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On Chains of Purely Transcendental Field Extensions.

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Abstract - Conditions are given under which the union of an ascending chain of purely transcendental extensions of a ground field k is again a purely transcendental extension of k.

In an attempt to solve a problem posed by Anderson and Ohm on semi-valuations of group rings [1], we came across the following problem. Suppose that $K_0 \subset K_1 \subset \ldots \subset K_n \subset \ldots$ is a countable ascending chain of fields K_n such that (i) each K_n is a purely transcendental extension of K_0, and (ii) K_n is algebraically closed in K_{n+1}, for each $n \geqslant 0$. Is the union $K = \bigcup_n K_n$ likewise a purely transcendental extension of K_0? Though the answer to this question was not needed in the solution of the Anderson-Ohm problem (cf. Bastos-Viswanathan [2]), we have found the question interesting enough to look into it.

In this note we give a complete answer to this question under the hypothesis that K is the field of quotients of a group algebra (of a torsion-free abelian group). We will prove in Theorem 1 that in this case the answer is in the affirmative. However, the situation changes if uncountable chains are admitted as is shown by Example 2.

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Our results rely heavily on well-known theorems on torsion-free abelian groups. The connection with abelian group theory can be utilized to establish certain properties of what we call almost purely transcendental extensions; see Theorem 3.

1. – The main result.

For a field k and a torsion-free abelian group A, the group algebra $k[A]$ is a domain. Its field of quotients is denoted by $k(A)$. If A is a free abelian group then $k(A)$ is a purely transcendental extension of k and a basis of A is a transcendence base for $k(A)|k$.

Our main goal is to prove the following theorem.

THEOREM 1. – *Let k be any field, A a torsion-free abelian group, and B a subgroup of A. Suppose that*

$$k(B) = K_0 \subset K_1 \subset \ldots \subset K_n \subset \ldots \qquad (n < \omega) \tag{1}$$

is an ascending chain of fields such that

(i) *K_n is a purely transcendental extension of K_0 $(n < \omega)$;*

(ii) *K_n is algebraically closed in K_{n+1} $(n < \omega)$;*

(iii) $K = \bigcup_{n<\omega} K_n = k(A)$.

Then A/B is a free abelian group and $k(A)|k(B)$ is a purely transcendental extension.

PROOF. The chain (1) gives rise to an ascending chain of abelian groups in the following way. Consider the multiplicative groups K_n^* of the fields $K_n \bmod K_0^*$. From (1) we obtain an ascending chain

$$(1) = K_0^*/K_0^* \subset K_1^*/K_0^* \subset \ldots \subset K_n^*/K_0^* \subset \ldots \qquad (n < \omega)\,. \tag{2}$$

Conditions (i)-(iii) translate into (i*)-(iii*) as follows.

(i*) All K_n^*/K_0^* are free abelian groups.

In fact, it is well known that a polynomial ring over any field with an arbitrary (even infinite) number of indeterminates is a UFD.

Therefore, the multiplicative group of its field of quotients mod that of the ground field is freely generated by a set of irreducible polynomials over K_0.

(ii*) K_n^*/K_0^* is a pure subgroup of K_{n+1}^*/K_0^* $(n < \omega)$.

This is obvious, since because of (ii) the m-th roots of elements of K_n which are contained in K_{n+1}^* are already in K_n^*. (Thus (ii) can be weakened by assuming only that K_n is closed under taking roots of elements in K_{n+1}.)

(iii*) We have $\bigcup_{n<\omega} (K_n^*/K_0^*) = K^*/K_0^*$.

A well-known theorem by Hill [5] states that the union of a countable ascending chain of free abelian groups where each member is a pure subgroup in its successor is itself free. Hence K^*/K_0^* is a free abelian group. Now observe that A/B is a subgroup of K^*/K_0^*; therefore it is a free abelian group.

If A/B is free, then B is a direct factor of A, $A = B \times F$ for a free subgroup F of A. It follows that $k(A) = [k(B)](F)$. Since the adjunction of a free abelian group is nothing else than a purely transcendental extension, the proof is completed. □

One wonders if the last theorem can be improved. As we have noticed in the introduction, we do not know if the condition $K = k(A)$ can be dropped. (Of course, the freeness of K^*/K_0^* follows, but this alone does not imply that $K|K_0$ is a purely transcendental extension.) But we do know that the countability of the chain (1) is relevant. In fact, the following example shows that Theorem 1 is in general not valid for uncountable chains of fields.

EXAMPLE 2. Let

$$A_0 = 0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots \qquad (\alpha < \omega_1)$$

be a continuous well-ordered ascending chain of countable free abelian groups A_α (continuity means $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for limit ordinals α and ω_1 is the first uncountable ordinal) where each A_α is pure in $A_{\alpha+1}$. Suppose that the set

$$E = \{\alpha < \omega_1 | A_{\alpha+1}/A_\alpha \text{ is not free}\}$$

is stationary in ω_1. (It is easy to construct such a chain, e.g. by taking a projective resolution of Q: $0 \to F_0 \to F_1 \to Q \to 0$ and embedding A_α in $A_{\alpha+1}$ in the very same way F_0 is embedded in F_1, for each $\alpha < \omega_1$.) It is well-known (see Eklof [3]) that under these circumstances, $A = \bigcup_{\alpha<\omega_1} A_\alpha$ is not a free abelian group.

Now, for any field k, define $K_\alpha = k(A_\alpha)$, Then the chain

$$K_0 \subset K_1 \subset \ldots \subset K_\alpha \subset \ldots \subset K = \bigcup_{\alpha<\omega_1} K_\alpha \qquad (\alpha < \omega_1)$$

satisfies conditions (i), (ii), (iii) above. Since A is not free, $K|K_0$ cannot be a purely transcendental extension. □

2. – Almost purely transcendental extensions.

A closer look at the preceding example reveals that, though the union K is not a purely transcendental extension of K_0, it is close to being one: every subfield of K that contains K_0 and has countable transcendence degree over K_0 can be embedded in a countably generated purely transcendental extension of K_0 in K.

Let us call an extension K of a field K_0 *almost purely transcendental* if the transcendence degree of K over K_0 is an infinite cardinal λ and every subextension L of K_0 in K with transcendence degree $< \lambda$ can be embedded in a purely transcendental extension of K_0 in K.

We are now going to verify (for the needed set-theoretical concepts, see [7]):

THEOREM 3. (*a*) *Let λ be an uncountable regular cardinal which is not weakly compact. Then every field K_0 has an almost purely transcendental extension K of transcendence degree λ which is not purely transcendental.*

(*b*) *If λ is a singular cardinal, then every almost purely transcendental extension of transcendence degree λ is necessarily purely transcendental.*

PROOF. (*a*) Gregory [4] has proved that for every regular, not weakly compact cardinal λ there is a non-free abelian group A of cardinality λ all of whose subgroups of cardinalities $< \lambda$ are free.

Choose $K = K_0(A)$ with such an A. Then K is not a purely transcendental extension of K_0, since K^*/K_0^* contains the non-free subgroup A. Now if L is a subextension in K, of transcendence degree $\varkappa < \lambda$ over K_0, then an easy cardinality argument shows that $L \subset K_0(B)$ for a subgroup B of A, of cardinality $\varkappa$. As B must be free, $K_0(B)$ is a purely transcendental extension of K_0 that contains L.

(*b*) This part was proved by Hodges [6], p. 218. It is a consequence of his version of Shelah's compactness theorem for singular cardinals. We refer to [6] for details. □

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