

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 83 (1990), p. 165-170

[<http://www.numdam.org/item?id=RSMUP_1990__83__165_0>](http://www.numdam.org/item?id=RSMUP_1990__83__165_0)

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The Cohomology Groups $H^1(\mathbf{P}^3 - \mathbf{P}^1, \mathcal{O}(m))$.

ANTONIO CASSA (*)

Introduction.

In my paper: A ring structure on $Z_0(\mathbf{C}^4)$ and an inverse twistor formula (cfr. [C]) are introduced with a sketch of proof, some isomorphism among the cohomology groups $H^1(\mathcal{U}, \mathcal{O}(-n-2))$ and the spaces $\mathcal{S}_s^n(C)$ of holomorphic functions on the cone

$$C = \{z \in \mathbf{C}^4: z_{00} \cdot z_{11} - z_{01} \cdot z_{10} = 0\}$$

with vanishing order at least n on a plane S of C .

The present article develops the proof using a procedure inspired by a method invented by J. Frenkel (cfr. [F]); the isomorphisms so obtained give new representations of the spaces of holomorphic solutions for the Dirac equations (cf. [C]).

Notations.

Let's fix the following notations:

$$H_j = \{(\omega, \pi) \in \mathbf{C}^4: \pi_j = 0\} \quad U_j = \mathbf{C}^4 - H_j \quad (\text{for } j = 0, 1)$$

$$L = H_0 \cap H_1, \quad U = U_0 \cap U_1, \quad \mathcal{U} = \{U_0, U_1\}$$

$$H'_j = \pi(H_j), \quad U'_j = \pi(U_j), \quad L' = \pi(L), \quad U' = \pi(U), \quad \mathcal{U}' = \{U'_0, U'_1\}$$

where: $\pi: \mathbf{C}^4 - \{0\} \rightarrow \mathbf{P}^3$ is the natural projection.

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1. - We are going to define a factor F_m of $Z^1(\mathcal{U}, \mathcal{O}(m)) = \mathcal{O}(m)(U_0 \cap U_1)$. Let's consider the two « extension » maps:

$$e_0: \mathcal{O}(m)(U) \rightarrow \mathcal{O}(m)(U_0) \quad \text{and} \quad e_1: \mathcal{O}(m)(U) \rightarrow \mathcal{O}(m)(U_1)$$

given by:

$$e_0 f(\omega, \pi) = \frac{1}{2\pi i} \cdot \int_{|t_1|=r_1} \frac{f(\omega, \pi_0, t_1)}{t_1 - \pi_1} \cdot dt_1 \quad (\text{for } |\pi_1| < r_1)$$

and

$$e_1 f(\omega, \pi) = \frac{1}{2\pi i} \cdot \int_{|t_0|=r_0} \frac{f(\omega, t_0, \pi_1)}{t_0 - \pi_0} \cdot dt_0 \quad (\text{for } |\pi_0| < r_0) .$$

Since $e_j f = f$ for $f \in \mathcal{O}(m)(U_j)$ composing the extensions with the restrictions to U we get the projections:

$$p_0, p_1: \mathcal{O}(m)(U) \rightarrow \mathcal{O}(m)(U) \quad (p_0^2 = p_1^2 = id)$$

with the properties:

$$\text{I) } p_0 \circ p_1 = p_1 \circ p_0,$$

$$\begin{aligned} \text{II) } (p_0 \circ p_1)(f)(\omega, \pi) &= \frac{1}{(2\pi i)^2} \cdot \oint_{|t_j|=r_j} \frac{f(\omega, t_0, t_1)}{(t_0 - \pi_0) \cdot (t_1 - \pi_1)} \cdot dt_0 \cdot dt_1 = \\ &= f(\omega, \pi) + \frac{1}{2\pi i} \cdot \oint_{|t_0|=R_0} \frac{f(\omega, t_0, \pi_1)}{t_0 - \pi_0} \cdot dt_0 - \frac{1}{2\pi i} \cdot \oint_{|t_1|=R_1} \frac{f(\omega, \pi_0, t_1)}{t_1 - \pi_1} \cdot dt_1 + \\ &\quad + \frac{1}{(2\pi i)^2} \oint_{|t_j|=R_j} \frac{f(\omega, t_0, t_1)}{(t_0 - \pi_0)(t_1 - \pi_1)} \cdot dt \end{aligned}$$

(the first equation for $0 < |\pi_j| < r_j$, the second for $|\pi_j| > R_j$).

The new projection

$$p = id - p_0 - p_1 + p_0 \circ p_1: \mathcal{O}(m)(U) \rightarrow \mathcal{O}(m)(U)$$

defines the subspace

$$F_m = \{f \in \mathcal{O}(m)(U) : pf = f\}.$$

Since

$$p(\omega_0^{p_0} \cdot \omega_1^{p_1} \cdot \pi_0^{q_0} \cdot \pi_1^{q_1}) = \begin{cases} 0 & \text{if } q_0 \geq 0 \text{ or } q_1 \geq 0, \\ \omega^p \cdot \pi^q & \text{if } q_0 < 0 \text{ and } q_1 < 0, \end{cases}$$

the space F_m contains all functions with a Laurent expansion:

$$f(\omega, \pi) = \sum_{i_0, i_1 \geq 0} a_{k,j} \cdot \frac{\omega_0^{k_0} \cdot \omega_1^{k_1}}{\pi_0^{j_0} \cdot \pi_1^{j_1}}.$$

2. - We are interested in F_m for the following:

THEOREM (2.1). The inclusion $F_m \xrightarrow{j} \mathcal{O}(m)(U) = Z^1(\mathcal{U}, \mathcal{O}(m))$ induces an isomorphism: $\hat{j}: F_m \rightarrow H^1(\mathcal{U}, \mathcal{O}(m)) (\simeq H^1(\mathcal{U}', \mathcal{O}(m)))$

PROOF. If $f \in B^1(\mathcal{U}, \mathcal{O}(m))$ then $f = f_1 - f_0$ with $f_j \in \mathcal{O}(m)(U_j)$. Since $p(f_1 - f_0) = 0$ then $f = pf = 0$.

For $f \in \mathcal{O}(m)(U)$ let's take $f' = pf$ in F_m , the difference $f - f' = \text{res}_v^{U_1}(e_1 f) - \text{res}_v^{U_0}(e_0 \text{res}_v^{U_1} e_1 f - e_0 f)$ is in $B^1(\mathcal{U}, \mathcal{O}(m))$.

THEOREM (2.2). Are equivalent:

- i) f of $\mathcal{O}(m)(U)$ is in F_m ;
- ii) $\lim_{\pi_j \rightarrow \infty} f(\omega_0, \omega_1, \pi_0, \pi_1) = 0$ for every ω and $\pi_{j \pm 1} \neq 0$ ($j = 0, 1$);
- iii) there exist homogeneous polynomials $A_j(\omega)$ of degree $j_0 + j_1 + m$ such that:

$$f(\omega, \pi) = \sum_{i_0, i_1 \geq 0} \frac{A_j(\omega)}{\pi_0^{i_0} \cdot \pi_1^{i_1}}.$$

PROOF. i) $\Rightarrow$ ii) For every f in F_m it holds the inequality:

$$|f(\omega, \pi)| \leq \frac{4 \cdot R_0 \cdot R_1}{|\pi_0| \cdot |\pi_1|} \cdot \max_{|t_j| = R_j} |f(\omega_0, \omega_1, t_0, t_1)| \quad (|\pi_j| > 2 \cdot R_j > 0).$$

ii) $\Rightarrow$ i) From:

$$f(\omega, \pi) = \frac{1}{2\pi i} \cdot \int_{|t_j|=S_j} \frac{f(\omega, \dots, t_j, \dots)}{t_j - \pi_j} dt_j - \frac{1}{2\pi i} \cdot \int_{|t_j|=R_j} \frac{f(\omega, \dots, t_j, \dots)}{t_j - \pi_j} \cdot dt_j$$

it follows (as $S_j \rightarrow \infty$):

$$f(\omega, \pi) = \frac{1}{(2\pi i)^2} \cdot \iint_{|t_j|=R_j} \frac{f(\omega, t_0, t_1)}{(t_0 - \pi_0) \cdot (t_1 - \pi_1)} \cdot dt.$$

$$\text{i) } \Rightarrow \text{iii) } A_j(\omega) = \frac{1}{(2\pi i)^2} \cdot \iint_{|t_k|=R_k} t_0^{j_0-1} \cdot t_1^{j_1-1} \cdot f(\omega_0, \omega_1, t_0, t_1) \cdot dt_0 \cdot dt_1.$$

$$\text{iii) } \Rightarrow \text{i) } p \left(\frac{\omega_0^{p_0} \cdot \omega_1^{p_1}}{\pi_0^{j_0} \cdot \pi_1^{j_1}} \right) = \frac{\omega_0^{p_0} \cdot \omega_1^{p_1}}{\pi_0^{j_0} \cdot \pi_1^{j_1}}.$$

3. - The map $\sigma: F_m \rightarrow F_{-m-4}$ defined by:

$$\sigma(f)(\omega, \pi) = \frac{1}{\omega_0 \cdot \omega_1 \cdot \pi_0 \cdot \pi_1} \cdot f\left(\frac{1}{\pi_0}, \frac{1}{\pi_1}, \frac{1}{\omega_0}, \frac{1}{\omega_1}\right)$$

is a well defined isomorphism.

Infact with some computation it is possible to prove:

$$|\sigma(f)(\omega, \pi)| \leq \frac{M(r, s)}{r_0 \cdot r_1 \cdot s_0 \cdot s_1} \quad (M = \max\{|f(u, v)| : |v_j| = 1/(2 \cdot s_j), |u_j| \leq 1/r_j\}).$$

THEOREM (3.1). For every $f \in F_m$ there exist functions $\{f_s\}$ in F_{-a} (for $p_0 + p_1 = |m + 2|$ and $p_0, p_1 \geq 0$) such that:

$$a) \quad f = \sum_s \frac{1}{\pi_0^{p_0} \cdot \pi_1^{p_1}} \cdot f_{s_0, s_1} \quad \text{if } m < -2,$$

$$b) \quad f = \sum_s \omega_0^{p_0} \cdot \omega_1^{p_1} \cdot f_{s_0, s_1} \quad \text{if } m > -2.$$

PROOF.

$$\begin{aligned} a) \quad f &= \sum_{h_0+h_1 \geq |m+2|} \frac{1}{\pi_0 \cdot \pi_1} \cdot \frac{A_h(\omega)}{\pi_0^{h_0} \cdot \pi_1^{h_1}} = \sum_p \frac{1}{\pi_0^{p_0} \cdot \pi_1^{p_1}} \cdot \sum_k \frac{A_h(\omega)}{\pi_0^{k_0+1} \cdot \pi_1^{k_1+1}}, \\ b) \quad f &= \sigma_{-m-4}(\sigma_m(f)) = \sigma_{-m-4} \left(\sum_p \pi_0^{-p_0} \cdot \pi_1^{-p_1} \cdot g_p \right) \end{aligned}$$

4. - THEOREM (4.1). Let C be the cone in $\mathbf{C}^4$ defined by:

$$C = \{z \in \mathbf{C}^4: z_{00} \cdot z_{11} - z_{01} \cdot z_{10} = 0\}$$

and let S, T be:

$$S = \{z \in \mathbf{C}^4: z_{10} = z_{11} = 0\}, \quad T = \{z \in \mathbf{C}^4: z_{01} = z_{11} = 0\}.$$

Denoted by $\mathcal{I}_S, \mathcal{I}_T$ the ideal sheaves of S and T in C it holds:

$$a) \quad h_n: \mathcal{I}_S^n(C) \rightarrow F_{-n-2}, \quad h_n(k) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \frac{1}{\omega_1^n} \cdot k \left(\frac{\omega_0}{\pi_0}, \frac{\omega_0}{\pi_1}, \frac{\omega_1}{\pi_0}, \frac{\omega_1}{\pi_1} \right)$$

is a well defined isomorphism for every $n \geq 0$.

$$b) \quad h'_n: \mathcal{I}_T^n(C) \rightarrow F_{-n-2}, \quad h'_n(k) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \pi_1^n \cdot k \left(\frac{\omega_0}{\pi_0}, \frac{\omega_0}{\pi_1}, \frac{\omega_1}{\pi_0}, \frac{\omega_1}{\pi_1} \right)$$

as a well defined isomorphism for every $n \geq 0$.

PROOF. a) $n = 0$ Taken

$$f = \sum_{l_0, l_1} \frac{A_l(\omega)}{\pi_0^{l_0+1} \cdot \pi_1^{l_1+1}} \quad \text{in } F_{-2} \quad (\deg(A_l) = l_0 + l_1)$$

let's consider the function

$$\begin{aligned} k(z) &= z_{00} \cdot z_{01} \cdot f(z_{00} \cdot z_{01}, z_{01} \cdot z_{10}, z_{01}, z_{00}) = \sum_l \left(\frac{z_{01}}{z_{00}} \right)^{l_1} \cdot A_l(z_{00}, z_{10}) = \\ &= \sum_l \left(\frac{z_{11}}{z_{01}} \right)^{l_1} \cdot A_l(z_{00}, z_{10}) = \sum_l \left(\frac{z_{00}}{z_{01}} \right)^{l_0} \cdot A_l(z_{01}, z_{11}) = \sum_l \left(\frac{z_{10}}{z_{11}} \right)^{l_0} \cdot A_l(z_{01}, z_{11}) \end{aligned}$$

is holomorphic on $C - \{0\}$ and then on all C (the space C is perfect, cfr. [BS] cor. 3.12 pag. 79); it holds: $h_0(k) = f$.

a) b) $n > 0$ follow from the previous case and theorem (3.1).

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Manoscritto pervenuto in redazione il 6 giugno 1989.