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The Equation \( y' = fy \) in \( C_p \) when \( f \) is Quasi-Invertible.

ALAIN ESCASSUT (*)

SUMMARY - Let \( K \) be a complete algebraically closed extension of \( C_p \). Let \( D \) be a clopen bounded infraconnected set in \( K \), let \( H(D) \) be the Banach algebra of the analytic elements on \( D \), let \( f \in H(D) \) and let \( \mathcal{S}(f) \) be the space of the solutions of the equation \( y' = fy \) in \( H(D) \). We construct such a set \( D \) provided with a \( T \)-filter \( \mathcal{F} \) such that there exists a quasi-invertible \( f \in H(D) \) such that \( \mathcal{S}(f) \) has non zero elements \( g \) which approach zero along \( \mathcal{F} \). In extending this construction we show that for every \( t \in \mathbb{N} \), we can make a set \( D \) and an \( f \in H(D) \) such that \( \mathcal{S}(f) \) has dimension \( t \). That answers questions suggested in previous articles.

I. Introduction and theorems.

Let \( K \) be an ultrametric complete algebraically closed field, of characteristic zero and residue characteristic \( p \neq 0 \).

Let \( D \) be an infraconnected bounded clopen set in \( K \) and let \( H(D) \) be the Banach algebra of the Analytic Elements on \( D \) (i.e., \( H(D) \) is the completion of the algebra \( R(D) \) for the uniform convergence norm on \( D \)) \([E_1, E_2, E_3, K_1, K_2, R]\).

Recall that a set \( D \) in \( K \) is said to be infraconnected if for every \( a \in D \) the mapping \( x \mapsto |x - a| \) has an image whose adherence in \( R \) is an interval; then \( H(D) \) has no idempotent different from 0 and 1 is and only if \( D \) is infraconnected \([E_2]\). On the other hand, an open set \( D \) is infraconnected if and only if \( f' = 0 \) implies \( f = ct \) for every \( f \in H(D) \)[E_6]. Let \( f \in H(D) \); we denote by \( \mathcal{S}(f) \) the differential equation \( y' = fy \) (where \( y \in H(D) \)) and by \( \mathcal{S}(f) \) the space of the solutions of \( \mathcal{S}(f) \).

In \([E_7]\) we saw that \( \mathcal{S}(f) \) has dimension 1 as soon as it contains

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a $g$ invertible in $H(D)$. If $H(D)$ has no divisor of zero, $S(f)$ doesn’t have dimension greater than one.

In [E$_8$] we saw that if the residue characteristic of $K$ is zero, then $S(f)$ never has dimension greater than one.

But when the residue characteristic $p$ is different from zero, in [E$_9$] we saw that there does exist infraconnected clopen bounded sets with a $T$-filter $\mathcal{F}[E_4]$ and an element $f$ annulled by $\mathcal{F}$ such that the solutions of $S(f)$ are also annulled by $\mathcal{F}$. Thanks to such $T$-filters, for every $n \in \mathbb{N}$ we could construct infraconnected clopen bounded sets $D$ with $f \in H(D)$ such that $S(f)$ has dimension $n$, and we even constructed sets $D$ with $f \in H(D)$ such that $S(f)$ is isomorphic to the space of the sequences of limit zero.

Thus [E$_8$] suggested that a situation where the solutions of $S(f)$ were not invertible in $H(D)$ should be associated to a non quasi-invertible element $f$, and so should be spaces $S(f)$ of dimension greater than one.

(Recall that $f$ is said to be quasi-invertible in $H(D)$ if it factorizes in the form $P(x)g(x)$ where $P$ is a polynomial the zeros of which are in $D$ and $g$ is an invertible element of $H(D))[E_1, E_2, E_3, E_4].$

Here we will prove this connection does not hold in constructing an infraconnected clopen bounded set $D$ with a $T$-filter $\mathcal{F}$ and a quasi-invertible element $f \in H(D)$ such that $S(f)$ has solutions strictly annulled by $\mathcal{F}$.

Next, for every fixed integer $t$, an extension of that construction will provide us with a set $D$ and a quasi-invertible $f \in H(D)$ such that $\dim S(f) = t$.

**THEOREM 1.** There exist an infraconnected clopen bounded set $D$ with a $T$-filter $\mathcal{F}$ and quasi-invertible elements $f \in H(D)$ such that $S(f)$ has solutions strictly annulled by $\mathcal{F}$ and $S(f)$ has dimension 1.

More precisely, we will concretely construct such a set $D$ and $f \in H(D)$ in Proposition B.

**THEOREM 2.** Let $t \in \mathbb{N}$. There exist an infraconnected clopen bounded set $D$ and quasi-invertible elements $f \in H(D)$ such that $\dim (S(f)) = t$.

Theorem 2 will also be proven by a concrete construction.

**REMARK.** We are not able to construct an infraconnected clopen bounded set $D$ with a quasi-invertible $f \in H(D)$ such that $S(f)$ has infinite dimension. By then, the following conjecture seems to be likely.
The equation $y' = fy$ in $C_p$ when $f$ is quasi-invertible

**Conjecture.** If $f$ is quasi-invertible, $S(f)$ has finite dimension.

The following Proposition A will demonstrate Theorem 1 by showing how to obtain the set $D$, the $T$-filter $\mathcal{T}$, and the element $f$.

**Proposition A.** Let $(b_m)_{m \in \mathbb{N}}$ be a sequence in $d^-(0, 1)$ such that $|b_m| < |b_{m+1}|$, and let $(p_m)_{m \in \mathbb{N}}$ be a sequence of integers in the form $p^{q_m}$ where $q_m$ is a sequence of integers satisfying

\begin{align*}
(1) & \quad \lim_{m \to \infty} q_m = +\infty, \\
(2) & \quad |p_1| > |p_m| \quad \text{whenever } m \geq 2, \\
(3) & \quad \lim_{m \to \infty} \frac{b_m}{b_{m+1}}^{p_{m+1}} = 0.
\end{align*}

Let $R$ be $\geq 1$, and let $D = d(0, R) \setminus \bigcup_{m = 1}^{\infty} d^-(b_m, |b_m|)$. For each $m \in \mathbb{N}^*$ let

$$h_m = \prod_{j=1}^{m} \frac{1}{(1 - x/b_j)^p} \in R(D).$$

Then the sequence $(h_m)$ converges in $H(D)$ to a limit $h$ that is strictly annulled by the increasing $T$-filter $\mathcal{T}$ of center 0 of diameter 1, and $h \in S(\mathcal{T})$.

The series $\sum_{m=1}^{\infty} p_j/(b_m - x)$ converges in $H(D)$ to a limit $f$ quasi-invertible in $H(D)$ and $h$ is a solution of $S(f)$.

**II. The proof of Proposition A**

The proof of proposition will use the following Lemma B.

**Lemma B.** Let $q$ and $n$ be two integers such that $C < n \leq p^q$. Then $|C_{p^q}^n| \leq p^{-q}/|n|$.

**Proof.** If $n$ is a multiple of some $p^k$, then $p^q - n$ is obviously multiple of $p^k$. Let $b$ be the bijection from $\{1, \ldots, n\}$ onto $\{p^q - n + 1, \ldots, p^q\}$ defined by $b(j) = p^q - j + 1$. By the last sentence, when $j$ is divisible by $p^k$, $b(j + 1)$ is also divisible by $p^k$ hence $|b(j + 1)| \leq |j|$ therefore $|(p_{q-1}(p_{q-2}) \ldots (p^{q-n+1})| \leq |(n - 1)!|$ and finally $|C_{p^q}^n| \leq p^{-q}/|n|$. 
PROOF OF PROPOSITION A. Since we have \( \lim_{m \to \infty} |b_m / b_{m+1}|^{p_{m+1}} = 0 \) we have
\[
\lim_{m \to \infty} (p_{m+1} \log |b_{m+1}/b_m|) = +\infty.
\]
Thus we can easily define a sequence of integers \( l_m \) such that \( \lim_{m \to \infty} (q_m - l_m) = +\infty \) and \( \lim_{m \to \infty} (p_{m+1} \log |b_{m+1}/b_m|) = +\infty. \)
We put \( t_m = p_{l_m}, \quad \omega_m = |p_m/t_m|, \quad \epsilon_m = |b_{m-1}/b_m|^t_m. \)
Then we have
\[
\lim_{m \to \infty} \omega_m = \lim_{m \to \infty} \epsilon_m = 0.
\]
As the holes of \( D \) are in the form \( d^-(b_m,|b_m|) \) it is easily seen that
\[
\left| \frac{1}{1 - x/b_j} \right|_D \leq 1.
\]
Let us consider \( |h_{m+1}(x) - h_m(x)| \) when \( |x| \geq |b_m| \). We have
\[
|h_m(x)| \leq \prod_{j=1}^{m-1} \frac{1}{|1 - x/b_j|^{p_j}} \leq \epsilon_m
\]
and in the same way \( |h_{m+1}(x)| \leq \epsilon_m \) hence
\[
|h_{m+1}(x) - h_m(x)| \leq \epsilon_m.
\]
Now let us consider \( h_{m+1}(x) - h_m(x) \) when \( |x| < |b_m| \) and let us put
\[
u(x) = \frac{1}{(1 - x/b_{m+1})^{p_{m+1}}} - 1 = -\sum_{j=1}^{p_{m+1}} \left( \begin{array}{c} p_{m+1} \\ j \end{array} \right) \left( \frac{x}{b_{m+1}} \right)^j.
\]
Then it is clear that \( |\nu(x)| \leq \max_{1 \leq j \leq p_m} \left| \left( \begin{array}{c} p_{m+1} \\ j \end{array} \right) \right| \cdot \left| \frac{b_m}{b_{m+1}} \right|^j \) and then for
\( 1 \leq j \leq t_{m+1} \), as \( |j| \geq |t_{m+1}|, \) we obtain
\[
\left| \left( \begin{array}{c} p_{m+1} \\ j \end{array} \right) \right| \leq \left| \frac{p_{m+1}}{t_{m+1}} \right| \] by Lemma B.
Now for \( j > t_{m+1} \) we see that
\[
\left| \frac{b_m}{b_{m+1}} \right|^j \leq \left| \frac{b_m}{b_{m+1}} \right|^{t_{m+1}} = \epsilon_m \text{ and then}
\]
every term \( \left( \begin{array}{c} p_{m+1} \\ j \end{array} \right) \left( - \frac{x}{b_{m+1}} \right)^j \) is upper bounded by \( \max(\omega_{m+1},\epsilon_m) \) and therefore \( |\nu(x)| \leq \max(\omega_{m+1},\epsilon_m) \) whenever \( x \in D \cap d(0,|b_m|) \).
Finally by (6) we see that \(\|h_{m+1} - h_m\|_D \leq \max(\omega_{m+1}, \varepsilon_m)\) hence the sequence \(h_m\) converges in \(H(D)\) to the convergent infinite product

\[
h(x) = \prod_{j=1}^{\infty} \frac{1}{(1 - x/b_j)^{p_j}}.
\]

By (3) and by the definition of \(D\) it is easily seen that the increasing filter \(\mathcal{F}\) of center 0, of diameter 1, is a T-filter and it is the only one T-filter on \(D[E_4]\).

On the other hand, by (5) we have \(|h(x)| \leq \varepsilon_m\) whenever \(x \in D \setminus \mathcal{D} - (0, |b_m|)\) and therefore \(h\) is clearly annihilated by \(\mathcal{F}\), and it is strictly annihilated by \(\mathcal{F}\) (because \(\mathcal{F}\) is the only T-filter on \(D\)), and \(h(x) = 0\) whenever \(x \in \mathcal{P}(\mathcal{F})\) hence \(h \in \mathcal{J}_0(\mathcal{F})\).

Now let us consider the series \(\sum_{j=1}^{\infty} p_j/(b_j - x)\). Since \(\lim_{m \to \infty} |p^m| = 0\), by (4) we see that series converge to a limit \(f \in H(D)\). Moreover, it is easily seen that \(\lim_{|x| \to 1^-} \sup_{x \in D} |p_j/(b_j - x)| = |p_j|\) for every \(j \in \mathbb{N}^*\), hence, by (2),

we have \(\sup_{x \to 1^-} |f(x)| = p_1\), hence \(f\) is not annihilated by \(\mathcal{F}\).

Since \(\mathcal{F}\) is the only T-filter, \(f\) is then quasi-invertible.

At last, we shortly verify that \(h\) is solution of \(\delta(f)\).

By Corollary of [E6] we know that \(h' \in H(D)\) and the sequence \(h_m'\) converges to \(h'\) in \(H(D)'\). On the other hand, it is easily seen that

\[
h_m' = \left(\sum_{j=1}^{m} \frac{p_j}{(1 - x/b_j)^{p_j}}\right)h_m = \left(\sum_{j=1}^{m} \frac{p_j}{b_j - x}\right)h_m
\]

hence

\[
\lim_{m \to \infty} h_m' = h \left(\sum_{j=1}^{\infty} \frac{p_j}{b_j - x}\right) = hf
\]

and therefore \(h\) is a solution of \(\delta(f)\), and that ends the proof of Proposition A.

III. The proof of Theorem 2.

**Lemma C.** Let \(q, n\) be integers such that \(0 < n < q\). Then \(|q! / n!| \leq p^{1-(q-n)/p}\).
PROOF. \( q!/n! \) has \( q - n \) consecutive factors. It is easily seen among these \( q - n \) factors, the number of them that are multiple of \( p \), is at least \( \text{Int}((q - n)/p) \) and therefore \( v(q!/n!) \geq \text{Int}((q - n)/p) > \frac{(q - n)}{p} - 1 \) and that ends the proof of Lemma C.

LEMMA D. Let \( R \in [p^{-1/p}, 1[ \), let \( \epsilon \in ]0, 1/p[ \) and let \( \varphi(x) = \sum_{n=0}^{+\infty} a_n x^n \) be a Laurent series convergent for \( |x| = R \), such that \( \sup |a_n| R^n = |a_q| R^q \) with \( q < 0 \). Then \( \varphi \) does not satisfy the inequality

\[
\left| \frac{\varphi'(x)}{\varphi(x)} - 1 \right| < \epsilon \quad \text{for all } x \in C(0, R).
\]

PROOF. We suppose \( \varphi \) satisfies (1) and we put \( M = |a_q| R^q \). By (1) it is easily seen that

\[
|na_n - a_{n-1}| R^{n-1} \leq \epsilon M \quad \text{for every } n \in \mathbb{Z}.
\]

If \( q = -1 \), relation (2) gives \( |a_{-1}|/R \leq \epsilon |a_{-1}|/R \) hence \( \varphi = 0 \). We will suppose \( q < -1 \) and we will prove that (3) \( |a_n| = |a_q (-n - 1)!|/|(-q - 1)!| \) for \( n = q + 1, q + 2, \ldots, -2, -1 \). Indeed, suppose it has been proven up to the range \( t \) with \( q \leq t < -1 \) and let us prove it at the range \( t + 1 \). By (2) we have

\[
|t+1)a_{t+1} - a_t| R^t \leq \epsilon |a_q| R^q \quad \text{hence} \quad |(t + 1)a_{t+1} - a_t| \leq \frac{\epsilon |a_q|}{R^{t-q}}
\]

hence by (3)

\[
|(t+1)a_{t+1} - a_t| \leq \frac{\epsilon |a_t| |(-q - 1)!|}{R^{t-q} |(-t - 1)!|}.
\]

Now by Lemma C we know that \( |(-q)!/(-t)!| \leq p^{1-\frac{q}{p}} \). Since \( R \geq p^{-1/p} \), we see that \( R^{t-q} \geq p^{(-q - 1)!}/p \); hence \( |(-q)!/(-t)!| \leq R^{t-q} \) and therefore \( \epsilon |(-q)!/(-t)!| \leq R^{t-q} \). Then by relation (4) we have

\[
|(t+1)a_{t+1} - a_t| < |a_t| \quad \text{hence} \quad |(t + 1)a_{t+1} - a_t| = |a_t|,
\]

and therefore

\[
|a_{t+1}| = \left| \frac{a_t}{t+1} \right| = \frac{|a_q| (-t - 2)!}{|(-t - 1)!|}
\]

so that relation (3) is proven at the range \( t + 1 \). It is then proven for every \( n \) up to \( -1 \). Then relation (2) for \( n = 0 \) gives us \( |a_{-1}| R^{-1} \leq \epsilon |a_q| R^q \), hence by (3) we have \( |a_q| |(-q - 1)!| \leq \epsilon R^{q+1} |a_q| \)

and therefore

\[
\epsilon |(-q - 1)!| R^{q+1} \geq 1
\]
but we know that \( R^{q+1}|(q-1)!| \leq p^{-(q+1)/p} \) when \( f \) is quasi-invertible hence (6) is impossible.

Lemma D is then proven.

The following lemma was given in [55], in constructing the «Produits Bicroulants» (twice collapsing meromorphic products).

**Lemma E.** Let \( \rho, \rho', \rho'', \rho \in R_+ \) with \( 0 < \rho < \rho'' < \rho \). There exist sequences \( (b_n')_{n \in \mathbb{N}} \) and \( (b_n'')_{n \in \mathbb{N}} \) in \( \Gamma(0, \rho', \rho'') \) with \( |b_n'| > |b_{n+1}'| \), \( |b_n''| < |b_{n+1}''| \), \( \lim_{n \to \infty} b_n'' = R'' \), such that, if we denote by \( D \) the set \( d(0, R) \backslash \left( \bigcup_{n=1}^{\infty} d^-(b_n', \rho) \cup \bigcup_{n=1}^{\infty} d^-(b_n'', \rho) \right) \) the algebra \( H(D) \) has an element \( \varphi \in H(D) \) satisfying \( \lim_{|x| \to \rho'} \varphi(x) = 1 \) and \( \lim_{|x| \to \rho} \varphi(x) = 0 \).

**Proof of Theorem 2.** Let \( \omega_1, \ldots, \omega_t \) be points in \( d(0, 1) \) such that \( \omega_1 = 0 \), \( |\omega_i - \omega_j| = 1 \) whenever \( i \neq j \). Let \( r \in ]0, 1[ \) and let \( (b_m)_{m \in \mathbb{N}} \) be a sequence in \( d^-(0, r) \) such that \( |b_m| < |b_{m+1}| \) and \( \lim_{m \to \infty} |b_m| = r \) and let \( (q_m)_{m \in \mathbb{N}} \) be a sequence of integers such that \( q_1 < q_m \) for all \( m > 1 \), \( \lim_{m \to \infty} q_m = +\infty \) and \( \lim_{m \to \infty} \prod_{j=1}^{m-1} |b_j/b_m|^{(p^q)} = 0 \). Let \( T_m = d^-(b_m, |b_m|) \), let \( p_m = p^{q_m} \) and let \( A = d^-(0, r) \backslash \left( \bigcup_{m=1}^{\infty} T_m \right) \).

It is easily seen that \( A \) admits a \( T \)-sequence \( (T_m, q_m) \) \( [S_1] \). Let \( \mathcal{F} \) be the increasing \( T \)-filter of center 0, of diameter \( r \) on \( A \). First we will construct an infraconnected clopen set included in \( d(0, 1) \), of diameter 1, satisfying the following conditions:

1. \( \Omega \cap d^-(0, r) = A \).
2. \( \Omega \) has an increasing \( T \)-filter \( \mathcal{F} \) of center 0, of diameter 1.
3. \( \Omega \) has a decreasing \( T \)-filter \( \mathcal{G} \) of center 0, of diameter \( R \in ]r, 1[ \).
4. The only \( T \)-filters of \( \Omega \) are \( \mathcal{F}, \mathcal{F}, \mathcal{G} \).
5. There exists \( \varphi \) and \( \psi \in H(\Omega) \backslash \{0\} \) such that
   \[
   \varphi(x) = 1, \quad \psi(x) = 0 \quad \text{for } x \in \Omega \cap d(0, R)
   \]
   and
   \[
   \varphi(x) = 0, \quad \psi(x) = 1 \quad \text{for } x \in \Omega \backslash d^-(0, 1).
   \]

Let \( \varphi \in ]0, f[ \). By Lemma E there exist sequences \( (\beta_n')_{n \in \mathbb{N}} \) and
\((\beta_n')_{n \in \mathbb{N}}\) in \(\Gamma(0, R, 1)\) such that
\[
R < |\beta_{n+1}'| < |\beta_n'|, \quad \lim_{n \to \infty} \beta_n' = R, \quad |\beta_n''| < |\beta_{n+1}''| < 1, \quad \lim_{n \to \infty} |\beta_n''| = 1
\]
and such that the set
\[
\Lambda = \sigma(0, 1) \setminus \left( \bigcup_{n=1}^{\infty} d^-(\beta_n', \varphi) \cup \bigcup_{n=1}^{\infty} d^-(\beta_n'', \varphi) \right)
\]
defines an algebra \(H(\Lambda)\) that contains elements \(\varphi\) satisfying \(\varphi(x) = 1\) for \(|x| \leq R, \varphi(x) = 0\) for \(|x| = 1\). Let us put \(\psi = 1 - \varphi\) and let \(\Omega\) be the set \(\Lambda \cup (\Lambda \setminus d^- (0, r))\).

\(\Omega\) has clearly three \(T\)-filter:
- the filter \(\mathcal{F}\) on \(\Lambda\)
- the increasing filter \(\mathcal{F}\) of center 0, of diameter 1 that strictly annuls \(\varphi\).
- the decreasing filter \(\mathcal{G}\) of center 0, of diameter \(R\) that strictly annuls \(\psi\).

It is easily seen these three \(T\)-filters are the only \(T\)-filters on \(\Omega\), and \(\Omega, \varphi, \psi\) are then defined.

Let \(f(x) = \left( \sum_{m=1}^{\infty} p_{m/1}(1-x/b_m) \right)\) and let \(f_1(x) = \varphi(x)f(x) + \psi(x)\).

Then \(f_1(x) = f(x)\) when \(x \in \Omega \cap d(0, R)\) and \(f_1(x) = 1\) when \(x \in \Omega \setminus d^- (0, 1)\). We can deduce that \(f_1\) is a quasi-invertible element in \(H(\Omega)\). Indeed, by Proposition B, \(f\) is not annulled by \(\mathcal{F}\) and by \(\mathcal{G}\), hence \(f_1\) is not annulled by \(\mathcal{F}\) and by \(\mathcal{G}\) either; on the other hand, as \(f_1(x) = 1\) when \(|x| = 1, f_1\) is not annulled by \(\mathcal{F}\); hence \(f_1\) is not annulled by any one of the three \(T\)-filters on \(\Omega\) so that it is quasi-invertible in \(H(\Omega)\).

By Proposition B \(\varepsilon(f_1)\) has a solution \(g_1 = \prod_{m=1}^{\infty} 1/(1-x/b_m)^{p_{m/1}}\).

Now, for each \(y = 2, \ldots, t\) let \(\Omega_y = \omega_y + \Omega = \{x + \omega_y | x \in \Omega\}\) and let \(f_j \in H(\Omega_j)\) defined by \(f_j(x + \omega_j) = f_1(x)\). In \(\Omega_j\) the equation \(\varepsilon(f_j)\) has a solution \(g_j\) defined by \(g_j(x + \omega_j) = g_1(x)\). Let \(D = \bigcap_{j=1}^{t} \Omega_j\) and let \(f(x) = \prod_{j=1}^{t} f_j(x) \in H(D)\). Obviously, \(f(x) = f_j(x)\) when \(|x - \omega_j| < 1\) and \(f(x) = 1\) when \(|\xi - \omega_l| = 1\) for every \(l = 1, \ldots, t\). Each one of the \(f_j\) is quasi-invertible in \(H(D)\) so that \(f\) is also quasi-invertible.
Now each $g_j$ ($1 \leq j \leq t$) is a solution of $\mathcal{S}(f)$. Indeed, when $|x - \omega_j| < 1$ we have $g_j'(x) = f_j(x)g_j(x) = f(x)g_j(x)$ and when $|x - \omega_j| = 1$, $g_j(x) = 0$.

On the other hand, the $g_j$ clearly have supports two by two disjointed, hence they are linearly independent, and that shows $\mathcal{S}(f)$ has dimension $\geq t$.

We will end the proof in showing that $\{g_1, \ldots, g_t\}$ generates $\mathcal{S}(f)$.

Log will denote the real logarithm function of base $p$. Let $v$ be the valuation defined in $K$ by $v(x) = -\log |x|$ when $x \neq 0$ and $v(0) = +\infty$.

When $A$ is an infraconnected set containing 0, and $f \in H(A)$ we put

$$v(f, \mu) = \lim_{v(x) \to \mu} v(f(x))[E_2, E_3, E_4].$$

For each $j = 1, \ldots, t$, let $D_j = d^-(\omega_j, 1) \cap D$ and $B_j = d^- (\omega_j, R)$; let $D' = D \setminus \bigcup_{j=1}^t D_j$. By definition of $f$ we see that $f(x) = 1$ for all $x \in D'$ and $d^- (x, 1) \subset D'$ for every $x \in D'$. Then it is well known that the equation $y' = y$ has no solution $y$ in $H(d^-(x, 1))$ but the zero solution. Let $h \in \mathcal{S}(f)$. For every $a \in D'$, the restriction of $h$ to $d^-(a, 1)$ is a solution of the equation $y' = y$ that belongs to $H(d^-(a, 1))$ hence we see that $h(x) = 0$ for all $x \in D'$. Since $D'$ is equal to $d(0, 1) \setminus \bigcup_{j=1}^t d^-(\omega, 1)$ we see that

$$v(h, 0) = +\infty.$$ 

Now let us consider $h(x)$ when $x \in B_1$.

Since $D_1 = \Omega \cap d^- (0, 1)$ the three $T$-filters $\mathcal{F}, \mathcal{G}, \mathcal{F}_2$ of $\Omega$ are secant to $D_1$ and they are the only $T$-filters on $D_1$. Then $\mathcal{F}$ is the only one $T$-filter on $B_1$ because $\mathcal{F}$ and $\mathcal{G}$ are not secant to $d(0, R)$. The algebra $H(B_1)$ has no divisor of zero. Consider the restriction $\mathcal{F}_1$ of to $D_1$ and the restriction $\mathcal{F}_1$ to $B_1$. In $H(B_1)$ the space $\mathcal{S}(\mathcal{F}_1)$ has dimension one by Theorem 3 of $[E_7]$, hence there exists $\lambda_1 \in k$ such that $h(x) = \lambda_1 g_1(x)$ whenever $x \in B_1$.

Since $g_1 \in \mathcal{S}_0(\mathcal{F})$, that implies $h(x) = 0$ whenever $x \in \Gamma(0, r, R)$ hence $v(h, -\log R) = +\infty$. We will deduce that $v(h, \mu) = +\infty$ whenever $\mu \in [0, -\log R]$.

Indeed, suppose this is not true. Then $h$ is strictly annihilated by an increasing $T$-filter of center 0, of diameter $> R$, hence $h$ is strictly an-
nullled by $\mathcal{F}$. Since $\lim_{|x| \to 1^{-}} \varphi(x) = \lim_{x \to 1^{-}} \varphi(x) = 1$, there exists $s \in \mathbb{R}$ such that
\[ h'(x) x \leq 1 \quad \text{for } x \in D \cap \Gamma(0, s, 1). \quad (7) \]

On the other hand, it is easily seen that $h(x)$ is equal to a Laurent series in each annulus $\Gamma(0, |b_n|, |b_{n+1}|)$ and for every $s < 1$ there exist intervals $[r', r''] \subset [s, 1]$ such that the function $v(h, \mu)$ is strictly decreasing in $[-\log r'', -\log r']$ and such that $h(x)$ is equal to a Laurent series $\sum_{n} a_n x^n$. Let $p \in [r', r'']$, since $v(h, \mu)$ is strictly decreasing in $[-\log r'', -\log r']$ there exists $q < 0$ such that $|a_q| \leq \sup_{n \in \mathbb{Z}} |a_n| |\rho^n|$. Then $h$ satisfies the hypothesis of Lemma D and relation (7) is impossible. But then $v(h, \mu) = +\infty$ for every $\mu \in [0, -\log r]$ It follows that $h(x) = 0$ for every $x \in \Gamma(0, R, 1)$ because if there existed a point $\alpha \in \Gamma(0, R, 1)$ with $h(\alpha) \neq 0$, $\alpha$ should be the center of an increasing $T$-filter that would annul $h$ but the unique $T$-filter of center $\alpha$ is $\mathcal{F}$ and we have just seen that $\mathcal{F}$ does not annul $h$.

Thus we have now proven that $h(x) = 0$ for all $x \in B_1$ such that $r \leq |x| < 1$. Since $g_1(x) = 0$ whenever $x \in \Gamma(0, r, 1)$, the relation $h(x) = \lambda_1 g_1(x)$ is then true in all $B_1$. In the same way, for each $j = 2, \ldots, t$, we can show there exists $\lambda_j \in K$ such that $h(x) = \lambda_j g_j(x)$ for every $x \in B_j$ and then $h(x) = \sum_{j=1}^{t} \lambda_j g_j(x)$ is true in $\bigcup_{j=1}^{t} B_j$, and of course in $D'$, hence it is true in all $D$. That finishes proving $\{g_1, \ldots, g_t\}$ is a base of $\mathcal{S}(f)$.

REFERENCES


The equation $y' = fy$ in $C_p$ when $f$ is quasi-invertible


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