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Some Commutativity Criteria. - II

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In [1] we were concerned with groups $G$ such that $XY = YX$ for all $n$-sets $X$ and $Y$ in $G$, the $P_n$-groups of that paper. Theorem B stated that all infinite $P_n$-groups are abelian, but of course some finite $P_n$-groups are non-abelian. Our aim here is to establish the best possible result in this direction:

**THEOREM 1.** (i) Every group in $P_n$ of order at least $2n$ is abelian.
(ii) For each $t$, every group of order $t$ is in $P_n$ whenever $n > t/2$.

The same sort of questions can be asked about other algebraic structures than groups. To show how different semigroups are in this context, we prove, with the obvious definitions:

**THEOREM 2.** A semigroup $S$ with identity is a non-commutative $P_2$-semigroup if and only if $S$ is the disjoint union $S = A \cup B$, where
(i) $|A| = 2$ and $A$ is a left zero or right zero semigroup,
(ii) $B$ is a commutative subsemigroup containing the identity $1$ of $S$,
(iii) $xy = yx = x$ for all $x \in A$, $y \in B$.

**PROOFS.** To prove Theorem 1, we first establish the simple fact that $P_n \lneq P_{n+1}$ for all $n$. Let $A$ be a group (or indeed a semigroup) in $P_n$, and $X$, $Y$ any subsets of cardinal $n + 1$ of $A$. Let $xy$ be any element in the product, with $x \in X$, $y \in Y$. Then $x$ is in an $n$-set $X_1$ contained in

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X, and similarly y is contained in an n-set Y in Y. Thus xy ∈ X_1 Y_1 = Y_1 X_1 ≤ YX, so that XY ≤ YX. The converse is obvious, and we have XY = YX. Because of this, all we need do now is show that groups of order 2n and 2n + 1 in \( P_n \) are abelian.

We shall do the case \(|G| = 2n\) as an example. Let G be a non-abelian group of order 2n, and let \( x_1, x_2 \) be non-commuting elements of G. Further, let \( X = \{x_1, x_2, \ldots, x_n\} \) be any n-set chosen in such a way that \( x_r x_1 \neq x_2 \) for each \( r = 1, 2, \ldots, n \). The choice is possible since \(|G| ≥ 2n\). Finally, set

\[
Y = G \setminus \{1, x_2^{-1} x_1, \ldots, x_n^{-1} x_1\} = \{y_1, y_2, \ldots, y_n\}, \text{ say}.
\]

To establish the theorem, it is enough to show that \( XY \neq YX \).

Clearly, \( x_1 \notin XY \) by the choice of Y. We shall show that \( x_1 \in YX \). We have

\[
x_1 \in YX \iff \exists i, j \quad \text{with } y_i = x_1 x_j^{-1},
\]

so that

\[
x_1 \notin YX \iff \forall i, j: x_1 x_j^{-1} \neq y_i
\]

\[
\iff \forall j: x_1 x_j^{-1} \notin Y
\]

\[
\iff \forall j \exists r: x_1 x_j^{-1} = x_r^{-1} x_1
\]

\[
\iff \forall j \exists r: x_j = x_r^{x_1}.
\]

Thus

\[
x_1 \in XY \iff \exists j \forall r: x_r^{x_1} \neq x_j.
\]

However \( x_2^{x_1} \neq x_j \) for all \( j \), so \( x_1 \in YX \), as required.

The proof for \(|G| = 2n + 1\) is more-or-less identical: just take \( Y = G \setminus \{1, x_2^{-1} x_1, \ldots, x_n^{-1} x_1, x_1\} \) instead.

For part (ii) of the theorem, take G of order \( t, n > t/2 \) and two n-sets \( X, Y \) in G. Then \( XY = G \) since for all \( g \) in G, \( gY^{-1} \cap X \neq \emptyset \) so that \( gy^{-1} = x \) for suitable \( y \in Y, x \in X \), and \( g = xy \). Similarly, \( G = YX \), so that \( G \in \mathcal{P}_n \), as required, and this completes the proof of Theorem 1.

As for Theorem 2, it is a matter of routine verification to show that semigroups with the structure indicated in the statement are \( P_2 \)-semigroups.

To prove the converse, let S be a non-commutative \( P_2 \)-semigroup with identity 1, and \( a, b \) two non-commuting elements of S. We show
first that \( \{a, b\} \) is a left or right zero semigroup. Since \( S \) is in \( P_2 \),
\[
\{1, a\} \{1, b\} = \{1, b\} \{1, a\},
\]
so
\[
\{1, a, b, ab\} = \{1, b, a, ba\}.
\]
There are three cases to consider. If \( ab = 1 \), we must have \( ba = a \) or \( ba = b \), since \( ab \neq ba \). If \( ba = a \), then \( b = b(ab) = (ba)b = ab = 1 \), a contradiction; while if \( ba = b \), we have \( 1 = ab = a(ba) = (ab)a = 1 \cdot a = a \), another contradiction. Thus \( ab \neq 1 \), and, symmetrically, \( ba \neq 1 \).

If \( ab = a \), then
\[
\{1, a, b\} = \{1, a, b, ba\}
\]
so that \( ba = b \) since \( ab \neq ba \). Then
\[
a^2 = (ab)a = a(ba) = ab = a,
\]
\[
b^2 = (ba)b = b(ab) = ba = b,
\]
and \( \{a, b\} \) forms a left zero semigroup.

Finally, if \( ab = b \), the same sort of argument shows that \( A := \{a, b\} \) is a left zero semigroup.

For the remainder of the proof we shall assume without loss that \( A \) is a left zero semigroup, that is, \( a^2 = ab = a, b^2 = ba = b \). We show first that every element \( c \) outside \( A \) must commute with \( a \) or \( b \). If not, \( \{a, c\} \) and \( \{b, c\} \) are both left or right zero semigroups, and thus there are four cases to consider.

1) \( c^2 = c = ca, ac = a, cb = c, bc = b \).

This is impossible, since
\[
\{a, b\} \{1, c\} = \{a, b\},
\]
\[
\{1, c\} \{a, b\} = \{a, b, c\}.
\]

2) \( c^2 = c = ca, ac = a, bc = c, cb = b \).

Here
\[
\{a, b\} \{b, c\} = \{a, b, c\},
\]
\[
\{b, c\} \{a, b\} = \{b, c\}.
\]

3) \( c^2 = c = ac, ca = a, bc = b, cb = c \).
Here
\[ \{a, b\}\{a, c\} = \{a, b, c\} , \]
\[ \{a, c\}\{a, b\} = \{a, c\} . \]

4) Finally in this part of the argument, \( c^2 = c = ac \), \( ca = a \), \( bc = b \), \( cb = b \). Here
\[ \{a, b\}\{a, c\} = \{a, b, c\} , \]
\[ \{a, c\}\{a, b\} = \{a, c\} . \]
Thus \( c \) must commute with \( a \) or \( b \), and it is no loss of generality if we assume that \( ac = ca \). If \( c \) does not commute with \( b \), we have two cases to consider, depending on the structure of \( \{b, c\} \). Recall that \( a^2 = ab = a \), \( b^2 = ba = b \).

1) \( bc = b \), \( c^2 = cb = c \).
Hence
\[ \{a, b\}\{a, c\} = \{a, ac, b\} , \]
\[ \{a, c\}\{a, b\} = \{a, ac, c\} . \]
Thus \( b = ac \) and \( c = ac \), which is a contradiction.

2) \( bc = c = c^2 \), \( cb = b \).
Here
\[ \{a, b\}\{a, c\} = \{a, ac, b, c\} , \]
\[ \{a, c\}\{a, b\} = \{a, ac, b\} . \]
Thus \( c = ac \) and we have
\[ \{a, b\}\{b, c\} = \{a, c, b\} , \]
\[ \{b, c\}\{a, b\} = \{b, c\} . \]
which is false.
Hence, thus far we have shown that \( c \notin \{a, b\} \Rightarrow ac = ca \), \( bc = cb \). Consider the following product:
\[ \{a, b\}\{a, c\} = \{a, b, ac, bc\} , \]
\[ \{a, c\}\{a, b\} = \{a, ac, bc\} . \]
This gives that \( b = bc \), since the other possibility, \( viz. \ b = ac \), means
that \( ab = ba \) since \( a \) commutes with \( c \). Similarly, \( a = ac \). This completes item (iii).

The next step is to show that \( cd = dc \) for all \( c, d \) outside \( \{a, b\} \). This is clear, since otherwise \( \{c, d\} \) can play the part of \( \{a, b\} \) in the argument to this point, and we would get \( c = ac = ca = a \).

The final step is to show that \( c, d \notin \{a, b\} \Rightarrow cd \notin \{a, b\} \). Suppose, without loss of generality, that \( cd = a \). Then \( (cd)b = ab = a \), \( c(db) = cb = b \), a contradiction which completes the proof.

We have no idea what happens with \( P_3 \)-semigroups with identity, nor with semigroups without identity. The arguments here are likely to be very cumbersome.

REFERENCES


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