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In [1] we were concerned with groups G such that $XY = YX$ for all n-sets X and Y in G, the P_n-groups of that paper. Theorem B stated that all infinite P_n-groups are abelian, but of course some finite P_n-groups are non-abelian. Our aim here is to establish the best possible result in this direction:

THEOREM 1. (i) Every group in P_n of order at least $2n$ is abelian.

(ii) For each t, every group of order t is in P_n whenever $n > t/2$.

The same sort of questions can be asked about other algebraic structures than groups. To show how different semigroups are in this context, we prove, with the obvious definitions:

THEOREM 2. A semigroup S with identity is a non-commutative P_2-semigroup if and only if S is the disjoint union $S = A \cup B$, where

(i) $|A| = 2$ and A is a left zero or right zero semigroup,

(ii) B is a commutative subsemigroup containing the identity 1 of S,

(iii) $xy = yx = x$ for all x in A, y in B.

PROOFS. To prove Theorem 1, we first establish the simple fact that $P_n \leq P_{n+1}$ for all n. Let A be a group (or indeed a semigroup) in P_n, and X, Y any subsets of cardinal $n+1$ of A. Let xy be any element in the product, with $x \in X$, $y \in Y$. Then x is in an n-set X_1 contained in

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X, and similarly y is contained in an n-set Y_1 in Y. Thus $xy \in X_1 Y_1 = Y_1 X_1 \leqslant YX$, so that $XY \leqslant YX$. The converse is obvious, and we have $XY = YX$. Because of this, all we need do now is show that groups of order $2n$ and $2n+1$ in P_n are abelian.

We shall do the case $|G| = 2n$ as an example. Let G be a non-abelian group of order $2n$, and let x_1, x_2 be non-commuting elements of G. Further, let $X = \{x_1, x_2, \ldots x_n\}$ be any n-set chosen in such a way that $x_r^{x_1} \neq x_2$ for each $r = 1, 2, \ldots, n$. The choice is possible since $|G| \geqslant 2n$. Finally, set

$$Y = G \setminus \{1, x_2^{-1} x_1, \ldots, x_n^{-1} x_1\} = \{y_1, y_2, \ldots, y_n\}, \text{ say}.$$

To establish the theorem, it is enough to show that $XY \neq YX$.

Clearly, $x_1 \notin XY$ by the choice of Y. We shall show that $x_1 \in YX$. We have

$$x_1 \in YX \Leftrightarrow \exists i, j \quad \text{with } y_i = x_1 x_j^{-1},$$

so that

$$\begin{aligned} x_1 \notin YX &\Leftrightarrow \forall i, j\colon x_1 x_j^{-1} \neq y_i \\ &\Leftrightarrow \forall j\colon x_1 x_j^{-1} \notin Y \\ &\Leftrightarrow \forall j\, \exists r\colon x_1 x_j^{-1} = x_r^{-1} x_1 \\ &\Leftrightarrow \forall j\, \exists r\colon x_j = x_r^{x_1}. \end{aligned}$$

Thus

$$x_1 \in XY \Leftrightarrow \exists j\, \forall r\colon \overset{x_1}{x_r} \neq x_j.$$

However $x_2^{x_1} \neq x_j$ for all j, so $x_1 \in YX$, as required.

The proof for $|G| = 2n+1$ is more-or-less identical: just take $Y = G \setminus \{1, x_2^{-1} x_1, \ldots, x_n^{-1} x_1, x_1\}$ instead.

For part (ii) of the theorem, take G of order t, $n > t/2$ and two n-sets X, Y in G. Then $XY = G$ since for all g in G, $gY^{-1} \cap X \neq \emptyset$ so that $gy^{-1} = x$ for suitable $y \in Y$, $x \in X$, and $g = xy$. Similarly, $G = YX$, so that $G \in P_n$, as required, and this completes the proof of Theorem 1.

As for Theorem 2, it is a matter of routine verification to show that semigroups with the structure indicated in the statement are P_2-semigroups.

To prove the converse, let S be a non-commutative P_2-semigroup with identity 1, and a, b two non-commuting elements of S. We show first that $\{a, b\}$ is a left or right zero semigroup. Since S is in P_2,

$$\{1, a\}\{1, b\} = \{1, b\}\{1, a\},$$

so

$$\{1, a, b, ab\} = \{1, b, a, ba\}.$$

There are three cases to consider. If $ab = 1$, we must have $ba = a$ or $ba = b$, since $ab \neq ba$. If $ba = a$, then $b = b(ab) = (ba)b = ab = 1$, a contradiction; while if $ba = b$, we have $1 = ab = a(ba) = (ab)a = 1 \cdot a = a$, another contradiction. Thus $ab \neq 1$, and, symmetrically, $ba \neq 1$.

If $ab = a$, then

$$\{1, a, b\} = \{1, a, b, ba\}$$

so that $ba = b$ since $ab \neq ba$. Then

$$a^2 = (ab)a = a(ba) = ab = a,$$

$$b^2 = (ba)b = b(ab) = ba = b,$$

and $\{a, b\}$ forms a left zero semigroup.

Finally, if $ab = b$, the same sort of argument shows that $A := \{a, b\}$ is a left zero semigroup.

For the remainder of the proof we shall assume without loss that A is a left zero semigroup, that is, $a^2 = ab = a$, $b^2 = ba = b$. We show first that every element c outside A must commute with a or b. If not, $\{a, c\}$ and $\{b, c\}$ are both left or right zero semigroups, and thus there are four cases to consider.

1) $c^2 = c = ca$, $ac = a$, $cb = c$, $bc = b$.

This is impossible, since

$$\{a, b\}\{1, c\} = \{a, b\},$$

$$\{1, c\}\{a, b\} = \{a, b, c\}.$$

2) $c^2 = c = ca$, $ac = a$, $bc = c$, $cb = b$.

Here

$$\{a, b\}\{b, c\} = \{a, b, c\},$$

$$\{b, c\}\{a, b\} = \{b, c\}.$$

3) $c^2 = c = ac$, $ca = a$, $bc = b$, $cb = c$.

Here

$$\{a, b\}\{a, c\} = \{a, b, c\},$$
$$\{a, c\}\{a, b\} = \{a, c\}.$$

4) Finally in this part of the argument, $c^2 = c = ac$, $ca = a$, $bc = b$, $cb = b$. Here

$$\{a, b\}\{a, c\} = \{a, b, c\},$$
$$\{a, c\}\{a, b\} = \{a, c\}.$$

Thus c must commute with a or b, and it is no loss of generality if we assume that $ac = ca$. If c does not commute with b, we have two cases to consider, depending on the structure of $\{b, c\}$. Recall that $a^2 = ab = a$, $b^2 = ba = b$.

1) $bc = b$, $c^2 = cb = c$.

Hence

$$\{a, b\}\{a, c\} = \{a, ac, b\},$$
$$\{a, c\}\{a, b\} = \{a, ac, c\}.$$

Thus $b = ac$ and $c = ac$, which is a contradiction.

2) $bc = c = c^2$, $cb = b$.

Here

$$\{a, b\}\{a, c\} = \{a, ac, b, c\},$$
$$\{a, c\}\{a, b\} = \{a, ac, b\}.$$

Thus $c = ac$ and we have

$$\{a, b\}\{b, c\} = \{a, c, b\},$$
$$\{b, c\}\{a, b\} = \{b, c\}.$$

which is false.

Hence, thus far we have shown that $c \notin \{a, b\} \Rightarrow ac = ca$, $bc = cb$. Consider the following product:

$$\{a, b\}\{a, c\} = \{a, b, ac, bc\},$$
$$\{a, c\}\{a, b\} = \{a, ac, bc\}.$$

This gives that $b = bc$, since the other possibility, *viz.* $b = ac$, means that $ab = ba$ since a commutes with c. Similarly, $a = ac$. This completes item (iii).

The next step is to show that $cd = dc$ for all c, d outside $\{a, b\}$. This is clear, since otherwise $\{c, d\}$ can play the part of $\{a, b\}$ in the argument to this point, and we would get $c = ac = ca = a$.

The final step is to show that $c, d \notin \{a, b\} \Rightarrow cd \notin \{a, b\}$. Suppose, without loss of generality, that $cd = a$. Then $(cd)b = ab = a$, $c(db) = cb = b$, a contradiction which completes the proof.

We have no idea what happens with P_3-semigroups with identity, nor with semigroups without identity. The arguments here are likely to be very cumbersome.

REFERENCES

[1] J. C. LENNOX - A. MOHAMMADI HASSANABADI - J. WIEGOLD, *Some commutativity criteria*, Rend. Sem. Mat. Univ. Padova, 84 (1990).

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