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On the Homogenization of Some Nonlinear Problems in Perforated Domains.

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0. - Introduction.

In this paper we study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of $\mathbb{R}^n$.

Let Ω_ε be a fixed bounded domain Ω from which a set T_ε of holes has been removed. The set T_ε is obtaied in the following way: let T a fixed set properly contained in the basic cell Y, let D_ε be the hole homotetic by ratio ε to T. Let us suppose to have a periodic distribution of period εY of D_ε. Then T_ε is the set of the holes of this periodic distribution contained in Ω_ε.

Roughly speaking, let us consider the problem:

$$(\mathcal{T}_\varepsilon)\qquad \begin{cases} -\operatorname{div} a(x/\varepsilon, Du) = f & \text{in } \Omega_\varepsilon\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ a(x/\varepsilon, Du)\cdot\nu = \varepsilon g(x, x/\varepsilon) & \text{on } \partial T_\varepsilon\,, \\ u \in H^{1,p}(\Omega_\varepsilon) & p > 1\,, \end{cases}$$

where ν denotes the exterior normal with respect to Ω_ε, $f \in L^{p'}(\Omega)$, g is Y-periodic in the second variable and $a(x, \xi)$ is a matrix periodic in x and satisfying suitable coerciveness and growth conditions in ξ.

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Our aim is to study the asymptotic behaviour of the solutions u_ε of (P_ε) as $\varepsilon \to 0$. Indeed, we prove that the « limit » problem of $(\mathcal{P}_\varepsilon)$ is:

$$(\mathcal{P}_0) \qquad \begin{cases} -\operatorname{div} b(Du) = f\theta + \mu_g(x) & \text{in } \Omega\,, \\ u \in H_0^{1,p}(\Omega) & p > 1\,, \end{cases}$$

with $\theta = |Y \setminus T|/|Y|$ and

$$\mu_g(x) = (1/|Y|)\int_{\partial T} g(x, y)\, d\sigma_{y'}\,.$$

Moreover the matrix $b(\xi)$ is given by an explicit formula.

As a consequence, we are able in particular to describe the « homogenized » operator of the problem:

$$\begin{aligned} &-\operatorname{div}\left(|Du|^{p-2}Du\right) = f && \text{in } \Omega_\varepsilon\,, \\ &u = 0 && \text{on } \partial\Omega\,, \\ &\left(|Du|^{p-2}Du\right)\cdot \nu = g(x, x/\varepsilon) && \text{on } \partial T_\varepsilon\,, \\ &u \in H^{1,p}(\Omega_\varepsilon) && p > 1\,. \end{aligned}$$

The homogenization theory for linear elliptic operators goes back to De Giorgi-Spagnolo [9], Bensoussan-Lions-Papanicolau [2], Sanchez-Palencia [15].

For Dirichlet nonlinear problems of the type

$$-\operatorname{div} a(x, u, Du) = f$$

some homogenization results for $p = 2$ were first given by Tartar [19] (see also Suquet [16]). By different techniques, for $p > 1$, homogenization results have been recently given in [11].

On the other hand the homogenization of some linear problems in perforated domains has been studied in [6] and in Cioranescu-Saint Jean Paulin [7] by using energy method.

1. – Statement of the problem.

Let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 2$, with a smooth boundary $\partial\Omega$. Introduce the representative cell $Y = [0, l_1) \times \ldots \times [0, l_n)$ and denote by T an open subset of Y, with smooth boundary ∂T, such that $\overline{T} \subset Y$. Denote by $\tau(\varepsilon\overline{T})$ the set of all traslated images $\varepsilon(k_1 + \overline{T})$, $k \in \mathbb{Z}^n$, $k_1 = (k_1 l_1, \dots, k_n l_n)$, $\varepsilon > 0$, of $\varepsilon\overline{T}$.

By this way $\mathbb{R}^n$ is periodically perforated by holes of ε-size.

We make the following assumption:

The holes $\tau(\varepsilon\overline{T})$ do not intersect the boundary $\partial\Omega$.

This assumption restricts the geometry of the open set Ω (example: Ω is a finite union of rectangles homothetic to the representative cell).

Define now the perforated domain:

$$\Omega_\varepsilon = \Omega \cap \{\mathbb{R}^n \setminus \tau(\varepsilon\overline{T})\} .$$

Hence, by the previous assumption it follows that:

$$\partial\Omega_\varepsilon = \partial\Omega \cap \partial T_\varepsilon$$

where T_ε is the subset of $\tau(\varepsilon\overline{T})$ contained in Ω.

The following notations are used in the following:

1) $Y^* = Y \setminus \overline{T}$;
2) $\theta = |Y^*|/|Y|$;
3) $|\omega| =$ the Lebesgue measure of ω (for any measurable set of $\mathbb{R}^n$);
4) $\chi(\omega) =$ the characteristic function of the set ω;
5) $\tilde{v} =$ the zero extension to the whole Ω, for any function v defined on Ω_ε;
6) $\langle f \rangle_E = (1/|E|)\int_E f(x)\,dx$, for $f \in L^1_{n,\mathrm{loc}}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ bounded open set of positive measure.

If $1 < p < +\infty$ and $p' = p/(p-1)$, we shall consider the following spaces:

$$H^{1,p}_{\mathrm{per}}(Y) =$$

$$= \{\, u(y) \in H^{1,p}(Y) \colon u \text{ has the same trace on the opposite faces of } Y \} ,$$

$$L^{p'}_{n,\ \mathrm{r}}(Y) = \{q(y) \in L^{p'}_n(Y) \colon \int_Y q \cdot Du \, dy = 0 \text{ for any } u \in H^{1,p}_{\mathrm{per}}(Y)\} .$$

Now, let $f \in L^{p'}(\Omega)$ and $g: \Omega \times Y \to \mathbf{R}$ verifying the following assumptions:

$(1.1)_1$ $g(x, \cdot)$ is Y-periodic and measurable for any $x \in \Omega$,

$(1.1)_2$ $g(x, \cdot) \in H^{1-1/p', p'}(\partial T)$,

$(1.1)_3$ $|g(x_1, y) - g(x_2, y)| \leqslant$
$\leqslant c(1 + |\varphi(y)|)\, \omega(|x_1 - x_2|)$ for any $y \in Y$ and $x_1, x_2 \in \Omega$,

where $\omega(t): [0, +\infty) \to [0, +\infty)$ is a bounded, concave and continuous function such that $\omega(0) = 0$ and $\varphi(y) \in L^{p'}(\partial T)$.

We shall consider the problem:

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\operatorname{div} a(x/\varepsilon, Du) = f & \text{in } \Omega_\varepsilon\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ a(x/\varepsilon, Du)\cdot \nu = \varepsilon g_\varepsilon(x) & \text{on } \partial T_\varepsilon\,, \end{cases}$$

where ν denotes the exterior normal with respect to Ω_ε, $g_\varepsilon(x) = g(x, x/\varepsilon)$ and $a(y, \xi)$ verifies the following structure conditions:

H_1) a is Y-periodic and measurable with respect to y,

H_2) for any y a.e. in $\mathbf{R}^n$ and any $\xi_1, \xi_2 \in \mathbf{R}^n$ then

if $p \geqslant 2$:

i) $|a(y, \xi_1) - a(y, \xi_2)| \leqslant \beta(1 + |\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|$

ii) $(a(y, \xi_1) - a(y, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^p$, $\alpha > 0$

or, if $1 < p \leqslant 2$:

j) $|a(y, \xi_1) - a(y, \xi_2)| \leqslant \beta|\xi_1 - \xi_2|^{p-1}$,

jj) $(a(y, \xi_1) - a(y, \xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2}$, $\alpha > 0$,

H_3) $a(y, 0) \in L_n^{p'}(Y)$.

Set

$$V_\varepsilon^p = \{\varphi \in H^{1,p}(\Omega_\varepsilon): \varphi = 0 \text{ on } \partial\Omega\}\,,$$

a variational solution of problem $(\mathcal{T}_\varepsilon)$ is a function $u_\varepsilon \in V^p_\varepsilon$ such that:

$$\int_{\Omega_\varepsilon} a(x/\varepsilon, Du_\varepsilon)\cdot D\varphi\, dx = \int_{\Omega_\varepsilon} f\varphi\, dx + \varepsilon \int_{\partial T_\varepsilon} g_\varepsilon \varphi\, d\sigma \tag{1.2}$$

for any $\varphi \in V^p_\varepsilon$.

REMARK 1.1. It is well known that, under the above hypotheses, the problem $(\mathcal{T}_\varepsilon)$ has a unique solution $u_\varepsilon \in V^p_\varepsilon$. ■

Finally, let us consider the problem:

$$\begin{cases} \int_{Y^*} a(y, Dv(y))\cdot D\varphi(y)\, dy = 0 & \forall \varphi \in H^{1,p}_{\mathrm{per}}(Y^*)\,, \\ v \in \xi\cdot y + H^{1,p}_{\mathrm{per}}(Y^*)\,, \end{cases} \tag{1.3}$$

where $\xi \in \mathbf{R}^n$ and

$H^{1,p}_{\mathrm{per}}(Y^*) =$
$= \{u(y) \in H^{1,p}(Y^*)\colon u$ has the same trace on the opposite faces of $Y\}$.

In the § 3 we will prove the convergence of the solutions $u_\varepsilon \in V^p_\varepsilon$ of $(\mathcal{T}_\varepsilon)$ to the solution of the «homogenized» problem:

$$\begin{cases} -\operatorname{div} b(Du) = f\theta + \mu_g(x) & \text{in } \Omega\,, \\ u \in H^{1,p}_0(\Omega)\,, \end{cases} \tag{$\mathcal{T}_0$}$$

where

$$\mu_g(x) = (1/|Y|)\int_{\partial T} g(x, y)\, d\sigma_y \tag{1.4}$$

and, for any $\xi \in \mathbf{R}^n$, if $v(y)$ is the solution of (1.3), b is defined by

$$b(\xi) = (1/|Y|)\int_{Y^*} a(y, Dv)\, dy\,. \tag{1.5}$$

2. – Preliminary results.

We recall some lemmas about the spaces introduced in § 1 and the existence of a family of extension-operators.

LEMMA 2.1. *If $q(y)$ is an element of $L^{p'}_{n,\mathrm{per}}(Y)$, then it can be extended by periodicity to an element of $L^{p'}_{n,\mathrm{loc}}(\mathbb{R}^n)$, still denoted by q, such that*

$$\operatorname{div}_x q = 0 . \quad \blacksquare$$

LEMMA 2.2. *Let f be an Y-periodic function of $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \leqslant p \leqslant +\infty$ and set*

$$f_\varepsilon(x) = f(x/\varepsilon) \quad \textit{for any } x \textit{ a.e. in } \mathbb{R}^n ,$$

then as $\varepsilon \to 0$:

$$f_\varepsilon \rightharpoonup \langle f \rangle \quad \textit{in } w\text{-}L^p_{\mathrm{loc}}(\mathbb{R}^n) \textit{ if } p < +\infty$$

$$f_\varepsilon \rightharpoonup \langle f \rangle \quad \textit{in } w^*\text{-}L^\infty(\mathbb{R}^n) \textit{ if } p = +\infty . \quad \blacksquare$$

For a proof of the previous lemmas one may see [16] annexe 2.

LEMMA 2.3 (see [7]). *There exists a linear continuous extension-operator $P \in \mathcal{L}(H^{1,p}(Y^*), H^{1,p}(Y))$ such that:*

$$\|D(P\varphi)\|_{L^p_n(Y)} \leqslant \|D\varphi\|_{L^p_n(Y^*)} \tag{2.1}$$

for any $\varphi \in H^{1,p}(Y^*)$. ■

LEMMA 2.4 (see [7]). *There exists a family P_ε of linear continuous extension-operators, $P_\varepsilon \in \mathcal{L}(V^p_\varepsilon, H^{1,p}_0(\Omega))$ verifying the following condition:*

$$\|D(P_\varepsilon\varphi)\|_{L^p_n(\Omega)} \leqslant \|D\varphi\|_{L^p_n(\Omega_\varepsilon)} \tag{2.2}$$

for any $\varphi \in V^p_\varepsilon$, where c is a constant independent of ε. ■

The previous lemmas are proved in the case $p = 2$ in [7]. The same argument can be used in the general case.

If $\gamma \in W^{1-1/p',p'}(\partial T)$ let us consider the linear form on $H^{1,p}(\Omega)$, $1/p + 1/p' = 1$:

$$\langle \mu_\gamma^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial T_\varepsilon} \gamma(x/\varepsilon) \varphi \, d\sigma$$

and

$$\mu_\gamma = (1/|Y|) \int_{\partial T} \gamma(x) \, d\sigma_y .$$

PROPOSITION 2.5 (see [6], [12]). *If $\gamma \in W^{1-1/p',p'}(\partial T)$, $1 < p' < +\infty$, there exists a unique solution $\psi_\gamma \in W^{2,p'}(Y^*)$ of the problem:*

$$(2.3) \qquad \begin{cases} -\Delta \psi_\gamma = -(|Y|/|Y^*|)\mu_\gamma & \text{in } Y^* , \\ \partial \psi_\gamma / \partial \nu = \gamma & \text{on } \partial T , \\ \psi_\gamma \ Y\text{-periodic} , & \\ \langle \psi_\gamma \rangle_{Y^*} = 0 . & \end{cases}$$

In particular if γ is a constant function, then $\psi_\gamma \in W^{1,\infty}(Y^*)$. ■

REMARK 2.6. It is easy to verify that the solution of problem (2.3) can be extended by periodicity to $\mathbb{R}^n \setminus \tau(\overline{T})$ and the function

$$\psi_\gamma^\varepsilon(x) = \psi_\gamma^\varepsilon(x/\varepsilon) \qquad \text{a.e. } x \in \Omega_\varepsilon$$

verify:

$$(2.4) \qquad \begin{cases} -\Delta \psi_\gamma^\varepsilon = -\varepsilon^{-2}(|Y|/|Y^*|)\mu_\gamma & \text{in } \mathbb{R}^n \setminus \tau(\varepsilon\overline{T}) , \\ \partial \psi_\gamma^\varepsilon / \partial \nu = \varepsilon^{-1} \gamma(x/\varepsilon) & \text{on } \partial \tau(\varepsilon \overline{T}) . \end{cases} \quad ■$$

Now, we can prove the following lemma that we'll use in the sequel:

LEMMA 2.7. *Let Q be an interval of $\mathbb{R}^n$, $S_\varepsilon = \tau(\varepsilon\overline{T}) \cap Q$ and $Q_\varepsilon = Q \setminus \overline{S}_\varepsilon$. If $\overline{S}_\varepsilon \cap \partial Q = \varphi$ then:*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\partial S_\varepsilon} \gamma(x/\varepsilon) \varphi_\varepsilon(x) \, d\sigma = \mu_\gamma \int_Q \varphi(x) \, dx$$

for any sequence $\{\varphi_\varepsilon\}$ of $H^{1,p}(\Omega)$ such that $\varphi_\varepsilon \to \varphi$ in $w\text{-}H^{1,p}(\Omega)$, $1 < p < +\infty$.

PROOF. Let ψ_γ be the solution of (2.3), then by remark 2.6 we have:

$$\varepsilon\int_{\partial S_\varepsilon}\gamma(x/\varepsilon)\varphi_\varepsilon(x)\,d\sigma = \tag{2.5}$$
$$= \varepsilon^2\int_{Q_\varepsilon} D_x\psi^\varepsilon_\gamma\cdot D_x\varphi_\varepsilon\,dx + (|Y|/Y^*|)\mu_\gamma\int_{Q_\varepsilon}\varphi_\varepsilon\,dx - \varepsilon^2\int_{\partial Q} D_x\psi\varphi^\varepsilon_\gamma\cdot\nu_i\,d\sigma\,.$$

Then by lemma 2.2 and by observing that $D_x\psi^\varepsilon_\gamma(x) = (1/\varepsilon)D_y\psi_\gamma(x/\varepsilon)$, passing to the limit as $\varepsilon\to 0$ we get the result. ■

REMARK 2.8. If h is a constant function, by using similar arguments as in the previous lemma, it can be found that

$$\varepsilon\int_{\partial S_\varepsilon}|\varphi_\varepsilon|\,d\sigma\leqslant c$$

(with c independent of ε) for any sequence $\{\varphi_\varepsilon\}$ bounded in $H^{1,1}(Q)$. ■

Let $g\colon \Omega\times\mathbb{R}^n\to\mathbb{R}$ verifying $(1.1)_1\div(1.1)_3$ and

$$\langle\mu^\varepsilon_g,\varphi\rangle = \varepsilon\int_{\partial T_\varepsilon} g(x, x/\varepsilon)\varphi\,d\sigma \qquad \forall\varphi\in H^{1,p}_0(\Omega)\,,\quad 1<p<+\infty\,.$$

LEMMA 2.9. *The measures* μ^ε_g, μ_g *are in* $H^{-1,p'}(\Omega)$ *and*

$$\mu^\varepsilon_g\to\mu_g \qquad \textit{in } s\text{-}H^{-1,p'}(\Omega)\,,$$

where μ_g *is given by* (1.4).

PROOF. It is enough to prove that

$$\langle\mu^\varepsilon_g,\varphi_\varepsilon\rangle\to\langle\mu_g,\varphi\rangle \tag{2.6}$$

if $\varphi_\varepsilon,\varphi\in H^{1,p}_0(\Omega)$ and $\varphi_\varepsilon\to\varphi$ in w-$H^{1,p}_0(\Omega)$.

Let us consider $\forall\nu\in\mathbb{N}$ a partition of $\mathbb{R}^n$ by intervals $Q_{i\nu}$ of side $2^{-\nu}l_i$. Since the holes T_ε do not intersect $\partial\Omega$, we can assume that $\forall\varepsilon>0$, T_ε does not intersect $\partial Q_{i\nu}$.

Let us denote by $x_{i\nu}$ and $\chi_{i\nu}$ respectively the center and the characteristic function of $Q_{i\nu}$. Set

$$g_\nu(x, y) = \sum_i \chi_{i\nu}(x) g(x_{i\nu}, y) .$$

We have

$$\varepsilon\int_{\partial T_\varepsilon} g(x, x/\varepsilon)\varphi_\varepsilon\, d\sigma = \varepsilon\int_{\partial T_\varepsilon}[g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)]\varphi_\varepsilon\, d\sigma + \varepsilon\int_{\partial T_\varepsilon} g_\nu(x, x/\varepsilon)\varphi_\varepsilon\, d\sigma .$$

We observe that by $(1.1)_3$ $\mu_g \in C^0(\bar{\Omega})$, then by lemma 2.7 we have:

$$(2.7) \qquad \lim_\nu \lim_{\varepsilon\to 0} \varepsilon\int_{\partial T_\varepsilon} g_\nu(x, x/\varepsilon)\varphi_\varepsilon(x)\, d\sigma = \lim_\nu \lim_{\varepsilon\to 0} \sum_i \varepsilon \int_{Q_{i\nu}\cap\partial T_\varepsilon} g(x_{i\nu}, x/\varepsilon)\varphi_\varepsilon(x)\, d\sigma =$$

$$= \lim_\nu \sum_i \int_{Q_{i\nu}\cap\Omega} \mu_g(x_{i\nu})\varphi(x)\, dx = \lim_\nu \int_\Omega \sum_i \chi_{i\nu}\mu_g(x_{i\nu})\varphi(x)\, dx =$$

$$= \int_\Omega \mu_g(x)\varphi(x)\, dx .$$

On the other hand we have:

$$(2.8) \quad \varepsilon\int_{\partial T_\varepsilon}[g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)]\varphi_\varepsilon(x)\, d\sigma \leqslant$$

$$\leqslant \varepsilon^{1/p'}\Big[\sum_i \int_{Q_{i\nu}\cap\partial T_\varepsilon} |g(x, x/\varepsilon) - g(x_{i\nu}, x/\varepsilon)|^{p'}\, d\sigma\Big]^{1/p'} \varepsilon^{1/p}\Big[\sum_i \int_{Q_{i\nu}\cap\partial T_\varepsilon} |\varphi_\varepsilon(x)|^p\, d\sigma\Big]^{1/p} .$$

By remark 2.8 we get

$$(2.9) \qquad \varepsilon^{1/p}\Big[\sum_i \int_{Q_{i\nu}\cap\partial T_\varepsilon} |\varphi_\varepsilon(x)|^p\, d\sigma\Big]^{1/p} \leqslant c .$$

Then by (2.8), (2.9) and $(1.1)_1 \div (1.1)_3$:

$$\varepsilon\int_{\partial T_\varepsilon}[g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)]\,\varphi_\varepsilon(x)\, d\sigma \leqslant c\omega(1/2^\nu)\Big[\varepsilon\int_{\partial T_\varepsilon}(1 + |\varphi(x/\varepsilon)|^{p'})\, d\sigma\Big]^{1/p'} \leqslant$$

$$\leqslant c'\,\omega(1/2^\nu)\Big[\varepsilon\int_{\partial T}(1 + |\varphi(y)|^{p'})\, d\sigma\Big]^{1/p'} .$$

Then passing to the limit as $\varepsilon \to 0^+$ and $\nu \to +\infty$ we obtain (2.6).

The proof is completed by using standard arguments concerning the duality application between $H_0^{1,p}(\Omega)$ and its dual. ■

We state now some lemmas about the structure properties of b. Using the same arguments of [11] one can prove:

LEMMA 2.10. *For any* $\xi \in \mathbb{R}^n$

$$|b(\xi)| \leqslant c(1 + |\xi|)^{p-1}$$

where $c = c(\alpha, \beta, p|Y|, \|a(y, 0)\|_{L_n^{p'}(Y)})$.

Further if v *is the solution of problem* (1.3) *we have*:

$$\int_{Y^*} |Dv(y)|^p \, dy \leqslant c(1 + |\xi|)^p \tag{2.10}$$

where $c = c(\alpha, \beta, p, |Y|, \|a(y, 0)\|_{L_n^{p'}(Y)})$. ■

LEMMA 2.1. $b(\xi)$ *is locally Holder* (*Lipschitz if* $p = 2$). ■

REMARK 2.12. We remark that the limit operator $b(\xi)$, as in the homogenization of Dirichlet problem (see [11]) may not verify the same structure conditions of $a(y, \xi)$.

In some special case, the Holder estimate on $b(\xi)$ can be improved (see [11]). ■

LEMMA 2.13. *For* $\xi_1, \xi_2 \in \mathbb{R}^n$ *we have*

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^p, \quad \text{if } p > 2 \tag{2.11}$$

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geqslant \alpha'|\xi_1 - \xi_2|^2(1 + |\xi_1| + |\xi_2|)^{p-2}, \quad \alpha' > 0, \quad \text{if } 1 < p \leqslant 2. \tag{2.12}$$

PROOF. Let us denote by v_1 and v_2 the solutions of problem (1.3) defining respectively $b(\xi_1)$ and $b(\xi_2)$.

Let us consider $u_i = v_i - \xi \cdot y$, $i = 1, 2$. Then $u_i(y)$ is an element of $H^{1,p}_{\text{per}}(Y^*)$ and so, by lemma 2.3, we can consider $Pu_i \in H^{1,p}_{\text{per}}(Y)$.

If we extend Pu_i by periodicity, the resulting function (still denoted by Pu_i) is in $H^{1,p}_{\rm loc}(\mathbb{R}^n)$.

Define

$$w_i^\varepsilon = \varepsilon Pu_i(x/\varepsilon) + \xi_i \cdot x\ , \qquad i = 1, 2\ ,$$

and for $\xi \in \mathbb{R}^n$

$$\tilde{a}(y, \xi) = \begin{cases} a(y, \xi) & \text{for } y \in Y^*\ , \\ 0 & \text{for } y \in \overline{T}\ . \end{cases} \tag{2.13}$$

It is easy to verify that

$$\begin{cases} w_i^\varepsilon \to \xi_i \cdot x & \text{in } w\text{-}H^{1,p}_{\rm loc}(\mathbb{R}^n)\ , \\ \tilde{a}(x/\varepsilon, Dw_i^\varepsilon) \to b(\xi_i) & \text{in } w\text{-}L^{p'}_{n,\rm loc}(\mathbb{R}^n)\ , \\ \operatorname{div} \tilde{a}(x/\varepsilon, Dw_i^\varepsilon) = 0\ , & \end{cases} \tag{2.14}$$

where the last relation is proved by using lemma 2.1.

If $p \geqslant 2$ from ii) of H_2), we get:

$$\alpha \int_{Y^*} \eta |Dw_1^\varepsilon - Dw_2^\varepsilon|^p \, dx \leqslant \int_Y \eta \big(\tilde{a}(x/\varepsilon, Dw_1^\varepsilon) - \tilde{a}(x/\varepsilon, Dw_2^\varepsilon), Dw_1^\varepsilon - Dw_2^\varepsilon\big)\, dx$$

where $\eta \in C_0^1(Y^*)$.

Then, passing to the limit as $\varepsilon \to 0$ and using (2.14), by the compensated compactness result of [12] we get

$$\alpha \int_{Y^*} \eta |\xi_1 - \xi_2|^p \leqslant \int_{Y^*} \eta \big(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2\big)\, dx\ .$$

Then from the arbitrarity of η, we deduce (2.11).

If $1 < p \leqslant 2$, from jj) of H_2) we have:

$$\sqrt{\alpha} \int_{Y^*} |Dw_1^\varepsilon - Dw_2^\varepsilon|\, dx \leqslant \Big(\int_Y \eta \big(\tilde{a}(x/\varepsilon, Dw_1^\varepsilon) - \tilde{a}(x/\varepsilon, Dw_2^\varepsilon), Dw_1^\varepsilon - Dw_2^\varepsilon\big)\, dx\Big)^{\frac{1}{2}} \cdot$$

$$\cdot \Big(\int_Y \eta (|Dw_1^\varepsilon| + |Dw_2^\varepsilon|)^{2-p}\, dx\Big)^{\frac{1}{2}}\ .$$

Then, passing to the limit as before, and remarking that by lemma 2.3

$$\int_Y |Dw_i^\varepsilon|^p\,dx \leqslant c\int_{Y^*} |Dv_i^\varepsilon|^p\,dy\,,$$

we can argue in a similar way to prove (2.12). ■

3. – Homogenization results.

THEOREM 3 1. *If* $a(x, \xi)$ *verifies the structure conditions* $H_1) \div H_3)$ *and* $g\colon \Omega\times Y \to \mathbb{R}$ *satisfies* $(1.1)_1 \div (1.1)_3$, *then for any* $f \in L^{p'}(\Omega)$ *the sequence* u_ε *of the solutions of problem* $(\mathcal{T}_\varepsilon)$ *verifies*:

$$P_\varepsilon u_\varepsilon \to u_0 \qquad \text{in } w\text{-}H_0^{1,p}(\Omega)\,,$$

$$\tilde{a}(x/\varepsilon, Du_\varepsilon) \to b(Du_0) \qquad \text{in } w\text{-}L_n^{p'}(\Omega)\,,$$

where u_0 *is the solution of problem* $(\mathcal{T}_0)$.

PROOF. We use the homogenization techniques of nonlinear operators introduced in [10], [11].

Let us denote by $P_\varepsilon u_\varepsilon$ the extension of u_ε given by lemma 2.4. By using lemma 2.9, it is very easy to verify that

$$\|P_\varepsilon u_\varepsilon\|_{H_0^{1,p}(\Omega)} \leqslant c$$

with c independent of ε.

Then by i) or j) of H_2), we get also:

$$\|\tilde{a}(x/\varepsilon, Du_\varepsilon)\|_{L_n^{p'}} \leqslant c$$

with c independent of ε.

Hence, up to a subsequence, we have

$$P_\varepsilon u_\varepsilon \to u_0 \qquad \text{in } w\text{-}H_0^{1,p}(\Omega)\,,$$

$$\tilde{a}(x/\varepsilon, Du_\varepsilon) \to a_0(x) \qquad \text{in } w\text{-}L_n^{p'}(\Omega)\,.$$

The theorem will be proved if we shaw that:

$$a_0(x) = b(Du_0) \quad \text{a.e. in } \Omega . \tag{3.1}$$

Indeed, using lemma 2.9 and the fact that

$$\chi_{\Omega_\varepsilon} \to \theta \quad \text{in } w^*\text{-}L^\infty(\Omega) \tag{3.2}$$

we obtain:

$$\int_\Omega a_0(x) D\varphi\, dx = \int_\Omega \theta f \varphi\, dx + \int_\Omega \mu_g \varphi\, dx \qquad \forall \varphi \in H_0^{1,p}(\Omega) .$$

Let us fix $\nu \in \mathbf{N}$ and denote by $\{Q_{i\nu}\}_i$ a partition of $\mathbf{R}^n$ as in the proof. of lemma 2.7. Then we define $I_\nu = \{i\colon Q_{i\nu} \subset \Omega\}$, $\Omega_\nu = \bigcup_{i \in I_\nu} Q_{i\nu}$. For any i let us consider $\langle Du_0 \rangle_{i\nu} = \langle Du_0 \rangle_{Q_{i\nu}}$. Then if $\chi_{i\nu}$ is the characteristic function of $Q_{i\nu}$, by the continuity of b (see lemma 2.11) we have, if $\nu \to +\infty$, that:

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \to b(Du_0(x)) \qquad \text{a.e. in } \Omega . \tag{3.3}$$

Moreover, from lemma 2.10 we have for any measurable set $E \subset \Omega$:

$$\int_E \Big| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \Big|^{p'} dx \leqslant c \int_E \Big(1 + \Big| \sum_{i \in I_\nu} \chi_{i\nu}(x) \langle Du_0 \rangle_{i\nu} \Big)\Big|^p dx .$$

So, from the equi-absolute continuity of the integral on the left-hand side and from (3.3) we deduce that:

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \to b(Du_0(x)) \quad \text{in } L_n^{p'}(\Omega) , \quad \text{as } \nu \to +\infty . \tag{3.4}$$

Let $v_{i\nu} \in \langle Du_0 \rangle_{i\nu} \cdot y + H^{1,p}_{\text{per}}(Y^*)$ the solution of problem (1.3) corresponding to $\langle Du_0 \rangle_{i\nu}$:

Then:

$$u_{i\nu} = (v_{i\nu} - \langle Du_0 \rangle_{i\nu} \cdot y) \in H^{1,p}_{\text{per}}(Y^*)$$

and, by lemma 2.3, $Pu_{i\nu} \in H^{1,p}_{\text{per}}(Y)$.

Set

$$w_{i\nu}(y) = Pu_{i\nu}(y) - \langle Du_0\rangle_{i\nu}\cdot y$$

and

$$w^{\varepsilon}_{i\nu}(x) = \varepsilon w_{i\nu}(x/\varepsilon)\ ,$$

by arguing as in the proof of lemma 2.13 we obtain:

$$\text{(3.5)}\qquad \begin{cases} w^{\varepsilon}_{i\nu} \to \langle Du_0\rangle_{i\nu}\cdot x & \text{in } w\text{-}H^{1,p}_{\text{loc}}(\mathbb{R}^n)\ ,\\ \tilde a(x/\varepsilon, Dw^{\varepsilon}_{i\nu}) \to b(\langle Du_0\rangle_{i\nu}) & \text{in } w\text{-}L^{p'}_{n,\text{loc}}(\mathbb{R}^n)\ ,\\ \operatorname{div}_x \tilde a(x/\varepsilon, Dw^{\varepsilon}_{i\nu}) = 0\ . \end{cases}$$

Using the periodicity of $Pu_{i\nu}$ and lemma 2.3, we have:

$$\sum_{i\in I_\nu}\int_{Q_{i\nu}} |Dw^{\varepsilon}_{i\nu}|^p\,dx \leqslant \sum_{i\in I_\nu} 2^{-\nu n}\varepsilon^n(1/\varepsilon + 2^\nu)^n \int_Y |Dw^{\varepsilon}_{i\nu}(y)|^p\,dy \leqslant$$

$$\leqslant c\sum_{i\in I_\nu} 2^{-\nu n}(1+\varepsilon^n 2^{\nu n})\left(\int_{Y^*}|Dv_{i\nu}|^p\,dy + |\langle Du_0\rangle_{i\nu}|\right)^p .$$

The from (2.10), writing the last term as an integral over Ω_ν we have:

$$\text{(3.6)}\qquad \sum_{i\in I_\varepsilon}\int_{Q_{i\nu}} |Dw^{\varepsilon}_{i\nu}|^p\,dx \leqslant c(1+\varepsilon^n 2^{\nu n})\int_\Omega (1+|Du_0|)^p\,dx\ .$$

Let now $\eta\in C^1_0(Q_{i\nu})$, $0\leqslant\eta\leqslant 1$, and extend it by periodicity to the whole $\mathbb{R}^n$.

Case $p\geqslant 2$. If $\varphi\in C^0_n(\bar\Omega)$, set $M_\varphi = \sup_\Omega|\varphi|$, from i) of H_2) we have:

$$\text{(3.7)}\qquad \left|\int_\Omega \tilde a(x/\varepsilon, Du_\varepsilon)\varphi\eta\,dx - \sum_{i\in I_\nu}\int_{Q_{i\nu}}\tilde a(x/\varepsilon, Dw^{\varepsilon}_{i\nu})\varphi\eta\,dx\right| \leqslant$$

$$\leqslant cM_\varphi|\Omega\setminus\Omega_\nu|^{1/p} + \left|\sum_{i\in I_\nu}\int_{Q_{i\nu}\cap\Omega_\varepsilon}[a(x/\varepsilon, Du_\varepsilon) - a(x/\varepsilon, Dw^{\varepsilon}_{i\nu})]\varphi\eta\,dx\right| \leqslant$$

$$\leqslant cM_\varphi|\Omega\setminus\Omega_\nu|^{1/p} + \sum_{i\in I_\nu}\int_{Q_{i\nu}\cap\Omega_\varepsilon} M_\varphi\,\eta\{(|Du_\varepsilon| + |Dw^{\varepsilon}_{i\nu}|)^{p-2}\cdot|Du_\varepsilon - Dw^{\varepsilon}_{i\nu}|\}\,dx \leqslant$$

$$\leqslant cM_\varphi|\Omega\setminus\Omega_\nu|^{1/p} + cM_\varphi^{p/(p-1)}\delta^{p/(p-1)}(1+\varepsilon^n 2^{\nu n}) +$$

$$+\ \delta^{-p}\sum_{i\in I_\nu}\int_{Q_{i\nu}\cap\Omega_\varepsilon}\eta|Du_\varepsilon - Dw^{\varepsilon}_{i\nu}|^p\,dx$$

where the last inequality is obtained by applying Young inequality with $\delta > 0$ and the estimate (3.5).

On the other hand from ii)

$$\sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} \eta|Du_\varepsilon - Dw^\varepsilon_{i\nu}|^p\,dx \leqslant$$
$$\leqslant \sum_{i\in I_\nu} \int_{Q_{i\nu}} \eta(\tilde a(x/\varepsilon, Du_\varepsilon) - \tilde a(x/\varepsilon, Dw^\varepsilon_{i\nu}), Du_0 - \langle Du_0\rangle_{i\nu})\,dx +$$
$$+ \sum_{i\in I_\nu} \int_{Q_{i\nu}} \eta(\tilde a(x/\varepsilon, Du_\varepsilon) - \tilde a(x/\varepsilon, Dw^\varepsilon_{i\nu}), (DP_\varepsilon u_\varepsilon - Dw^\varepsilon_{i\nu}) -$$
$$- (Du_0 - \langle Du_0\rangle_{i\nu}))\,dx\,.$$

Then, integrating by parts and using (3.5) and the fact that u_ε is the solution of $(\mathcal{T}_\varepsilon)$, we get:

$$\sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} \eta|Du_\varepsilon - Dw^\varepsilon_{i\nu}|^p\,dx \leqslant \tag{3.8}$$
$$\leqslant \sum_{i\in I_\nu} \int_{Q_{i\nu}} \eta(\tilde a(x/\varepsilon, Du_\varepsilon) - \tilde a(x/\varepsilon, Dw^\varepsilon_{i\nu}), Du_0 - \langle Du_0\rangle_{i\nu})\,dx +$$
$$+ \sum_{i\in I_\nu} \int_{Q_{i\nu}} [\eta f\chi_{\Omega_\varepsilon} - D\eta(\tilde a(x/\varepsilon, Du_\varepsilon) - \tilde a(x/\varepsilon, Dw^\varepsilon_{i\nu}))]\cdot$$
$$\cdot[(P_\varepsilon u_\varepsilon - u_0) - (w^\varepsilon_{i\nu} - \langle Du_0\rangle_{i\nu}\cdot x)]\,dx +$$
$$+ \langle \mu^\varepsilon_g, \eta[(P_\varepsilon u_\varepsilon - u_0) - (w^\varepsilon_{i\nu} - \langle Du_0\rangle_{i\nu}\cdot x)]\rangle\,.$$

Then, passing to the limit as $\varepsilon \to 0$ in (3.8), by (3.2), (3.5) and lemma 2.9, we obtain

$$\lim_{\varepsilon\to 0} \sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} \eta|Du_\varepsilon - Dw^\varepsilon_{i\nu}|^p\,dx \leqslant \tag{3.9}$$
$$\leqslant \sum_{i\in I_\nu} \int_{Q_{i\nu}} \eta(a_0(x) - b(\langle Du_0\rangle_{i\nu}), Du_0 - \langle Du_0\rangle_{i\nu})\,dx\,.$$

If we first pass to the limit in (3.7) as $\varepsilon \to 0$, then $\eta \to 1$, $\nu \to +\infty$, by (3.9) we get:

$$\left|\int_\Omega a_0(x)\varphi\,dx - \int_\Omega b(Du_0)\,\varphi\,dx\right| \leqslant cM_\varphi^{p/(p-1)}\delta^{p/(p-1)}$$

So, letting $\delta \to 0$, from the arbitrariety of φ we deduce (3.1).

Case $1 < p \leqslant 2$. In this case the proof is very similar to the previous case. Indeed, by using j) of H_2), we have:

$$\left|\int_\Omega \tilde{a}(x/\varepsilon, Du_\varepsilon)\varphi\eta\, dx - \sum_{i\in I_\nu} \int_{Q_{i\nu}} \tilde{a}(x/\varepsilon, Dw^\varepsilon_{i\nu})\varphi\eta\, dx\right| \leqslant$$

$$\leqslant cM_\varphi |\Omega\setminus\Omega_\nu|^{1/p} + \beta \sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} |Du_\varepsilon - Dw^\varepsilon_{i\nu}|^{p-1}|\varphi|\,\eta\, dx\,.$$

Then, using jj) we can control the last term:

$$\sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} |Du_\varepsilon - Dw^\varepsilon_{i\nu}|^{p-1}|\varphi|\,\eta\, dx \leqslant$$

$$\leqslant c\delta^{2/(3-p)} \sum_{i\in I_\nu} \int_{Q_{i\nu}\cap\Omega_\varepsilon} \left(|Du_\varepsilon| + |Dw^\varepsilon_{i\nu}|\right)^{(2-p)(p-1)/(3-p)} dx +$$

$$+ \delta^{-2/(p-1)} \sum_{i\in I_\nu} \int_{Q_{i\nu}} \eta\left(\tilde{a}(x/\varepsilon, Du_\varepsilon) - \tilde{a}(x/\varepsilon, Dw^\varepsilon_{i\nu}), DP_\varepsilon u_\varepsilon - Dw^\varepsilon_{i\nu}\right)\eta\, dx\,.$$

Hence, arguing as in the previous case we obtain the result. ■

An easy consequence of theorem 3.1 is the following result:

COROLLARY 3.2. *Assume that* $a(x, \xi)$ *verifies the structure conditions* $H_1) \div H_3)$ *and is homogeneous of degree* $p-1$ *with respect to* ξ.

Then for any $f \in L^{p'}(\Omega)$ *and* $g(x, y)$ *verifying* $(1.1_1) \div (1.1)_3$ *with* $\mu_g \neq 0$, *the function* $b(\xi)$ *is homogeneous of degree* $p-1$ *and the sequence* v_ε *of the solutions of the problem*:

$$\begin{cases} -\operatorname{div} a(x/\varepsilon, Dv_\varepsilon) = f & \text{in } \Omega_\varepsilon\,, \\ v_\varepsilon = 0 & \text{on } \partial\Omega\,, \\ a(x/\varepsilon, Dv_\varepsilon)\cdot\nu = g_\varepsilon & \text{on } \partial T_\varepsilon\,, \end{cases}$$

verifies:

$$\varepsilon^{1/(p-1)} P_\varepsilon v_\varepsilon \to v_0 \qquad \text{in } w\text{-}H_0^{1,p}(\Omega)\,,$$

$$\varepsilon\tilde{a}(x/\varepsilon, Dv_\varepsilon) \to b(Dv_0) \qquad \text{in } w\text{-}L_n^{p'}(\Omega)\,,$$

where v_0 *is the solution of the problem*:

$$-\operatorname{div} b(Dv_0) = \mu_g \quad \textit{in } \Omega_\varepsilon ,$$
$$v_0 \in H_0^{1,p}(\Omega) ,$$

with $b(\xi)$ *and* μ_g *given respectively by* (1.4), (1.5) *and* (1.3).

PROOF. The result follows by applying theorem 3.2 to the sequence $u_\varepsilon = \varepsilon^{1/(p-1)} v_\varepsilon$. ∎

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