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Extreme Elements of Finite p-Groups.

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In this paper we consider only finite p-groups. Such a group is said to *have equal centralizers*, if all non central elements have centralizers of the same size. It is said to *have independent centralizers*, if of any two centralizers, which are distinct and different from the whole group, none contains the other. These two classes (of which the second obviously contains the first) were first considered by N. Ito [I]. In [M1] it was shown, sharpening a result of [I], that if G has independent centralizers, then either G has an Abelian maximal subgroup, or $G/Z(G)$ is of exponent p. Another subclass of the groups with independent centralizers, namely groups *with Abelian centralizers* (each centralizer different from G is Abelian), was investigated by D. M. Rocke [R]. Recently, the aforementioned result of [M1] was reproved by L. Verardi for groups with equal centralizers [V] (this author seems to be unaware of [M1]), using a different method, which employes a certain inequality for centralizers. In Proposition 1 below, we generalize this inequality to all p-groups, and then consider the case of equality. This turns out to be a very strong assumption: in groups with equal centralizers it implies that the nilpotency class is two, and in general such a group is metabelian, usually of class at most p, the central factor group has Abelian centralizers, etc. (Proposition 2 and Theorem 3).

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The paper [V] contains also a result of H. Heineken, showing that elements centralizing G' in groups with equal centralizers must belong to the second centre. Proceeding by contradiction, this proof shows, on the way, that if one of the relevant elements is not in $Z_2(G)$, then its centralizer in the central factor group is just the image of the original centralizer. It turns out that the argument showing this remarkable property can also be generalized to all p-groups, for an appropriate type of an element—the extreme elements of the title—which are defined as elements whose centralizers have a certain maximality property. We open the paper by defining extreme elements and developing their properties. We can. e.g., generalize Heineken's result, by showing, that in groups with independent centralizers $C(G')$ is often contained in the p-th centre. Extreme elements are also applied in the proofs of the results mentioned in the previous paragraph.

An interesting feature of the situation described in Theorem 3 is, that one is led to consider two natural generalizations of earlier concepts, namely groups in which the centralizers have class at most 2, and groups in which the centralizers have at most two distinct sizes. We do not pursue this line here.

The notation is mostly standard. We usually write $C(x)$ for $C_G(x)$. When we say that an element *has a maximal centralizer*, we mean that the element is not central, and the centralizer is maximal relative to other centralizers, not necessarily among all subgroups. The *breadth* $b(x)$ of an element x is defined by $|G:C(x)| = p^{b(x)}$, and we denote $Z(x) = Z(C(x))$.

1. – Definition. An element x of a p-group G is called *extreme* if

a) $x \notin Z(G)$.

b) If $y = [x, t_1, \dots, t_k]$ is not central (for some elements $t_1, \dots, t_k$), then $C_G(y) = C_G(x)$.

Non central elements of $Z_2(G)$ are trivially extreme. As another example, let x be a non central element with a maximal centralizer among the elements of $C(G')$. Then $C(x) \lhd G$, so also $Z(x) \lhd G$. Therefore all commutators y as in *b*) are in $Z(x)$, so $C(y) \supseteq C(x)$, and the maximality of $C(x)$ shows that $C(y) = C(x)$, and x is extreme. In the same way we see that if x is an element whose centralizer is both maximal and normal, then x is extreme, but we shall see below that the extra generality of this example is only apparent, and the previous example is rather typical. If G has independent centralizers, all elements of $C(G')$ are of this type.

If x is extreme, and $x \in H \subseteq G$, then x is either central or extreme in H. Also, an element y as in b) of the definition is extreme.

An extreme element, which does not belong to $Z_2(G)$, will be called a *proper extreme* element.

If $G = G_1 \times G_2$, it is easy to see that a (non central) element $x = (x_1, x_2)$ is extreme if an only if it either belongs to $Z_2(G)$, or one of the components is extreme, and the other one is central. In particular, if G_1, say, contains proper extreme elements, and G_2 is not Abelian, then the extreme elements of G (together with $Z(G)$) do not form a subgroup.

P1. *If x is extreme, the $C(x)$ is a normal subgroup.*

PROOF. Let $t \in G$, and consider $x^t = x[x, t]$. If $[x, t] \in ZG)$, then certainly $C(x^t) = C(x)$. If $[x, t] \notin Z(G)$, then condition b) of the definition implies that $C(x) \subseteq C(x^t)$, and hence $C(x) = C(x^t)$. In either case, $C(x)^t = C(x^t) = C(x)$.

P2. *If x is extreme, there exists an element $z \in Z_2(G)$, such that x and z have the same centralizer. In particular, $x \in C(G')$.*

PROOF. Let $x \in Z_i(G)$, with i minimal. If $i > 2$, there exists an element t, such that $y = [x, t] \notin Z(G)$. Then $C(y) = C(x)$, and $y \in Z_{i-1}(G)$, so induction on i yields our claim.

COROLLARY. *If G has independent centralizers, the extreme elements are exactly the non central elements of $C(G')$.*

Property P2 justifies our previous claim, that the second example given after the definition of extreme elements is typical. In that example we considered elements of $C(G')$ with a maximal centralizer, and it is easy to see that the maximality is equivalent to all elements of $Z(x)$-$Z(G)$ having the same centralizer $C(x)$, so P2 shows that the definition of extreme elements just generalizes slightly this example.

P3. *If x is proper extreme, then $C_{G/Z(G)}(xZ(G)) = C_G(x)/Z(G)$.*

PROOF. Choose an element t such that $y = [x, t]$ is not central, and assume that $[x, u] \in Z(G)$. Then $[x, u, t] = 1$, and $[u, t, x] = 1$, by P2, so $[t, x, u] = 1$, i.e. $u \in C(y) = C(x)$.

P4. *If x is proper extreme, then $xZ(G)$ is extreme in $G/Z(G)$.*

PROOF. Part *a*) of the definition is assumed, and *b*) follows from P3, applied for both x and y.

P5. *If x is extreme, and $x \notin Z_i(G)$, then x centralizes $Z_i(G)$.*

PROOF. Apply P3 and P4 several times, until we reach $G/Z_{i-1}(G)$.

P6. *If x is a proper extreme element, then* $|Z(x):Z(G)| < p^{b(x)}$.

PROOF. As in the proof of P2, we may assume that $x \in Z_3(G)$. By P3, x has exactly $p^{b(x)}$ conjugates (mod $Z(G)$). These conjugates constitute a subgroup of $Z_2(G)/Z(G)$, and this subgroup is contained in the normal subgroup $Z(x)$, and does not contain x itself, hence our assertion.

Now if even $x \notin Z_3(G)$, we can assume $x \in Z_4(G)$, and use a similar argument, working (mod $Z_2(G)$), to get $|Z(x) \cap Z_3(G) \colon Z(x) \cap Z_2(G)| \geqslant p^{b(x)}$, etc. Thus we obtain

P7. *Let x be a proper extreme element, such that $x \notin Z_{i+1}(G)$. Then* $|Z(x):Z(G)| > p^{ib(x)}$.

On the other hand we have

P8. *Let x be as in* P7, *and let y not commute with x. Then* $b(y) > ib(x)$.

PROOF. First, let $x \in Z_3(G)$, so $i = 1$. Then we consider the commutators of x. All of these have the same centralizer as x, so they do not commute with y. Thus the subgroup that these elements form (mod $Z(G)$) acts regularly on the conjugacy class of y, and gives $p^{b(x)}$ distinct conjugates of y. All of these conjugates are congruent to y (mod $Z(G)$), but y^x is not, by P3. Thus y has more than $p^{b(x)}$ conjugates. For the general case, we get, by induction in $G/Z(G)$, that y has more than $p^{(i-1)b(x)}$ conjugates, which are distinct (mod $Z(G)$), while each of these has $p^{b(x)}$ conjugates congruent to it (mod $Z(G)$), by the previous argument.

COROLLARY. *In a group with equal centralizers, all extreme elements belong to $Z_2(G)$, and thus $C(G') = Z_2(G)$.*

(This is the result of Heineken referred to in the introduction).

P9. *A proper extreme element commutes with all elements whose breadth is not more than its own. In particular, a proper extreme element commutes with all extreme elements.*

Indeed, of any two proper extreme elements, one has a breadth at most as large as the other's, so they commute, and they commute also with elements of $Z_2(G)$, by P5.

COROLLARY. *Let G have independent centralizers. If $C(G') \neq Z_2(G)$, then $C(G')$ is Abelian.*

Indeed, in that case $C(G')$ is generated by its elements outside $Z_2(G)$, which are all proper extreme (Corollary to P2), and so commute with each other.

P10. *Let x be an extreme element satisfying $x \notin Z_k(G)$, for some k, and let $t_1 \notin C(x), \dots, t_k \notin C(x)$. Then $[x, t_1, \dots, t_k] \neq 1$.*

PROOF. If $k = 1$, this is obvious. Let $k > 1$. Then induction, using P3 and P4, shows that $y = [x, t_1, \dots, t_{k-1}] \notin Z(G)$, so $C(y) = C(x)$ and $[y, t_k] \neq 1$.

P11. *In a group G of exponent p, all extreme elements lie in $Z_{p-1}(G)$,*

PROOF. Let x be extreme, and let y not commute with x. Suppose that $x \in Z_c(G) - Z_{c-1}(G)$. Let $H = \langle x, y \rangle$. Letting all t_i be equal to y in P10, we see that $\operatorname{cl} H = c$, and that if $r \leqslant c$, then the commutator $z = [x, y, \dots, y]$, with $r - 1$ y's, lies in H_r but not in H_{r+1}. But for $r = p$, we know that z lies in H_{p+1} [H, III.9.7]. Thus $c < p$.

COROLLARY. *If G has independent centralizers, and does not contain an Abelian maximal subgroup, then $C(G') \subseteq Z_p(G)$.*

This follows from P11 by combining the facts that in such a group all elements of $C(G')$ are extreme, and that G/Z has exponent p.

2. – We now pass to the inequality mentioned in the introduction.

PROPOSITION 1. *Let x be a non central element of the p-group G, such that $C(x)$ is maximal, and let $y \notin C(x)$. Then $|Z(x) : Z(G)| \leqslant p^{b(y)}$. Equality holds if and only if $G = Z(x)\,C(y)$.*

PROOF. The maximality of $C(x)$ implies that, if $t \in Z(x) - Z(G)$, then $C(t) = C(x)$, and so $Z(x) \cap C(y) = Z(G)$. Thus $|Z(x):Z(G)| = |Z(x):Z(x) \cap C(y)| \leqslant |G:C(y)|$, with equality if and only if $G = Z(x)C(y)$.

We consider the case of equality in this Proposition, starting with groups with equal centralizers, for which the result is particularly simple.

PROPOSITION 2. *Let G be a group with equal centralizers, and suppose that $|Z(x):Z(G)| = p^{b(y)}$, for some non central elements x and y. Then G has class 2.*

PROOF. Of course, the equality holds for all elements y, because $b(y)$ is constant. In particular, let y be any element outside $C(x)$. Then $G = Z(x)C(y)$, so all conjugates of y are conjugates under $Z(x)$. It follows that, like y, none of these conjugates lies in $C(x)$. Thus the complement of $C(x)$ is a union of conjugacy classes. But then so is $C(x)$, i.e. $C(x) \lhd G$. By Heineken's result (Corollary to P8), $x \in Z_2(G)$, and, since all elements of $Z(x)$ have the same centralizer, $Z(x) \subseteq Z_2(G)$. We then have $[y, G] = [y, Z(x)] \subseteq [y, Z_2(G)] \subseteq Z(G)$, i.e. $y \in Z_2(G)$. Since the elements of type y, i.e. elements outside $C(x)$, generate G, we have $G = Z_2(G)$.

THEOREM 3. *Let x be an element with a maximal centralizer of the p-group G, let y have maximal breadth among all elements not commuting with x, and suppose that $|Z(x):Z(G)| = p^{b(y)}$. For simplicity, assume also that* $\mathrm{cl}\, G > 2$. *Then*

a) G is metabelian; indeed $Z(x)$ is a normal Abelian subgroup, with an Abelian factor group.

b) If $|G:C(x)| > p$, then $\mathrm{cl}\, G \leqslant p$, and $G/Z(G)$ has exponent p.

c) If $t \notin Z_2(G)$, then either $C(t) \subseteq C(x)$, or $|C(t)| = |C(y)|$. In either case, $\mathrm{cl}\, C(t) \leqslant 2$.

d) $G/Z(G)$ is a group with Abelian centralizers, in which the non central elements have at most two distinct breadths.

PROOF. Denote $b = b(y)$. By assumption, x commutes with all elements of breadth greater than b, and by Proposition 1, it commutes with all those of smaller breadth. Thus each element t outside

$C(x)$ has breadth b, and for each such element $G = Z(x)C(t)$. The argument of Proposition 2 shows that $C(x)$ is normal, hence so is $Z(x)$, and we have $[t, G] = [t, Z(x)] \subseteq Z(x)$. Since the elements t outside $C(x)$ generate G, we see that $G/Z(x)$ is Abelian. This proves a).

Note that as $C(x)$ is maximal, the normality of $C(x)$ shows that x is extreme. If $Z(x) \subseteq Z_2(G)$, we have above $[t, Z(x)] \subseteq Z(G)$, and $\mathrm{cl}\, G \leqslant 2$. Since we have excluded this case, $Z(x)$ is not contained in $Z_2(G)$, and we may as well assume $x \notin Z_2(G)$, so x is a proper extreme element.

Let $K = C(x) \cap C(y)$, then $K \lhd C(y)$, and K centralizes $Z(x)$, so $K \lhd G$. Next,

$$[K, G] = [K, Z(x)C(y)] = [K, C(y)] \subseteq G' \cap C(y) \subseteq Z(x) \cap C(y) = Z(G)$$

and thus $K \subseteq Z_2(G)$. The inclusion is proper, because $Z_2(G)$ also intersects $Z(x)$ outside $Z(G)$. As $Z_2(G) \subseteq C(x)$, by P5, we have $C(x) = Z_2(G)Z(x)$, $Z_2(G) = K(Z_2(G) \cap Z(x))$, and $\mathrm{cl}\, C(x) \leqslant 2$.

Let again $t \notin C(x)$. Then we have seen that $G = Z(x)C(t)$ and $Z(x) \cap C(t) = Z(G)$. But $C(t)' \subseteq C(t) \cap Z(x)$, so $\mathrm{cl}\, C(t) \leqslant 2$. Now let $t \in C(x) - Z_2(G)$. Write $t = uz$, with $u \in Z(x)$, $z \in K$. Then $u \notin Z_2(G)$, and so u is a proper extreme element. Suppose that $[s, t] = [s, u][s, z] \in Z(G)$, for some s, then $[s, u] \in Z(G)$, so P3 shows that $s \in C(x)$. In particular, $C(t) \subseteq C(x)$.

The last argument shows also, that in $H = G/Z(G)$, the subgroup $C = C(x)/Z(G)$ (which is Abelian) is the centralizer of each of its non central elements. Let $a = tZ(G) \in H - C$. Then this implies that $C_C(a) = Z(H)$. Since $G = C(t)C(x)$, we see that $C_H(a) = Z(H)$. $C_G(t)/Z(G)$ is Abelian. This proves d. Now [R, 3.13 and 3.16] shows that if $|G : C(x)| > p$, then $\mathrm{cl}\, G \leqslant p + 1$ and $\exp G/Z_2(G) = p$, but these claims can be improved. We start by employing a variation of the argument of [M1] to show that $\exp G/Z(G) = p$ (similar variations appear in [M2] and [M3]).

To this end, consider $Z(y)$. If $u \in Z(y) \cap K$, then u centralizes both $C(y)$ and $Z(x)$, and thus $u \in Z(G)$, so $Z(y) \cap C(x) \subseteq Z(y) \cap K = Z(G)$. Moreover, we have also $Z(t) \cap C(x) = Z(G)$ for any $t \notin C(x)$, as we have remarked that such a t behaves exactly like y. This in turn shows, that if $s \in Z(t) - Z(G)$, then $C(s) = C(t)$ and $Z(s) = Z(t)$. Therefore for any two elements t, s outside $C(x)$, either $Z(t) = Z(s)$ or $Z(t) \cap Z(s) = Z(G)$. Thus the subgroup $C(x)$, together with the various subgroups $Z(t)$, partition $H = G/Z(G)$.

Now suppose $\exp H > p$. By [B, 5.6], one of the components of the partition contains all elements of H of order greater than p, as well as $Z(H)$. This component is then $C(x)/Z(G)$. But in the metabelian group H, all elements of order greater than p generate a subgroup of index at most p [G], so if $|G\colon C(x)| > p$, we have $\exp G/Z(G) = p$.

Let $\operatorname{cl} G = c$. We have seen that $G' = [Z(x), G]$, and therefore $Z(x)$ is not contained in $Z_{c-1}(G)$, and we may as well assume that $x \notin Z_{c-1}(G)$. Then P10 shows that $[x, y, \ldots, y] \neq 1$, where y is repeated $c-1$ times, so $\operatorname{cl} \langle x, y\rangle = c$. But in the metabelian group H, of exponent p, the images of x and y generate a subgroup of class at most $p-1$, and thus $\operatorname{cl} \langle x, y\rangle \leqslant p$. This ends the proof.

COROLLARY 1. *Under the assumptions and notations of Theorem 3, the subgroup K is the core of $C(y)$, and $Z(K) = Z(G)$.*

PROOF. Since $C(x) = Z(x)K$, we have $Z(K) \subseteq Z(C(x))$, so $Z(K) = Z(G)$. Next, if L is a normal subgroup contained in $C(y)$, then P3 shows that $L \subseteq C(x)$, so $L \subseteq K$.

COROLLARY 2. *A group such as in Theorem 3 has independent centralizers if and only if $C(x)$ is Abelian, which is equivalent to $K = Z(G)$. In this case all centralizers are of class 2, and have exactly two distinct indices. Conversely, if all centralizers are of class 2, then the centralizers are independent.*

PROOF. If the centralizers are independent, then $C(x)$ is Abelian by the corollary to P9. Conversely, if $C(x) = Z(x)$ is Abelian, it is the centralizer of each of its non central elements, and this, together with Theorem 3 c), shows that the centralizers are independent, of class $\leqslant 2$, and of indices $p^{b(x)}$ or $p^{b(y)}$, which are distinct, by P8. Corollary 1 shows that $C(x)$ is Abelian if and only if $K = Z(G)$.

Now suppose that the centralizers are not independent, and let u be a non central element of K. Then $C = C(u)$ contains both $Z(x)$ and y, and therefore $C' \supseteq [Z(x), y] = [G, y]$. If C is of class 2, then $C' \subseteq C(y)$, so $[G, y] \subseteq C(y) \cap Z(x)$ and $y \in Z_2(G)$, which does not hold.

We note also, that the defining property of G in Theorem 3 is shared by G/Z and G/K, the last group having also Abelian centralizers, as $K \cap G' \subseteq K \cap Z(x) = Z(G)$, and so centralizers $(\operatorname{mod} K)$ and $(\operatorname{mod} Z(G))$ coincide.

Finally, we give the conditions that the abstract groups $Z(x)$ and $C(y)$ have to satisfy, in order for a group G of the relevant type to exist. We do not attempt the more difficult problem of describing « concretely » all such pairs, and thus « classifying » completely our groups. The following may rather be thought of as the first step in such a classification.

For this end, let us denote $A = Z(x)$, $B = C(y)$, $Z = N = Z(G)$, where we use Z if we consider $Z(G)$ as a subgroup of A, and N if we consider it as a subgroup of B, and let K be as above. Then the following properties hold:

I. A is an Abelian group with a subgroup Z.

II. B is a group with subgroups $N \subseteq K$, and $N \subseteq Z(B)$.

III. There is an isomorphism φ between Z and N.

IV. B acts on A, with kernel K, in such a way, that for each $b \in B - K$ we have $C_A(b) = C_A(B) = Z$.

V. $Z(K) = N$, but N is a proper subgroup of $Z(B)$.

(Condition V) holds in G, because y is in $Z(B)$). The group G is obtained from A and B by forming first the split extension of A by B, and then amalgamating Z and N according to φ.

Given any two groups A and B satisfying I) to IV), we can form an « amalgamated split extension » G as above. The given conditions imply that $C(x) = AK$, while $Z(x) = A$ follows from $Z(K) = N$. The second half of V gives us an element y such that $C(y) = B$. It remains to control the size of the centralizers. Let then $t = uv \in G$, with $u \in B$, $v \in A$.

VI. If $t \notin C(x)$ (i.e. $u \notin K$), then $[t, B] \subseteq [t, A]$.

These commutators are, of course, calculated inside the group G. Note that VI implies that B has class 2. Condition VI shows, that given any element b of B, we can find an element a in A, such that t commutes with ab. By IV, b determines a uniquely (mod Z), and so $|C(t)| = |C(y)|$. Thus all assumptions of Theorem 3 are guaranteed.

A more elegant condition than VI would be the same condition stated only for elements of B, because this can be formulated using only the groups A, B, Z, N, and the isomorphism φ. It is not clear to us if this weaker version of VI implies the full one.

To conclude, let us draw attention to two interesting special cases: when $K = N$, which is the case of independent centralizers (Corollary 2), and when N is a direct factor of B: the groups G/K and $G/Z_i(G)$ (for $i > 1$) are of this type.

REFERENCES

[B] R. BAER, *Partitionen endlicher Gruppen*, Math. Z., **75** (1961), pp. 333-372.

[G] J. A. GALLIAN, *The Hughes problem and generalizations*, Proc. Conf. Finite Groups (Park City, 1975), pp. 521-528, New York, 1976.

[H] B. HUPPERT, *Endliche Gruppen - I*, Berlin, 1967.

[I] N. ITO, *On finite groups with given conjugate types - I*, Nagoya Math. J., **6** (1953), pp. 17-28.

[M1] A. MANN, *Conjugacy classes in finite groups*, Israel J. Math., **31** (1978), pp. 78-84.

[M2] A. MANN, *Groups with small Abelian subgroups*, Arch. Math., **50** (1988), pp. 210-213.

[M3] A. MANN, *Some finite groups with large conjugacy classes*, in preparation.

[R] D. M. ROCKE, *p-groups with Abelian centralizers*, Proc. London Math. Soc., (3) **30** (1975), pp. 55-75.

[V] L. VERARDI, *On groups whose non central elements have the same finite number of conjugates*, Boll. U.M.I., (7) **2**-A (1988), pp. 391-400.

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