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Sobolev Theory for non Commutative Tori.

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SUNTO - In questa nota presentiamo un'estensione della teoria di (Hilbert-) Sobolev alle C^*-algebre dette tori non commutativi. Determiniamo, in particolare, una condizione sufficiente sull'indice dello spazio di Sobolev affinchè questo risulti essere anche una *-algebra di Banach.

Introduction.

The aim of the present note is to develop Hilbert-Sobolev theory in a non commutative context, namely for those C^*-algebras called non commutative tori ([15], [16]). These algebras naturally appear in many areas of mathematics and mathematical physics (for instance, they provide a theoretical framework for understanding the Quantum Hall Effect ([5])) and furnish the simplest examples of «non commutative manifolds» ([15], [16], [6], [7], [8]). Such extension of the «classical» theory (see e.g. [1], [13], [2]), besides having interest in itself, is needed, in particular, for the extension of the infinite dimensional manifold approach to moduli spaces of Yang Mills connections (e.g. [2], [10]) to these algebras ([18], [19]). The layout of this paper is the following: in the first two sections we recall some basic facts about non commutative tori (without aiming at completeness) and develop the analytic apparatus pertaining to them, introducing, for later purposes, five different but equivalent Sobolev norms, giving rise to (Hilbert-) Sobolev spaces H^l, $l \in \mathbb{N}$. In § 3 we prove Sobolev, Rellich and Maurin type theorems ([1]) and in § 4 we finally show that for $l > n$ (n being the «dimension» of the

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non commutative torus) H^l is a (non commutative) Banach *-algebra with unit, and H^m is a (two-sided) topological H^l-module for $l > n$, $0 \leq m \leq l$. We do not settle the question whether n is the optimal bound enforcing the result in general: certainly this is not the case for the commutative torus (which is of course a case covered by the theorem), for which the classical theory—which, however, highly relies on the commutativity of function algebras—gives $n/2$ (see [1], [2], [3]). Despite this apparent drawback, our approach is likely to work in a broader non commutative context, due to its «algebraic» character. These problems will be possibly tackled elsewhere. See also § 5 of this work, devoted to final remarks.

1. Non commutative tori.

In this section we recall basic material concerning non commutative tori. For full details we refer to the fundamental paper of M. A. Rieffel [16].

1.1. *The C^*-algebra structure.*

Let L be a real n-dimensional vector space and $D \subset L$, $D \simeq Z^n$ be a lattice. Let also ϑ be an element of $\Lambda^2 L^*$, i.e. a skew-symmetric bilinear form on L. The non commutative n-torus, denoted by A_ϑ, is the universal C^*-algebra with unit generated by unitary operators $\{U_x\}_{x \in D}$ on a Hilbert space H fulfilling the commutation relation

$$U_y U_x = e^{2\pi i \vartheta(x, y)} U_x U_y \equiv \lambda(x, y) U_x U_y \qquad x, y \in D .$$

Explicit representation of A_ϑ are naturally constructed ([16]). We shall consider the—quite explicit, in this case—GNS (Gelfand-Naimark-Segal) representation of A_ϑ arising from the normalized faithful trace τ discussed below. It is a faithful representation. A_ϑ is naturally viewed as a Z^n-cocycle C^*-algebra ([16]).

Let $\{e_j\}_{j=1,2,\dots,n}$ be a basis for L generating D, and set $U_{e_j} \equiv U_j$, $j = 1, 2, \dots, n$. It will be very convenient to introduce the multiindex $\boldsymbol{m} = (m_1, m_2, \dots, m_n) \in Z^n$, and let

$$U^{\boldsymbol{m}} := U_1^{m_1} \cdot U_2^{m_2} \cdot \ldots \cdot U_n^{m_n} . \tag{1.1}$$

We also set, for later use

$$|\boldsymbol{m}| := \sum_{k=1}^{n} |m_k| ; \qquad \boldsymbol{m}^2 := \sum_{k=1}^{n} m_k^2 .$$

Any element $a \in A_\vartheta$ can be uniquely expressed as

$$a = \sum_{m \in Z^n} a_m \cdot U^m , \qquad a_m \in \mathrm{C} . \tag{1.2}$$

Necessary and sufficient conditions for a sequence $\{a_m\}_{m \in Z^n}$ to yield an element $a \in A_\vartheta$ (through (1.2)) are not known.

An elementary but useful sufficient condition is $\{a_m\} \in l^1(\mathrm{Z}^n)$ since:

$$\|a\| = \|\sum_m a_m U^m\| \leqslant \sum_m |a_m| . \tag{1.3}$$

By $\|\cdot\|$ we mean the C^*-norm on A_ϑ or on any faithful representation thereof. Let us record, for later use, how the *-operation and the product in A_ϑ are reflected on the corresponding sequences associated to the elements involved (the vector space operations being obvious). The straightforward computation is based on the formulae:

$$(U^m)^* = (U^m)^{-1} = \sigma(m)\, U^m , \qquad U^m U^{m'} = \sigma(m, m')\, U^{m+m'} \tag{1.4}$$

where, if $\lambda_{ij} := \lambda(e_i, e_j)$,

$$\sigma(m) = \prod_{1 \leqslant i < j \leqslant n} \lambda_{ij}^{m_i m_j} ; \qquad \sigma(m, m') = \prod_{1 \leqslant i < j \leqslant n} \lambda_{ij}^{m_i' m_j}$$

Obviously $|\sigma(m)| = |\sigma(m, m')| = 1$, and, in particular:

$$\sigma(-m) = \sigma(m); \qquad \sigma(m, -m) = \sigma(m)^{-1} . \tag{1.5}$$

In view of (1.4), (1.5) we have

$$\begin{cases} (a^*)_m = \sigma(m)\, \bar{a}_{-m} , \\ (ab)_M = \sum_m \sigma(m, M - m)\, a_m\, b_{M-m} = \sum_m \sigma(m)^{-1} \cdot \sigma(m, M)\, a_m\, b_{M-m} , \end{cases} \tag{1.6}$$

A_ϑ is equipped with a unique faithful normalized trace τ (i.e. a faithful state on A_ϑ with $\tau(ba) = \tau(ab)$ for any A_ϑ), explicitly given by:

$$\tau(a) = a_0 . \tag{1.7}$$

Let $(H_\tau, \pi_\tau, \xi_\tau)$ be the GNS triple attached to τ: π_τ is the GNS representation of A_ϑ on the Hilbert space H_τ (stemming from the left regular representation of A_ϑ on itself) and ξ_τ is the GNS cyclic vector (see [14] for details). In the present case we have a completely explicit representation of these objects, which we now describe.

H_τ is the Hilbert space completion of A_ϑ with respect to the norm $\| \ \|_0$

induced by the scalar product

$$(1.8) \qquad \langle a, b\rangle := \tau(a^* b) \equiv \langle a, b\rangle_{H_\tau}; \qquad \|a\|_0^2 = \langle a, a\rangle .$$

Using (1.2), (1.6), (1.7), we have:

$$\langle a, b\rangle_{H_\tau} = \tau(a^* b) = (a^* b)_0 = \sum_{m} (a^*)_m b_{-m} \sigma(m, -m) =$$

$$= \sum_{m} \bar{a}_{-m} b_{-m} \sigma(m) \sigma(m, -m) = (\text{by } (1.5)),$$

$$\sum_{m} \bar{a}_m b_m = \langle a, b\rangle_{l^2(Z^n)} .$$

Hence H_τ can be identified with $l^2(Z^n)$. The elements $\{U^m\}_{m \in Z^n}$ are easily seen (using (1.4), (1.5)) to constitute an orthonormal basis in H_τ the cyclic vector ξ_τ is $U^0 = I$. The $l^2(Z^n)$ representation of U^m is $\{\delta_{m,m'}\}_{m' \in Z^n}$. $\pi_\tau(a)$ is simply a, viewed as an operator on H_τ, acting (on the left) via (1.6). Notice the formula

$$(1.9) \qquad \|b\|_0^2 \equiv \tau(b^* b) = \|b\xi_\tau\|_{H_\tau}^2 ,$$

(by the very definition of GNS vector).

We shall need the explicit formula for the matrix elements of b, viewed as a linear operator on H_τ. They read:

$$(1.10) \qquad (b)_{mM} \equiv \langle U^m, bU^M\rangle_{H_\tau} = \tau((U^m)^* bU^M) = \sigma(M)^{-1} \sigma(m, M)\, b_{m-M} ,$$

as an easy computation exploiting again (1.4), (1.5) shows.

1.2. *The differential structure.*

In order to perform analysis and differential geometry on $A_\mathcal{G}$, one has to select a norm dense *-subalgebra $A_\mathcal{G}^\infty$ of $A_\mathcal{G}$ (called the smooth algebra), playing the role of the smooth functions of the classical (i.e. commutative) case ([6], [7], [8], [16]). Here one considers a linear representation of $L \simeq \mathbb{R}^n$ (viewed as an abelian Lie algebra) on $A_\mathcal{G}$ via (unbounded) mutually commuting *-derivations $\{\delta_X\}_{X \in L}$:

$$(1.11) \qquad \delta_X U_x = i\langle X, x\rangle U_x \qquad x \in D$$

where $\langle\ ,\ \rangle$ is a metric for L for which $\{e_j\}_{j=1,\dots,n}$ is an orthonormal basis. We are discarding an unessential 2π factor in the definition of (1.11), with respect to the usual one. This action comes from a natural action of T^n on $A_\mathcal{G}$ but we shall not need this fact here.

Using obvious notations, (1.11) reads, in components:

$$(1.12) \qquad \delta_j U_k = i\delta_{jk} U_k .$$

Recall that by the very definition of *-derivation one has

$$\delta_X(ab) = \delta_X(a)\, b + a\delta_X(b) . \qquad \text{(Leibniz rule)},$$

$$\delta_X(a^*) = (\delta_X(a))^* \qquad \text{(*-property)},$$

(whenever these formulae make sense). In particular:

$$\delta_j^k U^m = (im_j)^k U^m . \tag{1.13}$$

τ is invariant with respect to δ_X, $X \in L$, i.e. $\tau\delta_X(a) = 0$, whenever $\delta_X(a)$ exists in A_ϑ.

A_ϑ^∞ is, by definition, the *-subalgebra (norm dense in A_ϑ) consisting of the smooth elements of A_ϑ with respect to the differential structure above. It is well known that $a \in A_\vartheta^\infty$ iff its associated sequence $\{a_m\}$ belongs to $\mathcal{S}(\mathbb{Z}^n)$ ($\mathcal{S}(\mathbb{Z}^n)$ denotes the space of Schwartz sequences, i.e. those sequences $\{a_m\}_{m \in \mathbb{Z}^n}$ for which $P(m)\, a_m \to 0$ if $|m| \to \infty$ with P any polynomial on $\mathbb{R}^n$).

We shall consider the Laplace operator on A_ϑ^∞ given by:

$$\Delta = - \sum_{j=1}^{n} \delta_j^2 , \tag{1.14}$$

Δ is a linear, unbounded, self adjoint, positive, elliptic operator on H_τ whose spectrum is

$$\sigma(\Delta) = \{m^2\}_{m \in \mathbb{Z}^n} . \tag{1.15}$$

Its kernel is one dimensional and is generated by $\xi_\tau = U^0 = I$.

Δ is diagonal with respect to the orthonormal basis $\{U^m\}$, $m \in \mathbb{Z}^n$, and its matrix elements read

$$(\Delta)_{mM} = m^2 \delta_{mM} . \tag{1.16}$$

Let $s \in \mathbb{N}$, $s \geqslant 1$. The operators $(I + \Delta)^s$, $I + \Delta^s$ have compact inverses: this renders them relevant for the introduction of Sobolev spaces for A_ϑ, which will be discussed in the next section.

Denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm of an operator on a Hilbert space.

We shall make use of the following elementary

(1.17) Lemma. i) Let

$$C_{h,s} := \sum_m \left(\sum_{j=1}^{n} |m_j|^h \right)^2 (1 + m^2)^{-s}, \qquad h = 1, 2, \ldots, k .$$

Then $C_{h,s} < \infty$ for any $h = 1, 2, \ldots, k$ iff $s > k + n/2$.

ii) $(I+\Delta)^{-s/2}$ is Hilbert-Schmidt iff $s>n/2$.

PROOF. On $\mathbf{R}^n$ any two norms are equivalent, whence in particular the Euclidean and the l^h-norm ($h\geq 1$) are such. This entails the existence of $C_1>C_2>0$ such that, for any $\boldsymbol{m}\in \mathbf{Z}^n$:

$$C_2(\boldsymbol{m}^2)^h \leq \left(\sum_{j=1}^{n} |m_j|^h\right)^2 \leq C_1(\boldsymbol{m}^2)^h . \tag{1.18}$$

Thus we are led to examine the integral

$$\int_0^\infty \frac{r^{2h} r^{n-1}}{(1+r^2)^s} dr \tag{1.19}$$

for $h=1,2,\ldots,k$, which converges iff $2s-2h-n+1>1$, i.e. iff $s>k+n/2$.

Moreover $C_{0,s}:=\sum_{\boldsymbol{m}}(1+\boldsymbol{m}^2)^{-s}=\|(I+\Delta)^{-s/2}\|^2_{\mathrm{HS}}$, by the same argument, is finite iff $s>n/2$. This ends the proof.

2. Sobolev spaces.

Let us introduce the following Sobolev norms on A_ϑ^∞ for $s\in\mathbf{N}$, $s\geq 1$.

$$\|a\|^2_{1,s}:=\langle a,(I+\Delta)^s a\rangle=\|(I+\Delta)^{s/2}a\|^2_0 ,$$

$$\|a\|^2_{2,s}:=\langle a,(I+\Delta^s)a\rangle=\|a\|^2_0+\|\Delta^{s/2}a\|^2_0 ,$$

$$\|a\|^2_{3,s}:=\|a\|^2_0+\sum_{j=1}^{n}\|\delta_j^s(a)\|^2_0 ,$$

$$\|a\|^2_{4,s}:=\sum_{j=1}^{n}\sum_{k=0}^{s}\|\delta_j^k(a)\|^2_0 ,$$

$$\|a\|^2_{5,s}:=\sum_{i,j=1,2,\ldots,n}\ \sum_{l=0}^{s}\sum_{k=0}^{l}\|\delta_i^{l-k}\delta_j^k(a)\|^2_0 .$$

(2.1) THEOREM i) The norms $\|\cdot\|_{j,s}$ $j=1,2,\dots,5$ are all equivalent.

ii) They are *-norms: $\|a^*\|_{j,s}=\|a\|_{j,s}$, $j=1,\dots,5$, $a\in A_\vartheta^\infty$.

iii) They do not depend on ϑ.

PROOF. i) $\|\cdot\|_{1,s}\sim\|\ \|_{2,s}$. Let $\Delta=\int_{\sigma(\Delta)}\lambda\, dE(\lambda)$ be the spectral resolution of the Laplacian on H_τ, $\sigma(\Delta)\subseteq R^+\cup\{0\}$. Then

$$\|a\|_{1,s}^2:=\int_{\sigma(\Delta)}(1+\lambda)^s d\langle a, E(\lambda)a\rangle\,,$$

$$\|a\|_{2,s}^2:=\int_{\sigma(\Delta)}(1+\lambda^s)\, d\langle a, E(\lambda)a\rangle\,.$$

Our assertion will be proven once we show the existence of $C>0$ such that

$$1+\lambda^s\geqslant C(1+\lambda)^s \qquad \lambda\geqslant 0,\quad s\geqslant 1\,,$$

(since we already have $(1+\lambda)^s\geqslant 1+\lambda^s$). Clearly we may assume $s\geqslant 2$. Let $F(\lambda):=(1+\lambda)^{-s}(1+\lambda^s)$, $\lambda\geqslant 0$. The absolute minimum of F occurs for $\lambda=1$ and equals 2^{1-s}. So take $C\leqslant 2^{1-s}$.

Notice that this particular equivalence does not depend on the specific structure of A_ϑ and thus holds in general provided the objects involved make sense.

$$\|\cdot\|_{2,s}\sim\|\ \|_{3,s}\,.$$

If $a=\sum a_m U^m$, then

$$\|a\|_{2,s}^2=\sum_m(1+(m^2)^s)|a_m|^2;\qquad \|a\|_{3,s}^2=\sum_m\left(1+\sum_{j=1}^n m_j^{2s}\right)|a_m|^2\,,$$

(using the definitions of $\|\cdot\|_{j,s}$, (1.13)-(1.16)).

We only have to show that there exists $C>0$ such that

$$1+\sum_{j=1}^n m_j^{2s}\geqslant C(1+(m^2)^s),\ m\in Z^n\,.$$

This is implied by the existence of $0<C<1$ such that

$$\sum_{j=1}^n m_j^{2s}\geqslant C(m^2)^s\,.$$

Now, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\|x\|_{l^s} = \left(\sum_{i=1}^{n} |x_i|^s\right)^{1/s} \geqslant C_1 \|x\|_{l^1} = C_1 \sum_{i=1}^{n} |x_i|$$

for some $C_1 \in (0,1)$. Letting $x = (m_1^2, m_2^2, \ldots, m_n^2)$ we have the assertion (setting $C = C_1^s$).

This equivalence explicitly depends on $A_{\mathscr{S}}$ as the next ones.

$$\| \ \|_{3,s} \sim \| \ \|_{4,s} \, .$$

It will suffice the show that there exists $C \geqslant n$ such that

$$C\left(\sum_{k=1}^{n} m_k^{2s} + 1\right) \geqslant \sum_{j=1}^{s} \sum_{k=1}^{n} m_k^{2j} + n\,,$$

which is implied by

$$C \sum_{k=1}^{n} m_k^{2s} \geqslant \sum_{j=1}^{s} \sum_{k=1}^{n} m_k^{2j} \, .$$

Now

$$\sum_{j=1}^{s} \sum_{k=1}^{n} m_k^{2j} = \sum_{k=1}^{n} m_k^{2s} (1 + m_k^{-2} + \ldots + (m_k^{-2})^{s-1}) \leqslant s \sum_{k=1}^{n} m_k^{2s}\,,$$

provided $|m_k| \geqslant 1$. But actually this inequality holds for any $\boldsymbol{m} \in \mathbb{Z}^n$. Taking $C \geqslant \max(n, s)$, we are done.

$$\|\cdot\|_{4,s} \sim \|\cdot\|_{5,s} \, .$$

One has $\|\cdot\|_{5,s} \geqslant \|\cdot\|_{4,s}$. Next, for any $i, j = 1, 2, \ldots, n$, $l = 1, 2, \ldots, s$, $k = 0, 1, \ldots, l$, we have:

$$\|\delta_i^{l-k} \delta_j^k (a)\|_0^2 = \sum_{\boldsymbol{m}} m_i^{2(l-k)} m_j^{2k} |a_{\boldsymbol{m}}|^2 \, .$$

But

$$m_i^{2(l-k)} m_j^{2k} \leqslant (\boldsymbol{m}^2)^l \qquad i, j = 1, 2, \ldots, n \, .$$

Thus

$$\sum_{i,j=1,2,\ldots,n} \|\delta_i^{l-k} \delta_j^k (a)\|_0^2 \leqslant n^2 \langle a, \Delta^l a \rangle\,,$$

whence

$$\sum_{i,j=1,2,\ldots,n} \sum_{l=1}^{s} \sum_{k=0}^{l} \|\delta_i^{l-k}\delta_j^k(a)\|_0^2 \leqslant$$

$$\leqslant n^2 \sum_{l=1}^{s}(1+l)\langle a, \Delta^l a\rangle \leqslant n^2(1+s)s\langle a, \Delta^s a\rangle .$$

Thus, for some $C > 0$

$$\|a\|_{4,s}^2 \leqslant \|a\|_{5,s}^2 \leqslant n^2(1+s)s\|a\|_{2,s}^2 \leqslant C\|a\|_{4,s}^2 ,$$

(since $\|\cdot\|_{2,s} \sim \|\cdot\|_{3,s} \sim \|\cdot\|_{4,s}$). This ends part i) of the proof.

ii) The norms above are of the form $\|a\|^2 := \langle a, Da\rangle$, where D is a (formally) self-adjoint operator on $A_\vartheta^\infty \subset H_\tau$, which is manifactured from *-derivations, whence

$$\|a^*\|^2 = \langle a^*, Da^*\rangle = \langle a^*, (Da)^*\rangle = \langle Da, a\rangle = \langle a, Da\rangle = \|a\|^2 .$$

iii) This is immediate from the explicit formulae defining the norms, which no longer involve λ (and ϑ). □

So we can form Hilbert Sobolev spaces H^s, $s = 1, 2, \ldots$, upon completing A_ϑ^∞ with respect to any norm $\|\cdot\|_{j,s}$, $j = 1, 2, \ldots, 5$. In the sequel we shall mainly work with $\|\cdot\|_{1,s}$, but implicit use will be made of other norms, when this proves convenient. Henceforth we omit the suffix j. Let ulso $H^0 \equiv H_\tau$. This explains the suffix 0 for the norm of H_τ.

REMARKS. i) In introducing Sobolev norms via a Laplace operator, we took inspiration from [3], see also [11]. We pursued this idea further in [19].

ii) An immediate calculation establishes $A_\vartheta^\infty = \bigcap_{s \geqslant 1} H^s$ ($\subset$ is obvious).

iii) An important consequence of part iii) of the previous theorem is that as far as the vector space structure of H^s is concerned, they do not differ from Sobolev spaces pertaining to the ordinary torus T^n (or rather, their Fourier-transformed ones). Hence, the theorems we prove in the following section hold unchanged in the A_ϑ case. Nevertheless we wish to discuss very simple «algebraic» proofs thereof, some features of which are likely to hold in a more general context.

3. Embedding theorems.

Let C^k, $k \in \mathbf{N}$, be the subspace of $A_\vartheta \equiv C^0$ consisting of those $a \in C^0$ for which $\delta_j^h(a) \in C^0$ for $1 \leqslant j \leqslant n$, $0 \leqslant h \leqslant k$, equipped with the (Banach) norm $\|a\|_{C^k} := \sum_{h=0}^{k} \sum_{j=1}^{n} \|\delta_j^h(a)\|$. An equivalent norm would be:

$$\|a\|_{C^k} := \max_{\substack{j=1,\dots,n\\ 0 \leqslant h \leqslant k}} \|\delta_j^h\{\alpha\}\|.$$

(3.1) THEOREM (Sobolev Embedding Theorem for A_ϑ). $H^s \subset C^k$ for $s > k + n/2$ and the embedding is continuous with respect to the natural topologies involved.

PROOF. Let $a = \sum_{\boldsymbol{m}} a_{\boldsymbol{m}} U^{\boldsymbol{m}}$, $a \in H^s$, s to be specified. If $1 \leqslant h \leqslant k$, we have

$$\sum_{j=1}^{n} \|\delta_j^h\{\alpha\}\| \leqslant \sum_{\boldsymbol{m}} \left(\sum_{j=1}^{n} |m_j|^h \right) |a_{\boldsymbol{m}}| \leqslant$$

$$\leqslant \left(\sum_{\boldsymbol{m}} (\boldsymbol{m}^2+1)^s |a_{\boldsymbol{m}}|^2 \right)^{1/2} \left(\sum_{\boldsymbol{m}} \left(\sum_{j=1}^{n} |m_j|^h \right)^2 (\boldsymbol{m}^2+1)^{-s} \right)^{1/2} = C_{h,s}^{1/2} \|a\|_s ,$$

i.e.

$$\|a\|_{C^k} \leqslant C \|a\|_s ,$$

for some finite C iff $s > k + n/2$, by Lemma (1.17). In particular, if $k = 0$ we get

$$\|a\| \leqslant C_{0,s}^{1/2} \|a\|_s . \tag{3.2}$$

for $s > n/2$. (3.2) will be important for the sequel.

(3.3) THEOREM. i) (Rellich Theorem for A_ϑ). The embedding $\mathfrak{J}\colon H^s \hookrightarrow H^t$ is compact for $s > t$, $s, t \geqslant 0$.

ii) (Maurin Theorem for A_ϑ). $\mathfrak{J}$ is actually Hilbert-Schmidt for $s - t > n/2$. See e.g. [1] for the classical proofs.

PROOF. i) Let $\varphi \in H^s$, then $\mathfrak{J}\varphi = \varphi \in H^t$ $(s > t)$ and

$$\|\varphi\|_t^2 = \|(I+\Delta)^{t/2}\varphi\|_0^2 = \|(I+\Delta)^{-(s-t)/2}(I+\Delta)^{s/2}\varphi\|_0^2 \leqslant \|(I+\Delta)^{-(s-t)/2}\|^2 \|\varphi\|_s^2$$

whence $\mathfrak{J}$ is continuous since $(I+\Delta)^{-(s-t)/2}$ is a compact, hence continuous, linear operator on H^0.

Now let $\{\varphi_n\}_{n=1,2,\ldots}$ be a bounded sequence in H^s: $\|\varphi_n\|_s = \|(I + \Delta)^{s/2}\varphi_n\|_0 < M$ for some $M > 0$, $\forall n \in \mathbb{N}$. Owing to the compactness of $(I+\Delta)^{-(s-t)/2}$, there exists a subsequence, again denoted by $\{\varphi_n\}_{n=1,2,\ldots}$ such that $\{(I+\Delta)^{t/2}\varphi_n\}_{n=1,2,\ldots}$ is Cauchy in $\| \ \|_0$, whence there exists $\psi \in H^0$ such that $(I+\Delta)^{t/2}\varphi_n \to \psi$, if $n \to \infty$. But $\|\varphi_n\|_0 \leq \|(I + \Delta)^{t/2}\varphi_n\|_0$, is also Cauchy in H^0, so $\exists \varphi \in H^0$ with $\varphi_n \to \varphi$, if $n \to \infty$. The operator $(I+\Delta)^{t/2}$ on H^0 is closed, so $\psi = (I+\Delta)^{t/2}\varphi$, i.e. $\varphi_n \to \varphi$ in H^t. Thus, a bounded sequence in H^s is transformed, up to passage to a subsequence, into a convergent one in H^t, i.e. $\mathfrak{J}$ is a compact embedding.

ii) Let $\{\xi_n\}_{n=1,2,\ldots}$ be an orthonormal basis in H^s with respect to which Δ is diagonal ($\Delta\xi_n = \lambda_n \cdot \xi_n$, $\lambda_n \geq 0$, provided the spectrum of Δ is discrete, which is obviously true in our setting): $\|(I+\Delta)^{s/2}\xi_n\|_0 = \|\xi_n\|_s = 1$. All we have to show is that

$$\|\mathfrak{J}\|_{\mathrm{HS}}^2 := \sum_{n=1}^{\infty} \|\xi_n\|_t^2 < \infty \qquad \text{iff } s - t > \frac{n}{2}.$$

But

$$\sum_{n=1}^{\infty} \|\xi_n\|_t^2 = \sum_{n=1}^{\infty} \|(I+\Delta)^{(t-s)/2}(I+\Delta)^{s/2}\xi_n\|_0^2 =$$

$$= \sum_{n=1}^{\infty} (I+\lambda_n)^{t-s} = \|(I+\Delta)^{(t-s)/2}\|_{\mathrm{HS}}^2 .$$

Now, in our case, $\lambda_n \to \lambda_{\boldsymbol{m}} = \boldsymbol{m}^2$, $\boldsymbol{m} \in \mathbb{Z}^n$ and $\|\mathfrak{J}\|_{\mathrm{HS}}^2 = C_{0,s-t} < \infty$ iff $s - t > n/2$, by Lemma (1.17). □

As already remarked, these proofs of Rellich and Maurin theorems might work in a broader context, provided one has detailed information on the spectrum of Δ in H^0.

4. The main result.

Our task is now to analyze the algebraic properties of the Sobolev spaces introduced above. The key lemma is the following:

(4.1) LEMMA. i) Let $a \in A_{\mathcal{S}}$, $b \in H^0$. Then

$$\|ab\|_0 \leq \|a\| \, \|b\|_0 . \tag{4.2}$$

ii) Let $a \in H^s$, $s > n/2$. Then

$$\|ba\|_0 \leq C_{0,s}^{1/2} \|b\|_0 \, \|a\|_s . \tag{4.3}$$

Under these assumptions, (4.2) implies

$$\|ab\|_0 \leqslant C_{0,s}^{1/2} \|b\|_0 \|a\|_s \,. \tag{4.4}$$

PROOF. i) Is straightforward and it is actually part of the general GNS theorem:

$$\|ab\|_0^2 = \|ab\xi_\tau\|_0^2 \leqslant \|a\|^2 \|b\xi_\tau\|_0^2 = \|a\|^2 \|b\|_0^2 \,.$$

ii) $\|ba\|_0 = \|ba\xi_\tau\|_{H_\tau}^2 = \|(I+\Delta)^{-s/2}(I+\Delta)^{s/2} a\xi_\tau\|_{H_\tau}^2 \leqslant$

$$\leqslant \|b(I+\Delta)^{-s/2}\|^2 \|(I+\Delta)^{s/2} a\xi_\tau\|_{H_\tau}^2 \leqslant \|b(I+\Delta)^{-s/2}\|_{\mathrm{HS}}^2 \|a\|_s^2 \,.$$

Thus, in order to finish the proof it is enough to show that $b(I+\Delta)^{-s/2}$ is Hilbert-Schmidt for $s > n/2$. Its matrix elements are the following:

$$(b(I+\Delta)^{-s/2})_{mM} = \sum_p (b)_{mp} (1+\boldsymbol{p}^2)^{-s/2} \delta_{pM} = \tag{4.5}$$

$$= b_{m-M}\,\sigma(\boldsymbol{M})^{-1}\sigma(\boldsymbol{m},\boldsymbol{M})(1+\boldsymbol{M}^2)^{-s/2} \,,$$

(by (1.10), (1.16)). So

$$\|b(I+\Delta)^{-s/2}\|_{\mathrm{HS}}^2 = \sum_{m,M} |b_{m-M}|^2 (1+\boldsymbol{M}^2)^{-s} = C_{0,s} \sum_m |b_m|^2 = C_{0,s}\|b\|_0^2 \,,$$

which, again by Lemma (1.17), is finite iff $s > n/2$. The last statement of ii) immediately follows from (3.2).

We are now ready to state

(4.6) THEOREM. i) Let $l \in \mathrm{N}$, $l > n$. Then H^l is a (non commutative) Banach *-algebra.

ii) If $l > n$ and $m \in \mathrm{N}$, $0 \leqslant m \leqslant l$, then H^m is a H^l-topological module.

PROOF. Let $f \in H^m$, $g \in H^l$ Let δ be any δ_j, $j = 1, 2, \ldots, n$. Then

$$\|\delta^m(fg)\|_0 = \Big\|\sum_{k=0}^m \binom{m}{k} \delta^k f \delta^{m-k} g\Big\|_0 \leqslant \sum_{k=0}^m \binom{m}{k} \|\delta^k f \delta^{m-k} g\|_0$$

and we also get an analogous inequality upon permuting f and g, which is disposed of in the same manner.

We wish to apply Lemma (4.1) to each term $\|\delta^k f \delta^{m-k} g\|_0$, $k = 0, 1, \ldots, m$. Now $\delta^k f \in H^{m-k}$, $\delta^{m-k} g \in H^{l-m+k}$ (reasoning with $\|\ \|_{5,s}$).

We ought to ensure that at least one of the non negative numbers $m-k$, $l-m+k$ is greater than $n/2$ (so that either $\delta^k f$ or $\delta^{m-k} g$ can play the role of a in Lemma (4.1)). But this is true provided $l > n$, since $m-k \leq n/2$, $l-m+k \leq n/2$ would give $l \leq n$. If either $\delta^k f$ or $\delta^{m-k} g$ is to be used as b in (4.3), (4.4), one has, respectively

$$\|\delta^k f\|_0 \leq H_1 \|f\|_m ; \qquad \|\delta^{m-k} g\|_0 \leq H_2 \|g\|_l$$

for positive constants H_1, H_2. The other term is estimated, respectively, by

$$\|\delta^{m-k} g\|_{l-m+k} \leq H_3 \|g\|_l \qquad or \qquad \|\delta^k f\|_{m-k} \leq H_4 \|f\|_m$$

for positive constants H_3, H_4. Reasoning similarly for any $j = 1, 2, \ldots, n$ we easily get, for an overall positive constant K (depending on n, l, m)

$$\|fg\|_m \leq K \|f\|_m \|g\|_l ; \qquad \|gf\|_m \leq K \|f\|_m \|g\|_l ,$$

which yields our desired result. If $m = l$, to obtain a Banach algebra norm ($\|fg\|_l \leq \|f\|_l \|g\|_l$) we only have to perform a rescaling of the Sobolev norm ($\| \ \|_l$ going to $K \| \ \|_l$). It will result, in general, that $\|I\|_l > 1$. The proof is complete.

5. Concluding remarks.

This note provides a first small step towards the extension of Sobolev theory to non commutative algebras. Its ideas can however be extended in many directions. A notable problem would be the development of general L^p-Sobolev theory and interpolations theory, which will require L^p-theory for von Neumann algebras (see e.g. [12], [9], and references therein).

This might settle, in particular, the Banach algebra optimal bound question. Another possibility would come from exploiting, in the non commutative torus case, the relationship between the product operation and Moyal *-products arising in quantization theory ([20], [17], [7]).

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