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An Application of Fractional Calculus.

R. K. RAINA (*)

SUMMARY - The object of this note is to show how fractional calculus approach can be employed to derive a formula expressing a generalized hypergeometric function in terms of a finite sum of lower order functions. Another representation of this formula is also given.

1. Introduction.

In his paper Karlsson [2] gave a formula for the generalized hypergeometric function ${}_pF_q$ with positive integral differences between certain numerator and denominator parameters in terms of a finite sum of lower order functions. This result [2, p. 270, eqn. (1)] is given by

$$(1.1) \quad {}_pF_q \left[\begin{matrix} b_1 + m_1, \dots, b_n + m_n, a_{n+1}, \dots, a_p; \\ b_1, \dots, b_n, b_{n+1}, \dots, b_q \end{matrix} ; z \right] = \\ = \sum_{j_1=0}^{m_1} \dots \sum_{j_n=0}^{m_n} C(j_1, \dots, j_n) z^{J_n} {}_{p-n}F_{q-n} \left[\begin{matrix} a_{n+1} + J_n, \dots, a_p + J_n; \\ b_{n+1} + J_n, \dots, b_q + J_n \end{matrix} ; z \right],$$

where

$$(1.2) \quad C(j_1, \dots, j_n) = \binom{m_1}{j_1} \dots \binom{m_n}{j_n} \cdot \frac{(b_2 + m_2)_{J_1} \dots (b_n + m_n)_{J_{n-1}} (a_{n+1})_{J_n} \dots (a_p)_{J_n}}{(b_1)_{J_1} \dots (b_n)_{J_n} (b_{n+1})_{J_n} \dots (b_q)_{J_n}},$$

and

$$(1.3) \quad J_n = j_1 + \dots + j_n,$$

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m_i ($i = 1, \dots, n$) are positive integers, and $n \leq \min(p, q)$, provided that $p \leq q$ ($|z| < 1$, if $p = q + 1$), and no denominator parameter is a negative integer or zero.

A simpler proof of (1.1) was given by Srivastava [7]. Our purpose in this paper is to extend (1.1) to the case when the differences m_i ($i = 1, \dots, n$) between the numerator and denominator parameters are arbitrary. This formula is obtainable in a straightforward manner by applying the results well known in the theory of fractional calculus. A new representation of the result is also considered.

2. Extension of (1.1).

For any bounded sequence of real or complex numbers $\{A_n\}$, define a function $f(z)$ by the power series

$$(2.1) \quad f(z) = \sum_{n=0}^{\infty} A_n z^n, \quad |z| < R.$$

A widely used basic formula in the fractional calculus is the formula

$$(2.2) \quad D_z^\beta \{z^{\alpha-1}\} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} z^{\alpha-\beta-1}, \quad \operatorname{Re}(\alpha) > 0,$$

valid for all values of β .

Also, for arbitrary β , the Leibniz rule is given by ([3, p. 263])

$$(2.3) \quad D_z^\beta \{u(z)v(z)\} = \sum_{n=0}^{\infty} \binom{\beta}{n} D_z^{\beta-n} \{u(z)\} D_z^n \{v(z)\},$$

where $u(z)$ and $v(z)$ are functions of the form $z^{a-1}h(z)$ and $z^{b-1}k(z)$, respectively, where $h(z)$ and $k(z)$ are analytic in the disc $|z| < \rho$, provided that $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(a+b) > 0$.

Let us write

$$(2.4) \quad z^{\alpha+\beta-1} f(z) = \sum_{n=0}^{\infty} A_n z^{\alpha+\beta+n-1},$$

and operate D_z^β on both the sides, then (2.4) in conjunction with (2.2) and (2.3) gives

$$(2.5) \quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n}{(\alpha)_n} A_n z^n = \sum_{n=0}^{\infty} \binom{\beta}{n} \frac{z^n}{(\alpha)_n} D_z^n \{f(z)\},$$

provided that $\operatorname{Re}(\alpha + \beta) > 0$, the arbitrary sequence of real or complex numbers is bounded, and $|z| < R$.

The fractional derivative operator D_z^β can be applied term-wise to the right side of the series of (2.4) in view of the result mentioned in a recent monograph of Srivastava and Monacha [9, p. 289, Theorem 2], under, of course, the hypotheses surrounding the above equation (2.5).

As a consequence of the relation (2.5), we now derive a formula expressing a generalized hypergeometric function in terms of a finite sum of lower order. To this end, if we specialize the arbitrary sequence $\{A_n\}$ by setting

$$(2.6) \quad A_n = \frac{\prod_{i=2}^p (a_i)_n}{n! \prod_{i=2}^q (b_i)_n}, \quad n \geq 0,$$

then (2.5) is seen (on slight adjustment of parameters) to yield the result

$$(2.7) \quad {}_pF_q \left[\begin{matrix} b_1 + \beta_1, a_2, \dots, a_p; \\ b_1, \dots, b_q \end{matrix} ; z \right] = \sum_{j_1=0}^{\infty} \binom{\beta_1}{j_1} \frac{(a_2)_{j_1} \dots (a_p)_{j_1}}{(b_2)_{j_1} \dots (b_q)_{j_1}} z^{j_1} \cdot {}_{p-1}F_{q-1} \left[\begin{matrix} a_2 + j_1, \dots, a_p + j_1; \\ b_2 + j_1, \dots, b_q + j_1; \end{matrix} z \right],$$

provided that $p \leq q$ (and $|z| < 1$, if $p = q + 1$), $\operatorname{Re}(b_1 + \beta_1) > 0$, and no denominator parameter is zero or a negative integer.

The repeated application of (2.7) to each of its R.H.S. when $a_i = b_i + \beta_i$ ($i = 2, \dots, n$) gives the result

$$(2.8) \quad {}_pF_q \left[\begin{matrix} b_1 + \beta_1, \dots, b_n + \beta_n, a_{n+1}, \dots, a_p; \\ b_1, \dots, b_n, b_{n+1}, \dots, b_q \end{matrix} ; z \right] = \\ = \sum_{j_1=0}^{\infty} \dots \sum_{j_n=0}^{\infty} \binom{\beta_1}{j_1} \dots \binom{\beta_n}{j_n} \frac{(b_2 + \beta_2)_{j_1} (b_3 + \beta_3)_{j_2} \dots (b_n + \beta_n)_{j_{n-1}}}{(b_1)_{j_1} (b_2)_{j_2} \dots (b_n)_{j_n}} \cdot \\ \cdot \frac{(a_{n+1})_{j_n} \dots (a_p)_{j_n}}{(b_{n+1})_{j_n} \dots (b_q)_{j_n}} z^{j_n} {}_{p-n}F_{q-n} \left[\begin{matrix} a_{n+1} + j_n, \dots, a_p + j_n; \\ b_{n+1} + j_n, \dots, b_q + j_n; \end{matrix} z \right],$$

provided that $p \leq q$ (and $|z| < 1$, if $p = q + 1$), n being any positive integer such that $n \leq \min(p, q)$, where J_n is given by (1.3), and $\operatorname{Re}(b_i + \beta_i) > 0$ for $i = 1, \dots, n$.

The result (2.8) immediately reduces to (1.1) if the arbitrary parameters β_i ($i = 1, \dots, n$) are replaced by the positive integers m_i ($i = 1, \dots, n$). It must be stated here that (2.8) is precisely the same result obtained earlier by Chakrabarti [1, p. 200, eqn. (1.4)] by following a different line of approach employing essentially the Vandermonde's convolution theorem [8, p. 243].

3. Another representation of (2.8).

Consider the generalized Leibniz rule due to Osler [5, p. 2, eqn. (3.1)] (see also [3, p. 264])

$$(3.1) \quad D_z^\alpha \{u(z)v(z)\} = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an + \varepsilon} D_z^{\alpha - an - \varepsilon} \{u(z)\} D_z^{an + \varepsilon} \{v(z)\},$$

where $0 < a \leq 1$, α and ε are arbitrary (real or complex) numbers. Then in view of a known special case of a fractional derivative formula due to Raina and Koul [6, p. 99, eqn. (7)] (see also [4, p. 374]), the following result can easily be arrived at:

$$(3.2) \quad {}_pF_a \left[\begin{matrix} b_1 + \beta_1, a_2, \dots, a_p; \\ b_1, b_2, \dots, b_a; \end{matrix} ; z \right] = \sum_{j_1=-\infty}^{\infty} c_1 \binom{\beta_1}{c_1 j_1 + \varepsilon_1} \frac{1}{(b_1)_{c_1 j_1 + \varepsilon_1} \Gamma(1 - c_1 j_1 - \varepsilon_1)} \cdot {}_pF_a \left[\begin{matrix} 1, a_2, \dots, a_p; \\ 1 - c_1 j_1 - \varepsilon_1, b_2, \dots, b_a; \end{matrix} ; z \right],$$

provided that $p \leq q$ ($p = q + 1$, if $|z| < 1$), $0 < c_1 \leq 1$, ε_1 is any arbitrary complex number, and none of the denominator parameters is a negative integer or zero.

When $c_1 = 1$, $\varepsilon_1 = 0$, (3.2) does not apparently seem to be reducible to (2.7). However, a slight simplification on the right side of

the resulting equation, or else, using the simple identity

$$(3.3) \quad \frac{1}{\Gamma(1-j)} {}_pF_q \left[\begin{matrix} 1, a_2, \dots, a_p ; \\ 1-j, b_2, \dots, b_q ; \end{matrix} \quad z \right] = \\ = \frac{(a_2)_j \dots (a_p)_j}{(b_2)_j \dots (b_q)_j} z^j {}_{p-1}F_{q-1} \left[\begin{matrix} a_2+j, \dots, a_p+j ; \\ b_2+j, \dots, b_q+j ; \end{matrix} \quad z \right],$$

makes (3.2) (when $c_1 = 1$ and $\varepsilon_1 = 0$) reducible to (2.7).

The repeated application of (3.2) for each $a_i = b_i + \beta_i$ ($i = 2, \dots, n$) yields the result

$$(3.4) \quad {}_pF_q \left[\begin{matrix} b_1 + \beta_1, \dots, b_n + \beta_n, a_{n+1}, \dots, a_p ; \\ b_1, \dots, b_n, b_{n+1}, \dots, b_q ; \end{matrix} \quad z \right] = \\ = \prod_{i=1}^n \left\{ \sum_{j_i=-\infty}^{\infty} c_i \binom{\beta_i}{c_i j_i + \varepsilon_i} \frac{1}{(b_i)_{c_i j_i + \varepsilon_i} \Gamma(1 - c_i j_i - \varepsilon_i)} \right\} \cdot \\ \cdot {}_pF_q \left[\begin{matrix} 1, \dots, 1, a_{n+1}, \dots, a_p \\ 1 - c_1 j_1 - \varepsilon_1, \dots, 1 - c_n j_n - \varepsilon_n, b_{n+1}, \dots, b_q ; \end{matrix} \quad z \right],$$

provided that $p \leq q$ (and for $p = q + 1$, $|z| < 1$), ε_i ($i = 1, \dots, n$) are arbitrary complex numbers, and $0 < c_i \leq 1$ ($i = 1, \dots, n$).

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