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Nilpotent Groups of Class Two that Can Appear as Central Quotient Groups.

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In this note we will be concerned with the following question: Suppose $C_p \times C_p = G' \subseteq Z(G)$. What can be said about $G/Z(G)$ if G is isomorphic to some central quotient group $H/Z(H)$ of a group H? The answer to the corresponding question for $|G'| = p$ is wellknown for a long time; it is $|G/Z(G)| = p^2$ (see for instance Beyl and Tappe [1; p. 233]).

The proof of the answer (Proposition 3) makes use of our knowledge about vector spaces with two alternating bilinear forms. The bounds obtained are strict for odd primes p; this is shown in the second section. In the third section we give an example of a group G such that

$$G_3 = G^p = 1\,, \qquad |G'| = |Z(G)| = p^n\,, \qquad |G/G'| = p^{2n+\binom{n}{2}}$$

and G is a central quotient. This shows at least quadratic growth for the upper bound of the rank of $G/Z(G)$ with growing rank of G'.

1. The bounds for $|G/Z(G)|$.

In what follows we will have to deal with vector spaces V with two alternating bilinear forms f_1, f_2 which are of a comparatively transparent structure: There are two linear combinations

$$g_1 = af_1 + bf_2 \quad \text{and} \quad g_2 = cf_1 + df_2$$

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and a basis $x_1, \ldots, x_m$ of V such that

$$g_1(x_{2t-1}, x_{2t}) = g_2(x_{2t}, x_{2t+1}) = 1 ,$$

$$g_1(x_{2t}, x_{2t+1}) = g_2(x_{2t-1}, x_{2t}) = 0 \quad \text{for all } t \leqslant \frac{m}{2} ,$$

$$g_1(x_i, x_j) = g_2(x_i, x_j) = 0 \quad \text{for } |i-j| > 1 .$$

For brevity we will call such a vector space a string with respect to g_1 and g_2. If $\dim V$ is odd, V is also a string with respect to any two different linear combinations of f_1, f_2; on the other hand if $\dim V$ is even, g_1 is fixed and g_2 can be changed to any other linear combination different from g_1. The reference to the respective bilinear forms will mostly be unnecessary and is then omitted.

A direct sum is called an orthogonal sum, if in addition the summands are orthogonal to each other with respect to all bilinear forms considered.

Proposition 1. If two alternating forms f_1, f_2 are defined on a finite dimensional vector space V, then V is the orthogonal sum

$$V = R \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_t ,$$

where every linear combination $rf_1 + sf_2 \not\equiv 0$ is nondegenerate on R and $X_1, \ldots, X_t$ are strings. Any two such decompositions of V are of the same form.

Proof. Following Scharlau [2], we can compare the finite dimensional vector space V possessing two alternating forms with a Kronecker module consisting of two spaces and two endomorphisms mapping the first space into the second: for the first space take a subspace which is maximal with respect to both forms reducing to the zero form on it, for the second take the dual of the respective quotient space. Scharlau proves [2; 3.*e*, Theorem p. 14] that the decomposition of such a vector space into an unrefineable orthogonal sum is unique up to isomorphism. For R we take the sum of such summands with all linear combinations $rf_1 + rf_2$ non-degenerate; the remaining summands correspond to Kronecker modules M_2, L_n, L_n^* as described in [2; p. 16], and these are strings in our sense.

PROPOSITION 2. Denote by V a finite-dimensional vector space with two alternating forms f_1 and f_2 such that every lineare combination $rf_1 + rf_2$ is non-degenerate on V. Then

(i) $\dim V$ is even, and at least 4.

(ii) If W is a subspace of V of codimension 1, then $W = T \oplus S$, where every linear combination $rf_1 + rf_2$ is non-degenerate on T while S is a string of odd dimension.

PROOF. Since f_1 is nondegenerate on V, $\dim V$ is even, for $\dim V = 2$ f_1 and f_2 are linearly dependent. On the other hand, $\dim W = \dim V - 1$ is odd, and every linear combination $rf_1 + sf_2$ is degenerate on W, so W is an orthogonal sum with at least one string, and one string of odd dimension. Since every linear combination $rf_1 + sf_2$ is nondegenerate on V, $(rf_1 + sf_2)(x, W) = 0$ has as space of solution a subspace of dimension $\dim V - \dim W = 1$ at most. This shows that there is at most one string, it contains all the solutions mentioned.

In the sequel we make use of the following well known fact: If G is a p-group of nilpotency class 2 and $G' = \langle a \rangle \times \langle b \rangle$ with $p^p = b^p = 1$, then the mapping

$$(x, y) \rightarrow [x, y] = a^{r(x,y)} b^{s(x,y)}$$

induces two alternating bilinear forms on $G/Z(G)$. This allows us to argue from vector spaces to groups and back. This argument can be found operating in Vishnevetskii [3], for instance.

PROPOSITION 3. If G is a finite group such that

$$C_p \times C_p = G' \subseteq Z(G)$$

and there is a group H such that $G = H/Z(H)$, then

$$p^2 < |G/Z(G)| < p^6 .$$

PROOF. The first part of the inequality $p^2 < G/Z(G)$ is obvious. For the other we begin with some preliminary statements. We assume that G is isomorphic to some quotient $H/Z(H)$ and deduce restrictions on G.

(*a*) If $G = UV$ and $[U, V] = 1$, then $U' \cap V' = 1$.

If, on the contrary, $G = UV$ with $[U, V] = 1$ and $U' \cap V' \neq 1$ and $G = H/Z(H)$, we choose a basis $u_1, \dots, u_r, v_1, \dots, v_s$ of G such that the elements u_i belong to U and the v_j's to V. Since $U \cap V \subseteq Z(G)$ we have $\langle u_1, \dots, u_r \rangle Z(G) = UZ(G)$ and $\langle v_1, \dots, v_s \rangle Z(G) = VZ(G)$.

The pre-image of the element x of G with respect to the mapping of H onto $G = H/Z(H)$ shall be denoted by $\tilde{x}$.

Now

$$[[v_i, v_j], u_k] = [[u_k, v_i], v_j]^{-1}[[v_j, u_k], v_i]^{-1} = 1 \qquad \text{for all } i, j, k .$$

The same holds if the roles of U and V are interchanged. Take

$$\prod [u_i, u_j]^{n_{ij}} = \prod [v_i, v_j]^{m_{ij}} = c \neq 1 .$$

Then $\tilde{c} \notin Z(H)$ and $[\tilde{c}, \tilde{v}_k] = [\tilde{c}, \tilde{u}_k] = 1$ for all k, a contradiction. So (a) is true.

(b) If $G = UV$ with $[U, V] = 1$ and $UZ(G) \neq Z(G) \neq VZ(G)$, then $G/G' = p^4$.

By (a) we have $U' \cap V' = 1$, and the hypothesis yields $U' \neq 1 \neq V'$. So both commutator subgroups U', V' have order p and $UZ(G)/Z(G)$ and $VZ(G)/Z(G)$ are elementary abelian of even rank. Using (a) again we see that both these quotient groups must be of order p^2 and (b) follows.

From now on we consider $G/Z(G)$ as a F_p-vector space V with two alternating forms, as outlined just before this Proposition. By (b) we have

(c) If $\dim V > 4$, there is no proper decomposition of V into an orthogonal sum.

We assume $\dim V = m > 5$. If there is a linear combination $rf_1 + sf_2$ $((r, s) \neq (0, 0))$ which is degenerate on V, then V is a string with respect to $rf_1 + sf_2 = g$ and one of f_1, f_2, say f. So there are generators $x_1Z(G), \dots, x_mZ(G)$ of $G/Z(G)$ and a, b of G' such

$$[x_{2t-1}, x_{2t}] = a , \quad [x_{2t}, x_{2t+1}] = b , \quad [x_h, x_k] = 1 \quad \text{for } |h - k| > 1 .$$

Using pre-images as before we have

$$[[\tilde{x}_1, \tilde{x}_2], \tilde{x}_k] = [[\tilde{x}_2, \tilde{x}_k], \tilde{x}_1]^{-1}[[\tilde{x}_k, \tilde{x}_1], \tilde{x}_2]^{-1} = 1 \quad \text{for } k > 3 ,$$

and, using the same argument,

$$[[\tilde{x}_1, \tilde{x}_2], \tilde{x}_k] = [[\tilde{x}_5, \tilde{x}_6], \tilde{x}_k] = 1 \quad \text{for } k \leqslant 3 ,$$
$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_k] = 1 \quad \text{for } k > 4 ,$$
$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_k] = [[\tilde{x}_4, \tilde{x}_5], \tilde{x}_k] = 1 \quad \text{for } k \leqslant 2 ,$$
$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3] = [[\tilde{x}_4, \tilde{x}_5], \tilde{x}_3] = [[\tilde{x}_5, \tilde{x}_3], \tilde{x}_4]^{-1}[[\tilde{x}_3, \tilde{x}_4], \tilde{x}_5]^{-1} = 1 ,$$
$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_2]^{-1}[[\tilde{x}_4, \tilde{x}_2], \tilde{x}_3]^{-1} = 1 ,$$

and neither $\tilde{a}$ nor $\tilde{b}$ are outside $Z(H)$, a contradiction. This shows

(*d*) If V is a string, $\dim V \leqslant 5$.

Assume now that V is not a string but all bilinear forms are nondegenerate on V and $\dim V \geqslant 6$. Consider a subspace W of codimension 1 of V; by Proposition 2 we know that W is the orthogonal sum of a completely nondegenerate part and a string. If $\dim V \geqslant 8$ either the orthogonal sum is nontrivial and $\tilde{a}$, $\tilde{b}$ commute with all elements of the pre-image of W, or W is a string of dimension 7 at least, with the same consequence. Since this holds for all W, this also holds for V, a contradiction. We have found

(*e*) $\dim V \geqslant 7$.

If $\dim V = 6$, each W must be a string by Proposition 2 (i). We choose a basis $x_1 Z(G), \ldots, x_6 Z(G)$ of $G/Z(G)$ and determine the maximal subgroups U_i of G such that $[U_i, x_i] = \langle b \rangle$.

We have corresponding subspaces W_i of codimension 1 of V. These subspaces are strings and allow a basis as a string such that x_i appears as the first basis element y_1. Now

$$[[\tilde{y}_1, \tilde{y}_2], \tilde{y}_1] = [[\tilde{y}_3, \tilde{y}_4], \tilde{y}_1] = 1 ,$$
$$[[\tilde{y}_2, \tilde{y}_3], \tilde{y}_1] = [[\tilde{y}_4, \tilde{y}_5], \tilde{y}_1] = 1 ,$$

and $\tilde{a}$, $\tilde{b}$ commute with every of the $\tilde{x}_i$, the final contradiction

(*f*) $\dim V < 6$,

and this proves the Proposition.

2. Construction of some groups H.

To show that Proposition 3 is in a sense bestpossible we construct groups H for the case

$$G' = Z(G) = C_p \times C_p\,, \qquad G^p = 1\,.$$

This excludes $p = 2$, where more scrutinous observations are necessary. In each case a basis of $H_3 \cap Z(H)$ will be given such that the order of this characteristic subgroup is maximal. It is not too difficult to determine all $T \subset H_3 \cap Z(H)$ such that $(H/T)/Z(H/T)$ is still isomorphic to G; for brevity we do not concern ourselves with this task.

Case A: $|G/G'| = p^3$.
Here we have

$$G = \left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} [x_1, x_2] = a,\ [x_2, x_3] = b \\ x_i^p = [[x_i, x_j], x_k] = [x_1, x_3] = 1 \end{array} \right\rangle .$$

In the notation as before we find

$$H_3 \cap Z(H) = \langle [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_1], [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3], [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2], [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_2], [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_3] = [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_1]^{-1}\rangle\,.$$

Case B: $|G/G'| = p^4$, the string case.
Then

$$G = \left\langle x_1, x_2, x_3, x_4 \;\middle|\; \begin{array}{l} [x_1, x_2] = [x_3, x_4] = a \\ [x_2, x_3] = b \\ x_i^p = [[x_i, x_j], x_k] = 1 \\ [x_i, x_j] = 1 \text{ for } |i - j| > 1 \end{array} \right\rangle .$$

and

$$H_3 \cap Z(H) = \left\langle \begin{matrix} [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_3] = [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_1]^{-1} \\ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2]^{-1} \\ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_2], \; [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3] \end{matrix} \right\rangle .$$

Case C: $|G/G'| = p^4$ and G is a direct product $\langle x_1, x_2 \rangle \times \langle x_3, x_4 \rangle$.
Then

$$H_3 \cap Z(H) = \langle [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_1], [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2], [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_3], [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_4] \rangle .$$

Case D: $|G/G'| = p^4$, completely nondegenerate case.
Here G can be described as a group with the galois field of order p^2 as operator domain, and $H_3 \cap Z(H) \leqslant p^4$.
The actual description would depend on the prime p.

Case E: $G/G' = p^5$.
Here

$$G = \left\langle x_1, x_2, x_3, x_4, x_5 \;\middle|\; \begin{matrix} [x_1, x_2] = [x_3, x_4] = a \\ [x_2, x_3] = [x_4, x_5] = b \\ x_i^p = 1 = [[x_i, x_j], x_k] \\ [x_i, x_j] = 1 \text{ for } |i-j| > 1 \end{matrix} \right\rangle .$$

and

$$H_3 \cap Z(H) = \langle [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_2]^{-1} \rangle .$$

(In particular H does not exist if $x_2^p = [x_3, x_4]$.)

REMARKS. (1) The quotient groups $H_2 \cap Z(H)/H_3 \cap Z(H)$ have orders bounded by p in Case A, p^4 in Cases B, C, D and p^8 in Case E.

(2) If $G = H/Z(H)$ and G is a p-group, if further $|\langle x \rangle| = p$, then also $G \times \langle x \rangle$ is a central quotient: Choose a maximal subgroup M of G and an element y such that $G = \langle M, y \rangle$, and form the extension K of $\langle x, z \rangle$ by H such that $z^p = 1 = [x, z]$, $[x, \tilde{y}] = z$, $[x, \tilde{t}] = [z, \tilde{t}] = 1$ for all $\tilde{t}$ in the pre-image $\tilde{M}$ of M in H. Now $K/Z(K)$ is isomorphic to $G \times \langle x \rangle$.

This shows that groups H do exist as constructed in this section as long as $G^p = 1$, even if $G' \not\subseteq Z(G)$.

3. An example for higher rank.

Consider

$$G = \left\langle \begin{array}{l|l} s_i, t_i, \; i \leqslant n & s_i^p = t_i^p = m_{ij}^p = 1 \\ m_{ij} = m_{ji}, \; i \neq j, \; i, j \leqslant n & [s_i, t_i] = [s_j, s_j, m_{ij}] = c_i \\ & [s_i, t_j] = [s_i, s_j] = [t_i, t_j] = 1 \text{ for } i \neq j \\ & [m_{ij}, t_k] = 1 \text{ for all } i, j, k \\ & [m_{ij}, s_k] = 1 \text{ for } i \neq k \neq j \\ & [m_{jk}, m_{55}] = 1 \text{ for all } i, j, u, v \\ & [[g_1, g_2], g_3] = 1 \text{ for all } g_1 \text{ in } G \end{array} \right\rangle$$

This group is isomorphic to a central quotient $H/Z(H)$ where

$$H_3 \cap Z(H) = \langle [[\tilde{s}_i, \tilde{t}_i], \tilde{s}_i] = [[\tilde{s}_j, \tilde{m}_{ij}], \tilde{s}_i] = \\ = [[\tilde{s}_i, \tilde{m}_{ij}], \tilde{s}_j] = [[\tilde{s}_j, \tilde{t}_j], \tilde{s}_j] \quad \text{for all } i, j \rangle .$$

This follows from the fact that the vector space corresponding to the subgroup $\langle t_i, s_i, m_{ij}, s_j t_j, Z(G) \rangle$ is a string.

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