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Representing the algebra of regulated functions as an algebra of continuous functions

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Representing the Algebra of Regulated Functions as an Algebra of Continuous Functions.

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SUMMARY - We give an intuitively appealing description of the character space of the algebra of regulated functions on a compact interval.

0. The character space of the algebra of (complex valued) regulated functions on a compact interval has been described by S. K. Berberian in the framework if Banach algebras (see [B]).

Avoiding here any Banach algebra theory we give a description of the same space as an ordered topological space: this description is thoroughly elementary and makes the space easy to work with. In fact, this representation of regulated functions as continuous functions is so natural that it is probably folklore in some mathematical circles. However, it does not seem to be widely known; and since regulated functions are useful in elementary analysis (see, e.g. [D]), every information concerning them should be considered interesting.

1. We consider the set $L = \mathbf{R} \times \{-1, 0, 1\}$ lexicographically ordered (by first differences; that is $(x, i) < (y, j)$ means $x < y$ or $x = y$ and $i < j$). Given elements $\alpha, \beta \in L$, $\alpha \leqslant \beta$, the interval they determine in L will be denoted $[\![\alpha, \beta]\!]$, $[\![\alpha, \beta[\![$, etc.; given $x \in \mathbf{R}$, the elements $(x, 1)$, $(x, -1) \in L$ are denoted by $x+$, $x-$, respectively, while

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$(x, 0)$ is identified with x; via this identification, $\mathbb{R}$ and all its subsets are also subsets of L; given a subset S of $\mathbb{R}$, we write $S+$ for $\{x+ : x \in S\} \subset L$, analogous being the meaning of $S-$. Note that $x- < x < x+$, that x is the immediate successor of $x-$, and that $x+$ is the immediate successor of x (in L).

We consider the interval $[\![a, b]\!]$ of L with the order topology (see, e.g. [E] for informations on ordered topological spaces); we have $[\![a, b]\!] = ([a, b[+) \cup ([a, b]) \cup (]a, b]-)$ as a disjoint union. We describe explicitly a neighborhood base at every point of $[\![a, b]\!]$:

1) at $x+$: sets of the form:

$$]\!]x, x + 1/n[\![= [\![x+, (x + 1/n)-]\!] =$$
$$= ([x, x + 1[+) \cup (]x, x + 1/n[) \cup (]x, x + 1/n]-) \qquad (n = 1, 2, 3, \ldots),$$

2) at $x-$: sets of the form:

$$]\!]x - 1/n, x[\![= [\![(x - 1/n)+, x-]\!] =$$
$$= ([x - 1/n, x[+) \cup (]x - 1/n, x[) \cup (]x - 1/n, x]-) \quad (n = 1, 2, 3, \ldots),$$

points $x \in [a, b]$ are isolated, $\{x\} =]\!]x-, x+[\![$.

Notice that the above neighborhoods are clopen (= open and closed).

Every subset of $[\![a, b]\!]$ has a supremum and an infimum in $[\![a, b]\!]$ (as it is trivial to prove using the order completeness of $\mathbb{R}$).

Thus $[\![a, b]\!]$ is compact ([E]). We have seen that $[\![a, b]\!]$ is a compact Hausdorff first countable 0-dimensional space, which has $[a, b]$ as a dense, open and discrete subspace.

Recall that a function $f\colon [a, b] \to \mathbb{C}$ is said to be regulated if the limits

$$(*) \qquad \lim_{t \to x,\, t > x} f(t),$$

$$(**) \qquad \lim_{t \to x,\, t < x} f(t),$$

exist in $\mathbb{C}$ for every $x \in [a, b[$, $x \in]a, b]$ respectively.

Write $\mathrm{Rg}([a, b])$ for the set of all (complex valued) regulated functions on $[a, b]$; clearly $\mathrm{Rg}([a, b])$ is a $\mathbb{C}$-algebra under pointwise operations. For every topological space X, denote by $C(X)$ the $\mathbb{C}$-algebra of all complex valued continuous functions on X.

THEOREM. *For every* $f \in \mathrm{Rg}([a, b])$ *define* $\hat{f}\colon [\![a, b]\!] \to \mathbb{C}$ *by*

$$\begin{aligned} \hat{f}(x) &= f(x) && \text{for } x \in [a, b]\,, \\ \hat{f}(x+) &= \lim_{t \to x,\, t > x} f(t) && \text{for } x \in [a, b[\,, \\ \hat{f}(x-) &= \lim_{t \to x,\, t < x} f(t) && \text{for } x \in]a, b]\,. \end{aligned}$$

Then $\hat{f} \in C([\![a, b]\!])$. *Moreover, the mapping* $f \to \hat{f}$ *is a (supremum) norm preserving isomorphism of* $\mathrm{Rg}([a, b])$ *onto* $C([\![a, b]\!])$, *whose inverse is the restriction map* $g \to g \,|\, [a, b]$.

PROOF. All the statements are immediate, or at least very easy to check; one can prove this «local» version: given a function $h\colon [a, b] \to \mathbb{C}$, and $x \in [a, b[$ (resp.: $x \in]a, b]$) h admits a continuous extension to the subspace $[a, b] \cup \{x+\}$ (resp.: $[a, b] \cup \{x-\}$) of $[\![a, b]\!]$ if and only if h admits a right (resp.: left) sided limit at x, this limit giving the value of the extension at $x+$ (resp. $x-$).

For the statement on norms, use the density of $[a, b]$ in $[\![a, b]\!]$.

2. From the preceding description of the character space of $\mathrm{Rg}([a, b])$ as an ordered space is very easy to deduce some of its properties. Since $[\![a, b]\!]$ is first countable it cannot be the Stone-Čech compactification of any of its dense subspaces (see, e.g. [GJ], Ch. 9). This extends a result in [B]. Clearly $[\![a, b]\!]$ has weight (= minimum cardinal of a base) c, the continuum, hence it cannot be metrizable (this result is also in [B]).

The subspace $[a, b[+$ (resp. $]a, b]-$) is homeomorphic to $[a, b[$ (resp. to $]a, b]$) with the right (resp. left) Sorgenfrey topology (see [E]): both are then hereditarily Lindelöf non metrizable spaces of weight c.

The subspace $T = ([a, b[+) \cup (]a, b]-) = [\![a, b]\!] \setminus [a, b]$, known as the «two arrows space», ([E], 3.10.C.), is then a compact 0-dimensional hereditarily Lindelöf space of weight c (hence T is also perfectly normal, meaning that every open set is a cozero set, [GJ]).

The entire space $[\![a, b]\!]$ is however not hereditarily Lindelöf, having a discrete open subset of power c ($[a, b]$); neither then can $[\![a, b]\!]$ be perfectly normal: cozero sets in $[\![a, b]\!]$ are exactly the F_σ-open sets; in particular a subset of $[a, b]$ is a cozero set in $[\![a, b]\!]$ if and only if it is at most countable.

Since $[\![a, b]\!]$ is compact with a clopen base, the subalgebra A_0 of $C([\![a, b]\!])$ consisting of functions with finite range is uniformly dense in $C([\![a, b]\!])$; in the isomorphism of the previous theorem, functions of A_0 correspond to step functions on $[a, b]$ (for this, observe that a clopen basis for $[\![a, b]\!]$ is the set C consisting of all singletons $\{x\}$, $x \in [a, b]$ and of all intervals of the form $[\![x+, y-]\!]$, $x \in [a, b[$, $y \in]a, b]$, $x < y$: every clopen partition of $[\![a, b]\!]$ corresponds to a «subdivision» of $[a, b]$ in the usual Riemann integration theory sense).

This is the well known fact that step functions are uniformly dense in the set of regulated functions; the (easy) direct proof of this fact makes use of the compactness of $[\![a, b]\!]$, without even mentioning this space (see [D]).

3. Some remarkable subalgebras of $\mathrm{Rg}([a, b])$ have remarkable quotients of $[\![a, b]\!]$ as their structure spaces (in the cases here considered the character space always coincides with the maximal ideal space, equipped with the hull-kernel topology; this last is called structure space of the algebra).

First consider $C([a, b]_{\mathrm{us}})$, where $[a, b]_{\mathrm{us}}$ denotes $[a, b]$ with the usual topology; dual to the inclusion $C([a, b]_{\mathrm{us}}) \hookrightarrow \mathrm{Rg}([a, b]) \leftrightarrow C([\![a, b]\!])$ we have the quotient map $p\colon [\![a, b]\!] \to [a, b]_{\mathrm{us}}$, which is essentially the restriction to $[\![a, b]\!] \subset L = \mathbb{R} \times \{-1, 0, 1\}$ of the first projection: that is $p(x) = x$, $p(x+) = x$, $p(x-) = x$, whenever this makes sense; notice that p is closed (obviously, since all spaces are compact T_2) but not open.

Next, let $\mathrm{Rg}_*([a, b])$ denote the subalgebra of regulated functions which are right continuous at every $x \in [a, b[$, and left continuous at b. Its character space is the subspace $T = [\![a, b]\!] \setminus [a, b]$ described in sect. 2; the map $\varrho_+\colon [\![a, b]\!] \to T$ which is dual to the inclusion of Rg_* into Rg is actually a retraction onto T, and is given by $\varrho_+(x) = x+$ for $x \in [a, b]$, $\varrho_+(b) = b-$: all this is easy to verify. Analogous statements for the subalgebra ${}_*\mathrm{Rg}([a, b])$ of regulated functions left continuous on $]a, b]$ and right continuous at a; the structure space is still T and the retraction $\varrho_-\colon [\![a, b]\!] \to T$ is defined by $\varrho_-(x) = x-$ for $x \in]a, b]$, $\varrho_-(a) = a+$. The space T has been described also by Hewitt in [H]. It is perhaps worth noting that T, being compact, has its subspace topology coinciding with the order topology it has as an ordered set.

Finally, we consider $\mathrm{Rm}([a, b])$, subalgebra of regulated functions whose discontinuities are all removable.

Its structure space is the quotient obtained from $[\![a, b]\!]$ by identifying $x+$ and $x-$, for every $x \in]a, b[$; this space is clearly homeomorphic to the « Alexandroff double » of $[a, b]_{us}$ (see [E], p. 173).

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