s(w(f)) = s\left(\inf_i (v(a_i) + i\delta)\right) = \inf_i (s(v(a_i)) + is(\delta)) = \\ &= \inf_i (v'(a_i) + is(\delta)) = v'_{(a, s(\delta))}(f). \end{aligned}$$

COROLLARY 2.2. Let $(a, \delta) \in K \times G_v$ and $(a', \delta') \in K \times G_{v'}$. The following assertions are equivalents:

- i) $w \leq w'$,
- ii) $s(\delta) = \delta'$ and $v'(a - a') \geq \delta'$.

PROOF. i) \Rightarrow ii) If $w \leq w'$, then according to Corollary 1.8b), one has $w' = sw$, and so according to Theorem 2.1, it follows: $v'_{(a', \delta')} = v'_{(a, s(\delta))}$. Then $s(\delta) = \delta'$ and $v'(a - a') \geq \delta'$.

The implication ii) \Rightarrow i) is obvious.

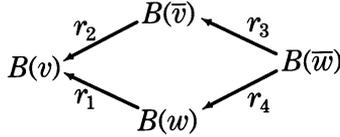
COROLLARY 2.3. Let K be an algebraically closed field and let $v, v' \in V(K)$ be such that $v \leq v'$. Let $s: G_v \rightarrow G_{v'}$ be such that $v' = sv$ and let $w = v_{(a, \delta)}$ be an r.t. extension of v to $K(X)$. Then:

- a) There exists only an extension w' of v' to $K(X)$ such that $w \leq w'$.
- b) The valuation w' , stated at the point a is a r.t. extension of v' to $K(X)$ and one has $w' = v'_{(a, s(\delta))}$.
- c) The valuation w is a maximal element in the ordered set of all extensions of v to $K(X)$.

The proof of the Corollary 2.3 follows by Corollary 1.8 and Theorem 2.1.

2. - Let (K, v) be a valued field. We fix an algebraic closure \bar{K} of K and denote by \bar{v} a fixed extension of v to \bar{K} . Let w be an extension of v to $K(X)$, and let \bar{w} be an extension of w to $\bar{K}(X)$ such that \bar{w} is an extension

of \bar{v} to $\bar{K}(X)$. (The existence of \bar{w} is given in [3, Proposition 2.1]). One has the following commutative diagram



where $r_i, i = 1, 2, 3, 4$ is the corresponding restriction mapping (see Section 1.2).

It is easy to see that w is an r.t. extension of v if and only if \bar{w} is an r.t. extension of \bar{v} . Moreover if w is an r.t. extension of v , then according to [2, Theorem 2.1] w can be described by v and a minimal pair of definition (a, δ) of \bar{w} . (Recall that a pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ is called *minimal* if for every $b \in \bar{K}$ such that $[K(b): K] < [K(a): K]$ one has $\bar{v}(a - b) < \delta$).

THEOREM 2.4. Let (K, v) be a valued field. Then any r.t. extension of v to $K(X)$ is quasiunramified.

PROOF. We shall utilize the notations given above. Since \bar{K} is algebraically closed, we know that $G_{\bar{v}}$ is divisible and is a divisible closure of G_v . Therefore \bar{v} is a divisible extension of v to \bar{K} . Moreover since $\bar{w} = \bar{v}_{(a, \delta)}$ for a suitable pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ then by the definition of $\bar{v}_{(a, \delta)}$ stated in 1, one see that $G_{\bar{v}} = G_{\bar{w}}$, and so \bar{w} is an unramified extension. Now according to Theorems 1.5 and 1.7, it follows that the restriction mappings r_2 and r_3 in the above diagram are bijective. Furthermore since $G_v \subseteq G_w \subseteq G_{\bar{w}} = G_{\bar{v}}$ and $G_{\bar{v}}$ is the divisible closure of G_v , then $G_{\bar{w}}$ is the divisible closure of G_w i.e. \bar{w} is a divisible extension of w . Thus by Theorem 1.7 it follows that the restriction mapping r_4 is bijective. Since $r_1 r_4 = r_2 r_3$ then r_1 is also bijective, and so w is a quasiunramified extension of v , as claimed.

COROLLARY 2.5. Let K be a field, let $v, v' \in V(K)$ be such that $v \leq v'$, let $s: G_v \rightarrow G_{v'}$ be a homomorphism of ordered groups such that $v' = sv$; and let w be a r.t. extension of v to $K(X)$. Let \bar{v} be an extension of v to \bar{K} and let $\bar{w} = \bar{v}_{(a, \delta)}$ be an extension of \bar{v} to $\bar{K}(X)$ whose restriction to $K(X)$ is just w . Assume that $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ is a minimal pair with respect to K . Then:

a) There exists an unique extension w' of v' to $K(X)$ such that $w \leq w'$.

b) The valuation w' at the point a) is an r.t. extension of v' to $K(X)$. Furthermore if $\bar{s}: G_{\bar{v}} \rightarrow G_{v'}$ is the homomorphism of ordered

groups uniquely defined by $s: G_v \rightarrow G_{v'}$, then $\bar{v}' = \bar{s}\bar{v}$ is an extension of v' to \bar{K} and $\bar{w}' = \bar{v}'_{(a, \bar{s}(\delta))}$ is an r.t. extension of \bar{v}' to $\bar{K}(X)$ whose restriction to $K(X)$ is just w' . Moreover $(a, \bar{s}(\delta))$ is a minimal pair with respect to K .

The proof is straightforward.

3. Some remarks on valuations.

1. - Let (K, v) be a valued field. As usual denote k_v the residue field of v and denote U_v the units group of O_v ; let $p_v: O_v \rightarrow k_v$ be the residue homomorphism, and let $\bar{p}_v: U_v \rightarrow k_v^*$, the restriction of p_v to U_v .

Let v' be a valuation on K such that $v' \leq v$, and let $s: G_{v'} \rightarrow G_v$ be the only homomorphism of ordered groups, such that $v = sv'$. Then there exists a unique homomorphism of group, $v'_1: k_{v'}^* \rightarrow \ker s$, such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & k_v^* & \xrightarrow{p_v} & U_v & \xrightarrow{i} & K^* \\
 (*) & \downarrow v'_1 & & & \swarrow v' & \downarrow v \\
 & \text{Ker } s & \xrightarrow{j} & G_{v'} & \xrightarrow{s} & G_v
 \end{array}$$

(i , and j are natural inclusions). It is known (see [5] or [8]) that v'_1 is a valuation on k_v and $G_{v'_1} = \ker s$. Let us denote:

$$A(v) = \{v' \in V(K) / v' \leq v\}.$$

According to the above considerations, we can define a mapping:

$$\psi_v: A_v(v) \rightarrow V(k_v), \quad \psi_v(v') = v'_1.$$

In valuation theory it is well known the following result:

PROPOSITION 3.1 ([5], [6], [8], [9]). The mapping ψ_v is bijective and for every $v' \in A(v)$ one has

$$\text{rank } v' = \text{rank } v'_1 + \text{rank } v$$

and the residue field $k_{v'_1}$ is canonically isomorphic to $k_{v'}$.

2. - Let (K, v) and (L, w) be valued fields such that $K \subseteq L$, and that w is an extension of v to L . Since the restriction mapping $r: V(L) \rightarrow$

$\rightarrow V(K)$ is order preserving, it induces a mapping, also denoted by r :

$$r: A(w) \rightarrow A(v).$$

Now, since the residue field k_v may be canonically identified to a subfield of k_w , then one may define a restriction mapping:

$$r': V(k_w) \rightarrow V(k_v).$$

With the above notation, one has the following commutative diagram, canonically defined:

$$\begin{array}{ccc} A(w) & \xrightarrow{r} & A(v) \\ \psi_w \downarrow & & \downarrow \psi_v \\ V(k_w) & \xrightarrow{r'} & V(k_v) \end{array}$$

Now, since ψ_v is a bijective mapping (Proposition 3.1) one has:

$$r = \psi_v^{-1} r' \psi_w.$$

From this equality it follows that r is a bijective mapping if and only if r' is also bijective. Particularly, if w is an inert extension of v . (i.e. the natural inclusion $k_v \rightarrow k_w$ is bijective) then r' is the identity mapping and so $r = \psi_v^{-1} \psi_w: A(w) \rightarrow A(v)$ is a bijective mapping. Therefore relative to inert extensions one has the following result:

PROPOSITION 3.2. Let L/K be an extension. Let $v, v' \in V(K)$ be such that $v' \leq v$ and let w be an inert extension of v to L . Then:

- There exists a unique extension w' of v' to L such that: $w' \leq w$.
- The valuation w' stated in a) is an inert extension of v' to L .
- The valuation w is a minimal element in the ordered set of all extensions of v to L .

PROOF. According to the above considerations, one has only to prove b). But this follows by commutative diagram:

$$\begin{array}{ccc} A(w) & \xrightarrow{r} & A(v) \\ \psi_w \downarrow & & \downarrow \psi_v \\ V(k_w) & \equiv & V(k_v) \end{array}$$

since one has $\psi_v(v') = \psi_w(w') = v'_1$, and, so according to Proposition 3.1, it results that $k_{v'_1} = k_{v'} = k_{w'}$.

3. - By the above diagram (*) it follows that v'_1 is trivial if and only if v and v' are identical. Particularly in the commutative diagram:

$$\begin{array}{ccc} A(w) & \xrightarrow{r} & A(v) \\ \psi_w \downarrow & & \downarrow \psi_v \\ V(k_w) & \xrightarrow{r'} & V(k_v) \end{array}$$

for an element $w' \in A(w)$ one has: $r(w') = v$ if and only if $\psi_v(r(w'))$ is trivial or equivalently $r'(\psi_w(w'))$ is trivial. Let us denote:

$$\begin{aligned} A'(w) &= \{w' \in A(w) / r(w') = v\}, \\ V'(k_w) &= \{w'_1 \in V(k_w) / r'(w'_1) \text{ is trivial}\}. \end{aligned}$$

Then we check that the bijective mapping ψ_w induces also a bijective mapping:

$$\psi'_w : A'(w) \rightarrow V'(k_w).$$

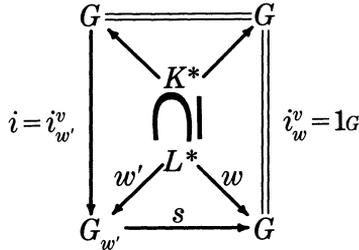
Let $w' \in A'(w)$ and let $w'_1 = \psi'_w(w')$. The link between the valuation w and w'_1 , is described by the following commutative diagram of groups and group homomorphisms:

$$\begin{array}{ccccccc} & & k_w^* & \xleftarrow{\bar{p}_w} & U_w \subseteq L^* & & \\ & & \downarrow w'_1 & & \downarrow w & & \\ 0 & \longrightarrow & G_{w'_1} & \xrightarrow{t} & G_{w'} & \xrightarrow{s} & G_w \longrightarrow 0 \end{array}$$

where the bottom row is an exact sequence of ordered groups and t and s are group homomorphisms of ordered groups.

THEOREM 3.3. Let w be an unramified extension of v to L , $G_v = G_w = G$, and let w' be another extension of v to L such that $w' \leq w$. (Hence $w' \in A'(w)$ and $w'_1 = \psi'_w(w') \in V'(k_w)$). Let us consider the group $G \times G_{w'}$ ordered lexicographically, and let $w'' : L^* \rightarrow G \times G_{w'}$, the mapping defined by: $w''(x) = (w(x), w'_1(p_w(x/a)))$, where $a \in K^*$ and $w(x) = v(a)$. Then w'' is a valuation on L equivalent to w' .

PROOF. According to the results of par. 1, 3, one has a commutative diagram



where i is a one to one homomorphism of ordered groups. From this diagram one has: $si = 1_G$. But then is known that the mapping:

$$\rho: G' \times G_{w_i} \rightarrow G_w$$

defined by: $\rho(g, g') = i(g) + t(g')$, is an isomorphism of groups. Now we assert that if $G \times G_{w_i}$ is ordered lexicographically, then ρ is in fact an isomorphism of ordered groups. Indeed, let $(g, g') \in G \times G_{w_i}$, be such that $(g, g') \geq 0 = (0, 0)$. If $g > 0$, then one has: $s\rho(g, g') = si(g) + st(g') = = g > 0$, i.e. $(g, g') > 0$; if $g = 0$, then one has $g' \geq 0$ and so, also, $\rho(g, g') = t(g') \geq 0$. Therefore we have proved that the mapping; $w'': L^* \rightarrow G \times G_{w_i}$, $w'' = \rho^{-1}w'$ is a valuation on L , equivalent to w' , and $G_{w''} = G \times G_{w_i}$. For every element $x \in L^*$, we may choose an element $a \in K^*$ such that $w(x) = v(a)$. Then $w(xa^{-1}) = w(x) - w(a) = = w(x) - v(a) = 0$, i.e. $xa^{-1} \in U_w$. But one has

$$i(w(x)) = i(v(a)) = w'(a)$$

and so

$$t(w'_1(\bar{p}_w(x/a))) = w'(x/a) = w'(x) - w'(a).$$

These equalities show that:

$$\begin{aligned} \rho(w(x), w'_1(\bar{p}_w(x/a))) &= i(w(x)) + t(w'_1(\bar{p}_w(x/a))) = \\ &= w'(a) + w'(x) - w'(a) = w'(x) \end{aligned}$$

and so:

$$w''(x) = (w(x), w'_1(\bar{p}_w(x/a))),$$

as claimed.

**4. Applications to the extensions of the valuations on K to $K(X)$.
The case K algebraically closed.**

In this section we shall assume that K is an algebraically closed field.

1. - Let v be a valuation on K . According to [4] an extension w of v to $K(X)$ will be called *residual algebraically* (briefly r.a.-extension) if k_w/k_v is an algebraic extension. Since K is assumed to be algebraically closed, then k_v is also algebraically closed, and so if w is an r.a.-extension of v to $K(X)$ then $k_w = k_v$. Hence every r.a. extension of v to $K(X)$ is inert.

According to the Proposition 3.2, one has:

PROPOSITION 4.1. Let $v, v' \in V(K)$ be such that $v' \leq v$, and let w be an r.a. extension of v to $K(X)$. Then:

a) There exists only an extension w' of v' to $K(X)$ such that $w' \leq w$.

b) The valuation w' stated in a) is an r.a. extension of v' to $K(X)$.

c) The valuation w is a minimal element in the set of all extensions of v to L .

Now let $v \in V(K)$. Since every extension of v to $K(X)$ is an r.t. extension or an r.a. extension, then according to Corollary 2.3 and Proposition 4.1 it follows:

COROLLARY 4.2. a) There is not three extensions w_1, w_2, w_3 of v to $K(X)$ such that:

$$w_1 < w_2 < w_3.$$

b) If w_1, w_2 are extensions of v to $K(X)$ such that $w_1 < w_2$ then w_1 is an r.a. extension of v and w_2 is an r.t. extension of v .

By Corollary 4.2 it results that the r.a. extensions of v to $K(X)$ are of two kinds according to the following definition: An r.a. extension w' of v to $K(X)$ is called of the *first kind* if there exists an extension w of v to $K(X)$ such that $w' < w$. (This w is necessary an r.t. extension of v to $K(X)$). An r.a. extension w of v to $K(X)$ is called of the *second kind* if it is not of the first kind. At the next point we shall give an explicit description of all r.a. extensions of v to $K(X)$ which are of the first kind.

2. – As usual let v be a valuation on K and let $w = v_{(a, \delta)}$ be an r.t. extension of v to $K(X)$. (As usual $(a, \delta) \in K \times G_v$ is a pair). Let $d \in K$ be such that $v(d) = \delta = w(X - a)$, and let us denote $t = \bar{p}_w((X - a)/d)$. It is plain that t is transcendental over k_v and that k_w is canonically isomorphic to $k_v(t)$. Hence we may write: $k_w = k_v(t)$. Let us consider the commutative diagram:

$$\begin{array}{ccc} A(w) & \xrightarrow{r} & A(v) \\ \psi_w \downarrow & & \downarrow \psi_v \\ V(k_w(t)) & \xrightarrow{r'} & V(k_v) \end{array}$$

where r and r' are the restriction mapping and ψ_w, ψ_v are the mappings defined at par. 3.1. From this diagram it follows that if $v' \in A(v)$ and $w' \in A(w)$, then w' is an extension of v' to $K(X)$ if and only if $\psi_w(w')$ is an extension of $\psi_v(v')$ to $k_v(t)$. If this is the case, then according to Proposition 3.1 it result that w' is an r.t. extension of v' to $K(X)$ if and only if $\psi_w(w')$ is an r.t. extension of $\psi_v(v')$ to $k_v(t)$.

Let $v' \in A(v)$ (hence one has: $v' \leq v$) and let $s: G_{v'} \rightarrow G_v$ be a homomorphism of ordered groups such that $v = sv'$. Let $\psi_v(v') = v'_1 \in V(k_v)$ and let $w'_1 = v'_{1(a'_1, \delta'_1)}$ (where $a'_1 = p_v(a_1)$, $a_1 \in O_v$ and $\delta'_1 \in G_{v'_1} = G_{\psi_v(v')} = \text{Ker } s$) be an r.t. extension of v'_1 to $k_v(t)$. Since ψ_w is a bijective mapping, there exists only a valuation $w' \in A(w)$ such that $\psi_w(w') = w'_1$. An explicit description of the valuation w' is given in the following result:

THEOREM 4.3. One has $w' = v'_{(a + da_1, v'(d) + \delta_1)}$.

PROOF. Let us denote $w' = v'_{(a + da_1, v'(d) + \delta'_1)}$. We shall prove that $w' \leq w$, hence $w' \in A(w)$ and that $\psi_w(w') = w'_1$. Indeed, one has:

$$s(v'(d) + \delta'_1) = s(v'(d)) + s(\delta'_1) = v(d) = \delta$$

and also:

$$v((a + da_1) - a) = v(da_1) = v(d) + v(a_1) \geq v(d) = \delta.$$

Thus, according to the Corollary 2.2 it follows that $w' \leq w$. Let $\psi_w(w') = w'_1$, and let

$$P(t) = p_v(a_0) + p_v(a_1)(t - a'_1) + \dots + p_v(a_n)(t - a'_1)^n$$

be an element of $k_v[t]$, where $a_i \in O_v$, $i = 0, \dots, n$.

Then we may write:

$$\begin{aligned}
 P(t) &= p_v(a_0) + p_v(a_1) p_w \left(\frac{X-a}{d} - a_1 \right) + \dots + p_v(a_n) p_w \left(\frac{X-a}{d} - a_1 \right)^n = \\
 &= p_w \left(a_0 + \frac{a_1}{d} (X - (a + da_1)) + \dots + \frac{a_n}{d^n} (X - (a + da_1))^n \right).
 \end{aligned}$$

Therefore one has:

$$\begin{aligned}
 w_1''(P(t)) &= w' \left(a_0 + \frac{a_1}{d} (X - (a + da_1)) + \dots + \frac{a_n}{d^n} (X - (a + da_1))^n \right) = \\
 &= \inf_i \left(v' \left(\frac{a_i}{d^i} \right) + i(v'(d) + \delta'_1) \right) = \inf_i (v'(a_i) - iv'(d) + iv'(d) + \delta'_1) = \\
 &= \inf_i (v'(a_i) + i\delta'_1) = \inf_i (v_1'(p_v(a_i)) + i\delta'_1) = w_1'(P(t)).
 \end{aligned}$$

These equalities shows that $\psi_w(w') = w_1'$, and so the proof of Theorem 4.3 is finished.

3. - Let, as usual, K be an algebraically closed field and let v be a valuation on K . In the following result are described all r.a. extensions of v to $K(X)$ which are of the first kind.

THEOREM 4.4. Let w' be a r.a. extension of v to $K(X)$ of the first kind. Then one has: $G_{w'} = G_v \times Z$ endowed with lexicographic ordering. Moreover there exists $(a, \delta) \in K \times G_v$ such that, if $P(X) \in K[X]$ is such that:

$$P(X) = a_0 + a_1(X - a) + \dots + a_n(X - a)^n$$

then

$$w'(P(X)) = \inf_i (v(a_i) + i\delta, i)$$

or

$$w'(P(X)) = \inf_i (v(a_i) + i\delta, -i).$$

PROOF. According to Proposition 4.1 a) there exists only an r.t. extension $w = v_{(a, \delta)}$ of v to $K(X)$ such that $w' < w$. Let $d \in K$ be such that $w(X - a) = v(d) = \delta$ and let $t = p_w((X - a)/d) \in k_w$. One knows that t is transcendental over k_v and $k_w = k_v(t)$ (see [2], Theorem 2.1). Let us con-

sider the mapping

$$\psi'_w: A'(w) \rightarrow V'(k_v(t))$$

(defined in Section 3) and let $w'_1 = \psi'_w(w')$. The valuation w'_1 of $k_v(t)$ is trivial on k_v , and so is defined by a polynomial $t - a_1$, $a_1 \in k_v$ or coincides with the valuation at infinity, defined by $1/t$.

Let us assume that w'_1 is defined by the polynomial $t - a_1$. Then $a_1 = p_v(b)$, $b \in O_v$ and so

$$t - a_1 = p_w\left(\frac{X - a}{d}\right) - p_v(b) = p_w\left(\frac{X - (a + bd)}{d}\right).$$

Now since $v(a + bd - a) = v(bd) = v(b) + v(d) \geq v(d) = \delta$, then $w = v_{(a, \delta)} = v_{(a + ad, \delta)}$. Hence we may substitute a to $a + bd$ and $t - a$, by t . Let:

$$f(X) = a_0 + a_1(X - a) + \dots + a_n(X - a)^n$$

be an element of $K[X]$. Then one has:

$$w(f(X)) = \inf_i (v(a_i) + i) = \inf_i (v(a_i) + iv(d)) = \inf_i v(a_i d^i) = v(a_{i_0} d^{i_0})$$

where i_0 is the smallest one for which this equality is verified. Then for every i , $0 \leq i \leq n$ one has:

$$v(a_i d^i) \geq v(a_{i_0} d^{i_0})$$

and

$$v(a_i d^i) > v(a_{i_0} d^{i_0}) \quad \text{if } i < i_0.$$

By these inequalities it follows that $a_i d^i / a_{i_0} d^{i_0} \in O_v$ for all i , and $p_v(a_i d^i / a_{i_0} d^{i_0}) = 0$ if $i < i_0$. Hence one has:

$$\begin{aligned} p_w\left(\frac{f(X)}{a_{i_0} d^{i_0}}\right) &= p_w\left(\frac{a_0 d^0}{a_{i_0} d^{i_0}} + \frac{a_1 d^1}{a_{i_0} d^{i_0}} \cdot \frac{X - a}{d} + \dots + \frac{a_n d^n}{a_{i_0} d^{i_0}} \left(\frac{X - a}{d}\right)^n\right) = \\ &= p_v\left(\frac{a_0 d^0}{a_{i_0} d^{i_0}}\right) + p_v\left(\frac{a_1 d}{a_{i_0} d^{i_0}}\right)t + \dots + \left(\frac{a_n d^n}{a_{i_0} d^{i_0}}\right)t^n = \\ &= t^{i_0} + p_v\left(\frac{a_{i_0+1} d^{i_0+1}}{a_{i_0} d^{i_0}}\right)t^{i_0+1} + \dots + p_v\left(\frac{a_n d^n}{a_{i_0} d^{i_0}}\right)t^n. \end{aligned}$$

These equalities show that

$$w_1' \left(p_w \left(\frac{f(X)}{a_{i_0} d^{i_0}} \right) \right) = i_0.$$

Therefore by Theorem 3.3, one has: $G_{w'} = G_v \times Z$ and:

$$\begin{aligned} w'(f(X)) &= \left(w(f(X)), w_1' \left(p_w \left(\frac{f(X)}{a_{i_0} d^{i_0}} \right) \right) \right) = \\ &= (\inf(v(a_i) + i\delta), i_0) = \inf(v(a_i) + i\delta, i). \end{aligned}$$

The last equality result since if one has: $v(a_{i_0}) + i_0\delta = v(a_i) + i\delta$, then $i_0 < i$.

Let us assume that w_1' is defined by $1/t$. Then choose i_0 the greatest index such that:

$$w(f(x)) = \inf_i (v(a_i d^i)) = v(a_{i_0} d^{i_0}).$$

Then $a_i d^i / a_{i_0} d^{i_0} \in O_v$ for all i , $0 \leq i \leq n$, and if $i > i_0$ then $p_v(a_i d^i / a_{i_0} d^{i_0}) = 0$. As above one check that:

$$p_w \left(\frac{f(X)}{a_{i_0} d^{i_0}} \right) = p_w \left(\frac{a_0 d^0}{a_{i_0} d^{i_0}} \right) + p_w \left(\frac{a_1 d}{a_{i_0} d^{i_0}} \right) t + \dots + p_w \left(\frac{a_{i_0-1} d^{i_0-1}}{a_{i_0} d^{i_0}} \right) t^{i_0-1} + t^{i_0}.$$

Therefore one has:

$$w_1' \left(p_w \left(\frac{f(X)}{a_{i_0} d^{i_0}} \right) \right) = -i_0.$$

Also, as above one has $G_{w'} \in G_v \times Z$ and:

$$\begin{aligned} w'(f(X)) &= \left(w(f(X)), w_1' \left(p_w \left(\frac{f(X)}{a_{i_0} d^{i_0}} \right) \right) \right) = \\ &= \left(\inf_i (v(a_i) + i\delta), -i_0 \right) = \inf_i (v(a_i) + i\delta, -i). \end{aligned}$$

The proof of Theorem 4.4 is complete.

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