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Pseudo-Closure-Operators.

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In the algebra of $R$-modules closure operators are used frequently; their common foundation is based as follows: let $M$ be an $R$-module, $\mathcal{L}(M)$ the complete lattice of the sub-modules of $M$, $\varphi: \mathcal{L}(M) \rightarrow \mathcal{L}(M)$ a mapping, written as

$$\varphi(N) = N^c \quad (N \in \mathcal{L}(M)),$$

such that

(i) $N_1 \subseteq N_2 \Rightarrow N_1^c \subseteq N_2^c$;
(ii) $N \subseteq N^c$;
(iii) $(N^c)^c = N^c$,

then $\varphi$ is called a closure operator on the lattice $\mathcal{L}(M)$. The submodule $N \in \mathcal{L}(M)$ is $\varphi$-closed if $N = N^c$.

A majority of the closure operators on $\mathcal{L}(M)$ can be defined by means of a Gabriel filter $\mathcal{F}$ on the ring $R$; the filter $\mathcal{F}$ defines a hereditary torsion functor $\tau = \tau_\mathcal{F}$, such that

$$\tau_\mathcal{F}(M) = \{ m \in M: \text{Ann}_R(m) \in \mathcal{F}\}$$

is the $\tau_\mathcal{F}$-torsion submodule of $M$. Any Gabriel filter $\mathcal{F}$ on the ring $R$ induces on the lattice $\mathcal{L}(M)$ a closure operator, defined by

$$N \mapsto N^c = \{ m \in M: (N; m) \in \mathcal{F}\}, \quad N \in \mathcal{L}(M).$$

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Then $N^e$ is a submodule of $M$, the $\tau_F$-closure (or the $\tau_F$-saturation) of $N \in \mathcal{L}(M)$. The submodule $N \in \mathcal{L}(M)$ is $\tau_F$-closed, if $N = N^e$, i.e. if and only if $\tau_F(M/N) = 0$.

The family of the $\tau_F$-closed submodules $N$ of $M$ is denoted by

$$\text{Sat}_F(M) = \{N \in \mathcal{L}(M) : N = N^e\} \ (1).$$

We find in the literature (see: Y. Miyashita [1]) a pseudo-closure operator, defined on the set $\Sigma(M)$ of all subsets of an $R$-module $M$, properly containing the element $\{0\} \in M$. For the subset $S \in \Sigma(M)$ we define the pseudo-closure $S^e$ of the subset $S \in \Sigma(M)$ by

$$(1) \quad S \mapsto S^e = \{0\} \cup \{m \in M : Rm \cap S \neq 0\},$$

and we set $0^e = 0$. Then we have for subsets $S, T \in \Sigma(M)$:

(i) $S \subseteq T \Rightarrow S^e \subseteq T^e$;

(ii) $S \subseteq S^e$;

(iii) $(S^e)^e = S^e$.

If $S$ is a subset of $M$, then $S^e$ is called the pseudo-closure of $S$, and $S$ will be called pseudo-closed, if $S = S^e$.

If $N$ is a submodule of the $R$-module $M$, then the pseudo-closure $N^e$ of the submodule $N$ is—in general—not a submodule of $M$.

**Example.** Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $N = \mathbb{Z}(2; 0)$; then $N_1 = \mathbb{Z}(1; 0)$ and $N_2 = \mathbb{Z}(1; \bar{1})$ are maximal essential extensions of $N$ in $M$. We have $N_1 \subseteq N_2 \subseteq N^e$, but $N^e$ is not a submodule of $N$; for $(1; 0) \in N_1 \subseteq N^e$, $(1; \bar{1}) \in N_2 \subseteq N^e$, but $(0; 1) = (1; \bar{1}) - (1; 0) \notin N^e$, since $Z(0; 1) \cap Z(2; 0) = (0; 0)$.

**Theorem 1.** If $N \in \mathcal{L}(M)$ then we have:

(i) the pseudo-closure $N^e$ of $N$ in $M$ contains every essential extension of $N$ in $M$;

(ii) if in particular $N^e$ is a submodule of $M$, then $N^e$ is the unique maximal essential extension of $N$ in $M$;

(1) See e.g. B. Stenström [2], p. 207.
(iii) If $N^e$ is the unique maximal essential extension of $N$ in $M$ for every submodule $N$ of $M$, then $M$ has the «intersection property» for essentially closed submodules of $M$ (i.e. the intersection of any collection of essentially closed submodules of $M$ is essentially closed in $M$).

**Proof.** (i) is a consequence of the definition (1).

(ii) If $N^e$ is a submodule of $M$, then $N^e$ cannot have a proper essential extension in $M$: i.e. then $N^e$ is essentially closed in $M$. That implies that $N^e$ is a complement of some submodule $N' \subseteq M$, and we may choose $N'$ in such a way, that $N'$ contains an element $0 \neq m_0 \in M \setminus N^e$. Then $Rm_0 \cap N^e = 0$ implies that $Rm_0 \cap N = 0$, i.e. $m_0 \notin N^e$, and thus $N^e$ is the unique maximal essential extension of $N$ in $M$.

(iii) If $N^e$ is a submodule of $M$ for all submodules $N \subseteq M$, then any submodule $N$ of $M$ has a unique maximal essential extension $\overline{N} = N^e$ in $M$, and that implies that $M$ has the «intersection property» for essentially closed submodules of $M$.

**Theorem 2.** If the $R$-module $M$ satisfies the «intersection property» for essentially closed submodules, then for every submodule $N \subseteq M$ the pseudo-closure $N^e$ is the unique maximal essential extension of $N$ in $M$.

Indeed, in this situation $N^e$ is the (unique) intersection of all essentially closed submodules of $M$ containing $N$.

**Theorem 3.** Let $R$ be a left Ore domain, $M$ a torsion-free $R$-module, and $N \neq 0$ a submodule of $M$; then:

(i) $N^e$ is the unique maximal essential extension of $N$ in $M$;

(ii) $M$ has the «intersection property» for essentially closed submodules.

**Proof.** If $0 \neq m \in N^e$, then for some $0 \neq r \in R$ we have $0 \neq rm \in N$. If $0 \neq r' \in R$, $Rr \cap Rr' \neq 0$ implies that $0 \neq r''r = r''r'$ for some $0 \neq r'' \in R$, $0 \neq r'' \in R$. Then $0 \neq r''rm = r''r'm \in N$, and, using the torsion-freeness of $M$, we have $0 \neq rm \in N^e$.

If $m_1, m_2$ are in $N^e$, then $0 \neq r_1m_1 \in N$, $0 \neq r_2m_2 \in N$ for some
Then $0 \neq r_1, r_2 \in R$. Then $Rr_1 \cap Rr_2 \neq 0$ implies that

$$0 \neq \varrho_1 r_1 = \varrho_2 r_2$$

for some $\varrho_1, \varrho_2 \in R$. Therefore

$$0 \neq \varrho_1 r_1 (m_1 + m_2) = \varrho_1 r_1 m_1 + \varrho_2 r_2 m_2 \in \mathcal{N};$$

i.e. $m_1 + m_2 \in \mathcal{N}$. Thus $\mathcal{N}$ contains with any $m \neq 0$ also $rm$, and with $m_1, m_2 \in \mathcal{N}$ also $m_1 + m_2 \in \mathcal{N}$. Thus $\mathcal{N}$ is a submodule of $\mathcal{M}$, and (by theorem 1 (ii)), this implies (i) and (ii).

**Corollary 4.** If $R$ is a commutative integral domain, $N \neq 0$ a submodule of the torsionfree $R$-module $\mathcal{M}$, then the pseudo-closure $\mathcal{N}$ of $\mathcal{N}$ is a submodule of $\mathcal{M}$, and $\mathcal{M}$ has the «intersection property» for essentially closed submodules.

From the theorems 1 and 2 it follows that the «intersection property» for essentially closed submodules of $\mathcal{M}$ is a necessary and sufficient condition therefore that the pseudo-closure $\mathcal{N}$ of any submodule $\mathcal{N}$ of $\mathcal{M}$ is a submodule of $\mathcal{M}$.

We will give some other examples of sufficient conditions for $R$ (resp. $\mathcal{M}$) in order that $\mathcal{M}$ has the «intersection property» for essentially closed submodules of $\mathcal{M}$. Therefore we define:

**Definition ($\beta$).** Let $\mathcal{N}$ be a submodule of $\mathcal{M}$; then the pair $(\mathcal{N}; \mathcal{M})$ satisfies the condition ($\beta$), if $1_{\mathcal{M}}$ is the only $R$-automorphism of $\mathcal{M}$ inducing $1_{\mathcal{N}}$.

The condition ($\beta$) of the pair $(\mathcal{N}; \mathcal{M})$ is equivalent with each of the following conditions:

($\beta'$) every $R$-endomorphism $f$ of $\mathcal{N}$ has at most one extension $f$ on $\mathcal{M}$;

($\beta''$) $\text{Hom}_R(\mathcal{M}/\mathcal{N}; \mathcal{M}) = 0$.

We give some examples:

1) If $\mathcal{N}$ is an essential submodule of the non-singular $R$-module $\mathcal{M}$, then $\mathcal{M}/\mathcal{N}$ is singular, i.e. $\text{Hom}(\mathcal{M}/\mathcal{N}; \mathcal{M}) = 0$, and the pair $(\mathcal{N}; \mathcal{M})$ satisfies ($\beta$).
2) If $M$ is a rational extension of the submodule $N$, then $\text{Hom}_R(M/N; M) = 0$, i.e. the pair $(N; M)$ satisfies $(\beta)$.

The following result proves that the property $(\beta)$—in a special sense—is a sufficient condition therefore that the pseudo-closure $N^c$ of a submodule $N$ of $M$ is a submodule of $M$.

**Theorem 5.** Let $N \neq 0$ be a submodule of $M$, $\bar{M}$ an injective hull of $M$; if we assume that the pair $(N; \bar{M})$ satisfies the condition $(\beta)$, then:

(i) $N$ has a unique maximal essential extension in $\bar{M}$;

(ii) $M$ has the «intersection property» for essentially closed submodules of $M$ if the condition $(\beta)$ holds for any pair $(N; \bar{M})$;

(iii) the pseudo-closure $N^c$ of a submodule $N$ is a submodule of $M$.

**Proof.** Let $N_1, N_2$ be two maximal essential extensions of $N$ in $M$, and $\bar{N}_1, \bar{N}_2$ the corresponding injective hulls of $N$ in $M$. Then $N_i = M \cap \bar{N}_i$ ($i = 1, 2$). Furthermore there exists an isomorphism $\varphi: \bar{N}_1 \cong \bar{N}_2$, $\varphi(n) = n$ ($\forall n \in N$). Let $N^*$ be an injective hull of a complement of $N$ in $\bar{M}$. Then $\bar{M} = \bar{N}_1 \oplus N^* \cong \bar{N}_2 \oplus N^*$. Define: $\alpha \in \text{End}_R(\bar{M})$ by $\alpha(\bar{N}_1) = \varphi(\bar{N}_1) = \bar{N}_2$, $\alpha(n^*) = n^*$ ($\forall n^* \in N^*$). Then $\alpha$ is an $R$-automorphism of $\bar{M}$, inducing $1_\bar{M}$. Since the pair $(N; \bar{M})$ satisfies the condition $(\beta)$, we have $\alpha = 1_\bar{M}$, i.e. $\bar{N}_1 = \bar{N}_2$, and therefore $N_1 = \bar{N}_1 \cap M = \bar{N}_2 \cap M = N_2$. Hence any submodule $N$ of $M$ has a unique maximal essential extension in $\bar{M}$. Then the proof of (ii) and of (iii) follows from theorem 1.

**Literature**


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