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Existence of Solutions for a Class of Nonconvex Differential Inclusions.

F. ANCONA - G. COLOMBO (*)

SUMMARY - We prove existence of solutions for the Cauchy problem

$$\dot{x} \in F(x) + f(t, x), \quad x(0) = \xi,$$

where F is upper semicontinuous and $F(x)$ is contained in the subdifferential $\partial V(x)$ of a convex continuous function V , while f is a Carathéodory single valued map.

1. Introduction.

It is well known that the initial value problem for the differential inclusion $\dot{x} \in F(x)$ may not have solutions when F is upper semicontinuous but has not necessarily convex values. Bressan, Cellina and Colombo have recently [2] given a condition ensuring existence for the Cauchy problem

$$(1) \quad \dot{x} \in F(x), \quad x(0) = \xi \in \mathbb{R}^n.$$

They assume F to be upper semicontinuous with values contained in the subdifferential ∂V of a convex function $V: \mathbb{R}^n \rightarrow \mathbb{R}$. This function permits to estimate the L^2 -norm of the derivatives of approximat-

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ing polygonals and to obtain their strong L^2 -convergence from the weak one.

We extend here the technique of [2] to the differential inclusion

$$\dot{x} \in F(x) + f(t, x),$$

where F is as in [2] and f is a Carathéodory single valued map. So we obtain a result that contains Peano's existence theorem as a particular case.

2. The result.

We consider the Cauchy problem

$$2) \quad \dot{x}(t) \in F(x(t)) + f(t, x(t)), \quad x(0) = \xi \in \mathbb{R}^n,$$

under the following assumptions:

i) F is an upper semicontinuous multifunction from $\mathbb{R}^n$ into the compact nonempty subsets of $\mathbb{R}^n$ (i.e. for every x and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $F(x') \subseteq F(x) + \varepsilon B$, where B is the unit ball of $\mathbb{R}^n$);

ii) there exists a convex continuous function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$F(x) \subseteq \partial V(x) \quad \text{for every } x \in \mathbb{R}^n,$$

where $\partial V(x)$ denotes the subdifferential of V at x ;

iii) $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Carathéodory, i.e. for every $x \in \mathbb{R}^n$, $t \mapsto f(t, x)$ is measurable, for a.e. $t \in \mathbb{R}$, $x \mapsto f(t, x)$ is continuous and there exists $m \in L^2(\mathbb{R})$ such that

$$|f(t, x)| \leq m(t) \quad \text{for a.e. } t \in \mathbb{R}, \quad \text{for all } x \in \mathbb{R}^n.$$

We recall that, under the assumption i), F satisfies ii) if and only if it is cyclically monotone [2].

By a solution of our Cauchy problem we mean an absolutely continuous function x which satisfies (2) a.e. On the space of solutions

we consider the H^1 topology, which coincides with the topology induced by the sup norm on x and the L^2 norm on $\dot{x}$.

The following is our existence result.

THEOREM. Let F and f be maps satisfying i), ii) and iii). Then there exists $T > 0$ such that on $[0, T]$ the Cauchy problem (2) admits a nonempty set of solutions, which is compact in the $H^1(0, T)$ topology.

PROOF. We first define a family of approximate solutions similar to a construction of Tonelli [4, vol. 1 p. 42-45/vol. 2 p. 129-130] and then prove that a subsequence converges to a solution of (2).

By i) there exist $R > 0$ and $M > 0$ such that for every $x \in B(\xi, R)$ and for every $y \in F(x)$ we have $|y| \leq M$ [1, Proposition 1.1.3]; by iii) there exists $T > 0$ such that $\int_0^T (m(t) + M) dt < R$.

We define on $[0, T]$ a sequence of approximate solutions x_n :

$$x_n(0) = \xi,$$

$$x_n(t) = x_n\left(i\frac{T}{n}\right) + \int_{iT/n}^t f\left(s, x_n\left(i\frac{T}{n}\right)\right) ds + \left(t - i\frac{T}{n}\right) y_i,$$

$$i = 0, \dots, n-1, \quad t \in \left[i\frac{T}{n}, (i+1)\frac{T}{n}\right],$$

where $y_i \in F(x_n(iT/n))$.

Set, for $t \in [iT/n, (i+1)T/n[$, $i = 0, \dots, n-1$,

$$(3) \quad f_n(t) = f\left(t, x_n\left(i\frac{T}{n}\right)\right), \quad g_n(t) = y_i.$$

Then, $|x_n(t) - \xi| \leq \int_0^t |f_n(s) + g_n(s)| ds \leq \int_0^T (m(s) + M) ds < R$, by our choice of T . Moreover, for all $t, t' \in [0, T]$,

$$|x_n(t') - x_n(t)| \leq \left| \int_t^{t'} |\dot{x}_n(s)| ds \right| \leq \left| \int_t^{t'} (m(s) + M) ds \right|,$$

so that the sequence $(x_n(\cdot))_n$ is equiuniformly continuous. Notice also that $\int_0^T |\dot{x}_n(s)|^2 ds = \int_0^T |f_n(s) + g_n(s)|^2 ds$ and therefore the sequence

$(\dot{x}_n(\cdot))_n$ is bounded in $L^2(0, T)$. Hence there exists a subsequence, still denoted by $(x_n)_n$, and an absolutely continuous function $x: [0, T] \rightarrow \mathbb{R}^n$ such that x_n converges to x in the sup norm topology and $\dot{x}_n$ converges to $\dot{x}$ in the weak topology of L^2 .

Since $(f_n(\cdot))_n$ converges to $f(\cdot, x(\cdot))$ in L^2 and, for $t \in]i(T/n), (i+1)T/n[$, $i = 0, \dots, n-1$,

$$(4) \quad \lim_{n \rightarrow \infty} d\left((x_n(t), \dot{x}_n(t) - f_n(t)), \text{graph}(F)\right) \leq \lim_{n \rightarrow \infty} \left| x_n(t) - x_n\left(i \frac{T}{n}\right) \right| = 0,$$

by Theorem 1.4.1 in [1] we obtain that x is a solution of the convexified differential inclusion

$$(5) \quad \dot{x} \in \text{co}(F(x)) + f(t, x), \quad x(0) = \xi.$$

By our assumption ii) we then have that

$$(6) \quad \dot{x}(t) - f(t, x(t)) \in \partial V(x(t)) \quad \text{for a.e } t \in [0, T].$$

Since the maps $t \mapsto x(t)$ and $t \mapsto V(x(t))$ are absolutely continuous, we obtain from Lemma 3.3 in [3, p. 73] and (6) that $(d/dt)(V(x(t))) = \langle \dot{x}(t), \dot{x}(t) - f(t, x(t)) \rangle$ a.e. on $[0, T]$; therefore,

$$(7) \quad V(x(T)) - V(\xi) = \int_0^T |\dot{x}(s)|^2 ds - \int_0^T \langle \dot{x}(s), f(s, x(s)) \rangle ds.$$

On the other hand, notice that, by (3),

$$\begin{aligned} \dot{x}_n(t) - f_n(t) &= y_i \in \partial V\left(x_n\left(i \frac{T}{n}\right)\right) \\ &\text{for } t \in \left]i \frac{T}{n}, (i+1) \frac{T}{n}\right[, \quad i = 0, \dots, n-1, \end{aligned}$$

and so the properties of the subdifferential of a convex function imply, for every $t \in]iT/n, (i+1)T/n[$,

$$\begin{aligned} V\left(x_n\left((i+1) \frac{T}{n}\right)\right) - V\left(x_n\left(i \frac{T}{n}\right)\right) &\geq \\ &\geq \left\langle \dot{x}_n(t) - f_n(t), x_n\left((i+1) \frac{T}{n}\right) - x_n\left(i \frac{T}{n}\right) \right\rangle; \end{aligned}$$

since this last expression equals

$$\begin{aligned} \left\langle y_i, \int_{iT/n}^{(i+1)T/n} \dot{x}_n(s) ds \right\rangle &= \int_{iT/n}^{(i+1)T/n} \langle y_i, \dot{x}_n(s) \rangle ds = \int_{iT/n}^{(i+1)T/n} \langle \dot{x}_n(s) - f_n(s), \dot{x}_n(s) \rangle ds = \\ &= \int_{iT/n}^{(i+1)T/n} |\dot{x}_n(s)|^2 ds - \int_{iT/n}^{(i+1)T/n} \langle f_n(s), \dot{x}_n(s) \rangle ds, \end{aligned}$$

by adding we obtain

$$(8) \quad V(x_n(T)) - V(\xi) \geq \int_0^T |\dot{x}_n(s)|^2 ds - \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle ds.$$

The convergence of $(f_n)_n$ in L^2 -norm and of $(\dot{x}_n)_n$ in the weak topology of L^2 implies that

$$\lim_{n \rightarrow \infty} \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle ds = \int_0^T \langle f(s, x(s)), \dot{x}(s) \rangle ds.$$

By passing to the limit for $n \rightarrow \infty$ in (8) and using the continuity of V , a comparison with (7) yields

$$\|\dot{x}\|_2^2 \geq \limsup \|\dot{x}_n\|_2^2;$$

since, by the weak lower semicontinuity of the norm,

$$\|\dot{x}\|_2^2 \leq \liminf \|\dot{x}_n\|_2^2,$$

we have that $\|\dot{x}\|_2^2 = \lim_{n \rightarrow \infty} \|\dot{x}_n\|_2^2$, i.e. $\dot{x}_n$ converges to $\dot{x}$ strongly in $L^2(0, T)$ [5, p. 124]. Hence there exists a subsequence $\dot{x}_n$ which converges pointwisely a.e. to $\dot{x}$. Recalling (4), we have that

$$d\left(\left(x(t), \dot{x}(t) - f(t, x(t))\right), \text{graph}(F)\right) = 0 \quad \text{for a.e. } t \in [0, T];$$

since the graph of F is closed [1, p. 41],

$$\dot{x}(t) \in F(x(t)) + f(t, x(t)) \quad \text{a.e.},$$

and so problem (2) does have solutions.

Let now $(x_n)_n$ be a sequence of solutions of (2). Using the same argument as for the approximate solutions we obtain that there exist an absolutely continuous function x and a subsequence $(x_n)_n$ such that x_n converges to x in C and $\dot{x}_n$ converges to $\dot{x}$ weakly in L^2 . Both x and the x_n are solutions of the convexified differential inclusion (5) and so formula (7) holds for x as well as for the x_n . By passing to the limit we obtain that $\|\dot{x}\|_2^2 = \lim_{n \rightarrow \infty} \|\dot{x}_n\|_2^2$ and so, by the same arguments as before, x is a solution of (2).

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