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Topological Characterization of Certain Classes of Lattices.

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Abstract - Let L be a distributive lattice with 0 and 1, and let Spec L be the set of all proper prime ideals of L. Spec L can be endowed with two topologies, the spectral topology and D-topology. In this paper, it is proved that there is a bijection from the set of all σ-ideals of L to the set of all D-open subsets of Spec L. Let Max L and Min L be the sets of maximal ideals and minimal prime ideals of L respectively. We prove that the two topologies coincide on Spec L, Max L, and Min L if and only if L is a boolean, stonian, and normal lattice respectively.

Throughout this paper a lattice means a distributive lattice with 0 and 1. Cornish [3] introduced σ-ideals in lattices and proved a lot of their basic properties. For any lattice L, let Spec L denotes the set of all proper prime ideals of L. This set can be given the hull-kernal topology or what is called also the spectral topology, where open sets are of the form $D(I) = \{P \in \text{Spec}\, L\colon I \nsubseteq P\}$, for details, see Brezuleanu and Diacanescu [1] and Johnstone [6]. Exactly, as in the case of commutative rings with unity, we can define another topology on Spec L called the D-topology. In the case of commutative rings with unity, see Lazard [7] and DeMarco [4]. A subset X of Spec L is called S-stable if for any $P, Q \in \text{Spec}\, L$, whenever $P \subseteq Q$ and $P \in X$,

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$Q \in X$. Trivially, the set of all spectrally open S-stable subsets of $\operatorname{Spec} L$ defines a topology on $\operatorname{Spec} L$ called the D-topology. In fact, it is a subtopology of the spectral topology on $\operatorname{Spec} L$, and open sets in the D-topology will be called D-open. For any $x \in L$, we denote by $x^* = \{y \in L: x \wedge y = 0\}$. Recall that an ideal I of L is called a σ-ideal if for all $x \in L$, $I \vee x^* = L$, i.e. $\exists x_1 \in I$ and $y \in x^*$ such that $1 = x_1 \vee y$. For more details about σ-ideals, see Cornish [3] and Georgescu and Voiculescu [5]. In this paper, our aim is to study the relationship between σ-ideals in a lattice L and D-open subsets of $\operatorname{Spec} L$. Then we characterize those lattices for which the two topologies coincide on $\operatorname{Spec} L$, $\operatorname{Max} L$ and $\operatorname{Min} L$, where $\operatorname{Max} L$ and $\operatorname{Min} L$ are the spaces of maximal ideals and minimal prime ideals of L respectively.

LEMMA 1. *A spectrally open set $D(I)$ is S-stable if and only if I is a σ-ideal of the lattice L.*

PROOF. Assume I is a σ-ideal of L. Let $P, Q \in \operatorname{Spec} L$ such that $P \subseteq Q$ and $P \in D(I)$. So $I \nsubseteq P$. Hence there exists $x \in I$ and $x \notin P$. Since I is a σ-ideal, there exist $x_1 \in I$ and $y \in x^*$ such that $1 = x_1 \vee y$. Therefore $y \in P \subseteq Q$. Because $1 = x_1 \vee y$, $x_1 \notin Q$. Consequently, $I \nsubseteq Q$, i.e. $Q \in D(I)$.

Conversely, assume that $D(I)$ is a spectrally open S-stable set. We proceed by contradiction, so assume I is not a σ-deal. Thus there exists $x \in I$ such that $x^* \vee I \neq L$. Using Zorn's lemma, there exists a maximal ideal M of L such that $x^* \vee I \subseteq M$. Let $\mathfrak{S} = \{J: J$ is an ideal of L such that $x \notin J$, $J \subseteq M\}$. This set contains the ideal $\{0\}$. By Zorn's lemma, there exists a prime ideal P of L in $\mathfrak{S}$. This is a maximal element in S, see Simmons [8]. Since $x \in I$, $P \in D(I)$. Thus $D(I)$ is not S-stable, which contradicts the assumption. Hence I is a σ-ideal of L.

Recall that an element $x \in L$ is called a complemented element in L if there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, y is called a complement for x. It is well known that if x has a complement in a distributive lattice then the complement is unique. We will denote the complement of x in L, if it exists, by x'.

Brezuleanu and Diaconescu [1] proved that any ideal I is the intersection of all prime ideals of L that contain I. So, $D(I) = D(J)$ if and only if $I = J$. So, we get the following theorem.

THEOREM 2. Let L be a lattice. The mapping $I \mapsto D(I)$ is a bijection from the set of all σ-ideals of L to the set of all D-open subsets of Spec L.

Now we characterize those lattices for which the spectral topology and the D-topology coincide on Spec L. First, we give the following easy lemma.

LEMMA 3. Let L be a lattice. Then $D(I)$ is spectrally clopen (i.e. open and closed) if and only if $D(I) = D(x)$ for some complemented elements x in L.

PROOF. Since $D(I) \cap D(J) = D(I \cap J)$ and $D(I) \cup D(J) = D(I \vee J)$, we get $D(I) \cap D(J) = \emptyset$ and $D(I) \vee D(J) = L$ if and only if $I \cap J = \{0\}$ and $I \vee J = L$. Now, $I \vee J = L$ if and only if $1 = x \vee y$ for some $x \in I$ and $y \in J$. Since $x \wedge y \in I \cap J$, $x \wedge y = 0$. Thus x is a complemented element in L. Moreover, because L is distributive for every $z \in I$, $z = z \wedge (x \vee y) = z \wedge x$ since $z \wedge y = 0$. Thus I is a principal ideal generated by x.

The converse is trivial.

THEOREM 4. Let L be a lattice. Then the spectral topology and D-topology coincide on Spec L if and only if L is a boolean lattice (a lattice every element of which has a complement).

PROOF. Assume that the two topologies coincide. Let $x \in L$. Then (Dx) is D-open. So, by lemma 1, the ideal generated by x, $(x]$, is a σ-ideal. Hence $(x] \vee x^* = L$, i.e. $\exists x_1 \in (x]$ and $y \in x^*$ such that $x_1 \vee y = 1$. Hence $x \vee y = 1$ and $x \wedge y = 0$. So x is a complemented element. Therefore L is a boolean lattice.

Conversely, assume L is boolean. Let I be an ideal of L, and $x \in I$. Then there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, i.e. $I \vee x^* = L$. So I is a σ-ideal of L.

Let Min L be the set of all minimal prime ideals of the lattice L. In the following theorem we characterize the lattices L for which the spectral topology and the D-topology coincide on Min L. Recall that a lattice L is called stonian if for every $x \in L$, x^* is generated by a complemented element in L. Stonian lattices were discussed extensively in literature and infact they form on interesting class of lattices.

THEOREM 5. A lattice L is stonian if and only if the spectral topology and the D-topology coincide on $\mathrm{Min}\, L$.

PROOF. Assume that L is a stonian lattices. Let $x \in L$, and $D_M(x) = D(x) \cap \mathrm{Min}\, L$. Since L is stonian, there exists a complemented element y in L such that $x^* = [y)$, the ideal generated by y. Now, we claim that $D_M(y')$, where y' is the complement of y.

Let $P \in D_M(x)$, then $x \notin M$. Because $x \wedge y = 0$, $y \in P$. Thus $y' \notin P$, i.e. $P \in D_M(y')$. For the other way around, let $P \in D_M(y')$, then $y' \notin P$, and so $y \in P$. Thus $x^* \subseteq P$. Since P is a minimal prime ideal of L, $x \notin P$, see Speed [9], i.e. $P \in D_M(x)$. Clearly, $D(y')$ is a D-open set. Hence $D_M(x)$ is a D-open set in $\mathrm{Min}\, L$. But $D_M(I) = \bigcup_{x \in I} D_M(x)$, so $D_M(T)$ is a D-open set in $\mathrm{Min}\, L$.

Conversely, assume that the two topologies coincide on $\mathrm{Min}\, L$. If $x \in L$, then $D_M(x)$ is a D-open set in L. So by lemma 1, there exists a σ-ideal I of L such that $D_M(x) = D_M(I)$. Consider

$$V(x^*) = \{P \in \mathrm{Spec}\, L \colon x^* \subseteq L\{$$

$$= \{P \in \mathrm{Spec}\, L \colon \exists Q \in \mathrm{Min}\, L \in x \notin Q \text{ and } Q \subseteq P\{\,,$$

see Simmons [8].

Because $D(I)$ is S-stable and $D_M(x) = D_M(I)$

$$D(I) = \{P \in \mathrm{Spec}\, L \colon \exists Q \in \mathrm{Min}\, L \in x \notin Q, \text{ and } Q \subseteq P\}\,.$$

Thus $V(x^*)$ is open. Since $V(x^*) = \mathrm{Spec}\, L\text{--}D(x)$, $V(x^*)$ is clopen. Thus $V(x^*)$ is a clopen set in the spectral topology on L. So there exists a complemented element b in L such that $x^* = (b]$, the ideal generated by b in L.

A lattice is called normal if every prime ideal in L is contained in a unique maximal one. For more details about normal lattices, set Johnstone [6] and Simmons [8]. It should be noted that normality of the lattice in the sense of Cornish [2] is the dual concept of our normality and vice versa. Finally, we prove a theorem characterizing lattices for which the spectral topology and the D-topology coincide on $\mathrm{Max}\, L$.

THEOREM 6. For a lattice L, the spectral topology and the D-topology coincide on $\operatorname{Max} L$ if and only if L is a normal lattice.

PROOF. Assume that L is a normal lattice, and let $x \in L$. For each prime ideal P of L, denote by M_P the unique maximal ideal of L containing P. For each open set $D(I)$ in the spectral topology, let $D^M(I) = D(I) \cap \operatorname{Max} L$. Consider $D^M(x)$, and let

$$V = \{P \in \operatorname{Spec} L \colon M_P \in D^M(x)\}$$
$$= \{P \in \operatorname{Spec} L \colon x \notin M_P\} .$$

To show that $D^M(x)$ is a D-open set in $\operatorname{Max} L$, assume $P_1 \subseteq P_2$, $P_1 \in V$, and $P_2 \in \operatorname{Spec} L$. So $P_2 \subseteq M_{P_1}$ because L is a normal lattice. Thus $M_{P_2} = M_{P_1}$. Therefore $M_{P_2} \in D^M(x)$, i.e. $P_2 \in V$. Consequently, V is an S-stable set. Because $V \cap \operatorname{Max} L = D^M(x)$, $D^M(x)$ is a D-open set in Mas L, and thus every spectrally open set in $\operatorname{Max} L$ is D-open. So the two topologies coincide on $\operatorname{Max} L$.

Conversely, assume that the two topologies coincide on $\operatorname{Max} L$. Let P be a prime ideal that is contained in two dist maximal ideals of L, say M_1 and M_2. Thus there exists $a \in M_1$ and $a \notin M_2$, i.e. $M_1 \notin D^M(a)$ and $M_2 \in D^M(a)$. Because $D^M(a)$ is D-open in $\operatorname{Max} L$, there exists a D-open subset of $\operatorname{Spec} L$ such that $D^M(a) — D^M(I)$. Since $M_2 \in D^M(a)$, $M_2 \in D^M(I)$ and so $I \not\subseteq M_2$ is S-stable and $P \subseteq M_1$, $M_1 \in D(I)$. Thus $M_1 \in D^M(a)$, which contradicts the fact that $a \in M_1$. Therefore L is a normal lattice.

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