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DĂNUŢ MARCU

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Note on the Gammoids Arising from Undirected Graphs.

DĂNUȚ MARCU (*)

1. Introduction.

In this note, we consider strict gammoids, which arise from undirected graphs. We exhibit a minimal example of a strict gammoid, which cannot arise in this way and we interpret the Ingleton and Piff's characterization (see [1]) of the strict gammoids for the undirected case. In a directed graph $D = (V, F)$, we say that $X \subseteq V$ is *linked into* $Y \subseteq V$, if there exists a set of mutually disjoint paths in D, whose set of the initial vertices is X and whose set of the terminal vertices is a subset of Y. Given $A, B \subseteq V$, the collection of all subsets of A, which can be linked into B, is a special type of matroid, known as a *gammoid*. In the case when $A = V$, the gammoid is said to be *strict*. This concept translates naturally to an undirected graph G. One can either replace paths by undirected paths, in the definitions, or one can regard G as a directed graph, in which each of its edges $\{u, v\}$ is replaced by two directed edges uv and vu. This latter comment was made by Woodall in [3] and he called (strict) gammoids arising from undirected graphs, *undirected* (*strict*) gammoids. In [3], Woodall gave an example of a strict gammoid, which was not an undirected gammoid, and, in this note, we exhibit a minimal such example and, in passing, we interpret the Ingleton and Piff's characterization of the strict gammoids for the undirected case.

(*) Indirizzo dell'A.: Str. Pasului 3, Sect. 2, 70241 Bucharest, Romania.

2. The main results.

THEOREM 1. *Any matroid of rank* $\leqslant 2$ *is an undirected strict gammoid.*

PROOF. Let M be the matroid in question, let V be its underlying set and let X be those points of V, which form independent singletons in M. Then, the relation «$\sim$», defined on X by

$$x \sim y \text{ if } x = y \text{ or if } \{x, y\} \text{ is a circuit of } M, \text{ with } x \neq y\,,$$

is easily seen to be an equivalence relation on X. Let its distinct equivalence classes be $[x_1], \ldots, [x_n]$ and let G be the undirected graph with the vertex set V and the edge set given by

$$E = \big\{\{x_i, x_j\}: 1 \leqslant i < j \leqslant n\big\} \cup \Big(\bigcup_{1 \leqslant i \leqslant n} \big\{\{x_i, x\}: x \in [x_i]\big\} \Big)\,.$$

Then, it is straightforward to check that M consists precisely of those subsets of V linked into B in G, where B is any subset of $\{x_1, \ldots, x_n\}$ of cardinality equal to the rank of M. Hence, M is an undirected strict gammoid. □

A *transversal* of a family of sets $\mathcal{A} = (A_1, \ldots, A_n)$ is a set of n elements $\{x_1, \ldots, x_n\}$, with $x_i \in A_i$, for each i. A *partial transversal* of $\mathcal{A}$ is a transversal of some subfamily of $\mathcal{A}$. It is well known that the set of the partial transversal of $\mathcal{A}$ form a matroid, and one arising in this way is called a *transversal matroid.* In this case, $\mathcal{A} = (A_1, \ldots, A_n)$ is a *presentation* of the matroid and it is well known that a transversal matroid of rank n has a presentation of a family consisting of precisely n sets. Of the many presentations of a transversal matroid M one is called a *minimal presentation,* if it uses the smallest number of sets possible and if none of the sets used can be replaced by a proper subset to give another presentation of M. Now, a family $\mathcal{A} = (A_1, \ldots, A_n)$ will be called *symmetric,* if there exist distinct $x_1, \ldots, x_n$, with $x_i \in A_i$, for $1 \leqslant i \leqslant n$ and $x_i \in A_j$ implies that $x_j \in A_i$, for $1 \leqslant i,\ j \leqslant n$. A transversal matroid will be called *symmetric,* if it possesses such a presentation. So, for example, a transversal matroid of rank $\leqslant 2$ is symmetric, a minimal presentation providing the required symmetric presentation. For, if (A_1, A_2) is a minimal presentation of a matroid, then it is easy to check that neither A_i is a subset of the other. Hence, there exist $x_1 \in A_1 \setminus A_2$ and $x_2 \in A_2 \setminus A_1$, from which the symmetry is clear.

THEOREM 2. *The duals of the undirected strict gammoids are precisely the symmetric transversal matroids.*

PROOF. In [1], Ingleton and Piff have shown that the duals of the transversal matroids are precisely the strict gammoids. More particularly, it follows, from a version of their result in [2, pag. 217], that if M (on the set V) has the presentation $\mathcal{A} = (A_1, \dots, A_n)$ and a transversal $\{x_1, \dots, x_n\}$, with $x_i \in A_i$ for each i, and if $D = (V, F)$ is the directed graph given by

$$F = \bigcup_{1 \leqslant i \leqslant n} \{\{x_i, x\} : x \in A_i \setminus \{x_i\}\} ,$$

then $X \subseteq V$ is linked into $B = V \setminus \{x_1, \dots, x_n\}$ if and only if $V \setminus X$ contains a transversal of $\mathcal{A}$. It is therefore easy to check that, in the special case when $\mathcal{A}$ is symmetric (and the x_i's are chosen accordingly), the same result holds for the corresponding undirected graph. Hence, the dual of a symmetric transversal matroid is an undirected strict gammoid. Conversely, if the dual of M is an undirected strict gammoid and consists of the sets linked into B in the undirected graph $G = (V, E)$, then, from the same result referred to above, it can be deduced that $V \setminus B$ has exactly n distinct elements $x_1, \dots, x_n$ and that M is the transversal matroid with the presentation $\mathcal{A} = (A_1, \dots, A_n)$, where

$$A_i = \{x_i\} \cup \{x\colon \{x_i, x\} \in E\} , \quad 1 \leqslant i \leqslant n .$$

It is clear that $\mathcal{A}$ is symmetric, and the result follows. □

We have remarked above that the transversal matroids of rank $\leqslant 2$ are symmetric, and we now see that sufficiently small transversal matroids of rank 3 are also symmetric.

THEOREM 3. *A transversal matroid of rank* 3, *on a set of* 6 *or fewer points, is symmetric.*

Proof. Let M be the matroid in question and let (A_1, A_2, A_3) be a minimal presentation of M. Then, in particular, if

$$|A_1 \cup A_2 \cup A_3| = m(\leqslant 6)\,,$$

it follows that

$$(1) \qquad |A_i| \leqslant m - 2 \leqslant 4\,, \quad \text{for each } i$$

and

$$(2) \qquad A_i \nsubseteq A_j\,, \quad \text{if } i \neq j\,.$$

Now, in cases, we exhibit a symmetric presentation of M.

Case 1. $A_1 \nsubseteq A_2 \cup A_3$, $A_2 \nsubseteq A_1 \cup A_3$ and $A_3 \nsubseteq A_1 \cup A_2$.

In this case, of course, there exist

$$x_1 \in A_1 \setminus (A_2 \cup A_3)\,, \quad x_2 \in A_2 \setminus (A_1 \cup A_3) \quad \text{and} \quad x_3 \in A_3 \setminus (A_1 \cup A_2)$$

and it is clear that (A_1, A_2, A_3) is symmetric.

Case 2. $A_1 \subseteq A_2 \cup A_3$, $A_2 \nsubseteq A_1 \cup A_3$ and $A_3 \nsubseteq A_1 \cup A_2$.

In this case, there exist

$$x_2 \in A_1 \setminus A_3 = (A_1 \cap A_2) \setminus A_3 \quad \text{and} \quad x_3 \in A_1 \setminus A_2 = (A_1 \cap A_3) \setminus A_2\,.$$

If there exists $x_1 \in A_1 \cap A_2 \cap A_3$, then the symmetry of (A_1, A_2, A_3) is clear. So, we may assume that $A_1 \cap A_2 \cap A_3 = \phi$, such that $|A_1| = |A_1 \cap A_2| + |A_1 \cap A_3|$. If $|A_1 \cap A_2| \geqslant 2$, then there exist distinct $x_1', x_2' \in A_1 \cap A_2 = (A_1 \cap A_2) \setminus A_3$ and $x_3' \in A_3 \setminus (A_1 \cup A_2)$ and, again, the symmetry of (A_1, A_2, A_3) is clear. So, finally, we may suppose that $|A_1 \cap A_2| \leqslant 1$ and, similarly, that $|A_1 \cap A_3| \leqslant 1$. Then, using (2), it is easy to see that there exist four elements $x_1'', x_2'', x_3'', x_4''$, such that $A_1 = \{x_1'', x_2''\}$, $\{x_1'', x_3''\} \subseteq A_2 \setminus A_3$ and $\{x_2'', x_4''\} \subseteq A_3 \setminus A_2$. If we now replace the element x_1'' of A_2 by x_2'', we get a symmetric presentation of M, with representatives x_1'', x_3'' and x_4''.

Case 3. $A_1 \subseteq A_2 \cup A_3$, $A_2 \subseteq A_1 \cup A_3$ and $A_3 \nsubseteq A_1 \cup A_2$.

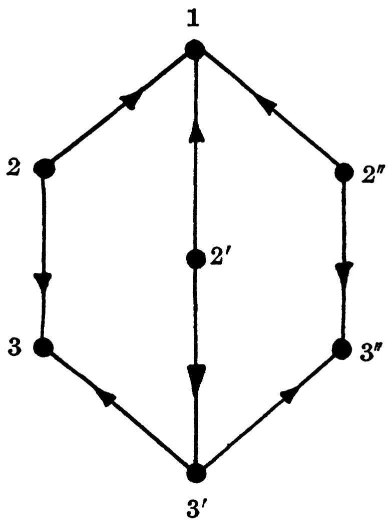
In this case, there exist distinct $x_1, x_2 \in (A_1 \cup A_2 \cup A_3)\setminus A_3 = A_1\setminus A_3 = (A_1 \cap A_2)\setminus A_3$ and $x_3 \in A_3\setminus(A_1 \cup A_2)$ and, again, the symmetry of (A_1, A_2, A_3) is clear.

Case 4. $A_1 \subseteq A_2 \cup A_3$, $A_2 \subseteq A_1 \cup A_3$ and $A_3 \subseteq A_1 \cup A_2$.

It is not difficult to see that, in this case, any subset of $A_1 \cup A_2 \cup A_3$, which has cardinality at most three and is dependent, must be contained in two of the sets A_1, A_2 and A_3 and be disjoint from the third. But then, (1) and (2) lead to a contradiction, in this particular case. Hence, every subset of $A_1 \cup A_2 \cup A_3$, of cardinality at most three, is in M and so M has the symmetric presentation $(A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3)$. □

It is immediate, from the above results, that a strict gammoid, which is not an undirected gammoid, must be of rank at least 3 and on a set of at least 7 elements; below, we present such a gammoid of rank precisely 3 and on a set of precisely 7 elements.

Example. A minimal strict gammoid, which is not an undirected gammoid. Let M be the strict gammoid of sets linked into $1, 3, 3''$, in the directed graph illustrated in the figure:



Then, the circuits of M, of cardinality 3, are precisely $\{1, 2, 3\}$, $\{1, 2', 3'\}$, $\{1, 2'', 3''\}$ and $\{3, 3', 3''\}$; all other sets of cardinality $\leqslant 3$ are independent. This example (and the verification below that M is not an undirected gammoid) is not dissimilar to Woodall's, in [3].

Assume that M is an undirected gammoid, consisting of the subsets of $\{1, 2, 3, 1', 2', 3', 1'', 2'', 3''\} (\subseteq V)$ linked into a set B of cardinality 3, in the undirected graph $G = (V, E)$. Then, since $\{3, 3', 3''\}$ is a circuit of M, it follows, from the Menger's Theorem, that there exist $x, y \in V$, such that every path from $\{3, 3', 3''\}$ to B, in G, uses at least one of x and y. This means that, in addition, every path from $\{3, 3', 3''\}$ to $\{1, 2, 1', 2', 1'', 2''\}$ uses at least one of x and y, since, for example, the existence of a path from 3 to 1 avoiding x and y, together with the independence of $1, 3', 3''$, would imply the existence of a path from 3 to B avoiding x and y. Now, let us call a path from v to $\{x, y\}$, which meets $\{x, y\}$ only at its terminal vertex, a *v-x path* or a *v-y path*, depending upon which member of $\{x, y\}$ it uses. Then, since $\{3, 1, 2'\} \in M$ but $\{3', 1, 2'\} \notin M$, it follows that either there exists a 3-x path but no $3'$-x path or that there exists a 3-y path but no $3'$-y path. Let us assume the former. A similar argument, applied to $\{3', 1, 2\} \in M$ and $\{3, 1, 2\} \notin M$, shows that there exists a $3'$-y path but no 3-y path. Similar arguments, with respect to the pairs $3, 3''$ and $3', 3''$, show that there exists no $3''$-x path and no $3''$-y path and, hence, no path from $3''$ to B.

This contradiction shows that M is not an undirected gammoid.

REFERENCES

[1] A. W. Ingleton - M. J. Piff, *Gammoids and transversal matroids*, J. Comb. Theory Ser. B, **15** (1973), pp. 51-68.

[2] D. J. A. Welsh, *Matroid Theory*, Academic Press, London - New York, 1976.

[3] D. R. Woodall, *The induction of matroids by graphs*, J. London Math. Soc., **10** (1975), pp. 27-35.

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