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On a Hamilton-Jacobi Equation.

GIOVANNI BASSANELLI (*)

SUMMARY - We prove a theorem of existence and uniqueness for the problem:

$$\frac{\partial}{\partial u} \varphi \frac{\partial}{\partial v} \varphi + \sum a_{jk}(u, v, z) \frac{\partial}{\partial z_j} \varphi \frac{\partial}{\partial z_k} \varphi = 0$$

with data: $\varphi(u, v, z, \eta) = z \cdot \eta$ for $uv = 0$, on a strip $[0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \dot{\mathbf{R}}^n$ where $A = (a_{jk})$ is a C^∞ , $n \times n$, negative definite real matrix.

1. Introduction.

Let $A = A(u, v, z)(u, v \geq 0, z \in \mathbf{R}^n, n \geq 1)$ be an $n \times n$ matrix. In this note we shall study the following problem:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial u} \varphi \frac{\partial}{\partial u} \varphi + \left\langle A \frac{\partial}{\partial z} \varphi, \frac{\partial}{\partial z} \varphi \right\rangle = 0, \\ \varphi(u, v, z, \eta) = z \cdot \eta, & \text{if } uv = 0, \end{cases}$$

with $\eta \in \dot{\mathbf{R}}^n$. First we want to show, briefly, the reason for our interest in (1.1).

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The formulation of quantum field theories in light-cone coordinates is useful in some subjects of physics (see [4], [6], [9]) and such a reformulation involves a study of the most important equations (Klein-Gordon, Dirac, etc.) in this frame. For example, let us consider the following Cauchy problem:

$$(1.2) \quad \begin{cases} (\partial_{uv}^2 - \Delta_z) f = 0 & \text{in } [0, U] \times \mathbf{R} \times \mathbf{R}^n, \\ f(0, v, z) = g(v, z), \end{cases}$$

(see [5]), this is a natural problem for the wave equation in light-cone coordinates. For a more general operator

$$P = \partial_{uv}^2 + \sum a_{jk} \partial_{z_j z_k}^2 + \sum b_j \partial_{z_j} + c,$$

with smooth coefficients, which are constant outside a compact subset of $\mathbf{R}_u \times \mathbf{R}_v \times \mathbf{R}_z^n$, and $A = (a_{jk})$ negative definite (there is no restriction assuming that A is real and symmetric), the corresponding Cauchy problem is

$$(1.3) \quad \begin{cases} Pf = 0 & \text{in } [0, U] \times \mathbf{R} \times \mathbf{R}^n, \\ f(0, v, z) = g(z, u). \end{cases}$$

This is a characteristic Cauchy problem and if, as $v \rightarrow -\infty$, g satisfies some growth conditions, then it is « well-posed » (see [1]). A formal computation performed on (1.2) suggests that, in order to construct a parametrix for (1.3), we must look for an operator of the form

$$(Eg)(u, v, z) = \\ = g(v, z) - \int_{-\infty}^v \int_{\eta} \exp[i\varphi^{\pm}(u, v, v', z, \eta)] a^{\pm}(u, v, v', z, \eta) g(v', \eta) d\eta dv'.$$

By a well known argument (see e.g. [2], [7]) φ^+ and φ^- have to be real and homogeneous functions of degree 1 in η , and they must satisfy equation (1.1) with the boundary condition

$$\varphi^{\pm}(u, v, v', z, \eta) = z \cdot \eta, \quad \text{if } u(v - v') = 0.$$

But actually v' is only a parameter, which, at present, we can overlook; thus we have exactly problem (1.1).

Our main result is the following *Theorem* (see Theorem 6.1): there exists $U > 0$ such that (1.1) has two solutions $\varphi^\pm(u, v, z, \eta)$, real and homogeneous of degree 1 in η , defined on the strip $[0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \mathbf{R}^n$, which are smooth for $uv > 0$, and with a suitable regularity at the boundary; every solution φ , satisfying the above conditions, coincides with φ^+ or φ^- on each connected component of $[0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \mathbf{R}^n$.

In order to prove this theorem we shall study the Hamilton-Jacobi problem (2.1) which, as we point out in § 3, is very different from the classical case, and it seems us to be of some interest.

2. Firstly we show that problem (1.1) can be reduced to a Hamilton-Jacobi equation, then we shall study the associated bicharacteristic curves.

Let $A = A(u, v, z) \in C^\infty(\bar{\mathbf{R}}^+ \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n)$ be a $n \times n$ real symmetric matrix, negative definite and constant outside a compact subset of $\bar{\mathbf{R}}^+ \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n$. By means of

$$x_1 = u + v, \quad x_2 = u - v, \quad x_{2+j} = z_j \quad (j = 1, \dots, n),$$

and writing $x' = (x_3, \dots, x_{2+n})$, $x = (x_1, x_2, x')$, $\psi(x, \eta) = \varphi(u, v, z, \eta)$, $B(x) = A(u, v, z)$, problem (1.1) becomes

$$\begin{cases} \left(\left(\frac{\partial}{\partial x_1} \psi \right)^2 - \left(\frac{\partial}{\partial x_2} \psi \right)^2 + \left\langle B(x) \frac{\partial}{\partial x'} \psi, \frac{\partial}{\partial x'} \psi \right\rangle \right) = 0, \\ \psi(|x_2|, x_2, x', \eta) = x' \cdot \eta. \end{cases}$$

This suggests that we fix $\eta \in \mathbf{R}^n$, from now on, and consider the following equations

$$(2.1) \quad \frac{\partial}{\partial x_1} \psi \pm \left[\left(\frac{\partial}{\partial x_1} \psi \right)^2 - \left\langle B(x) \frac{\partial}{\partial x'} \psi, \frac{\partial}{\partial x'} \psi \right\rangle \right]^{\frac{1}{2}} = 0$$

with data at the boundary:

$$(2.2) \quad \psi(|x_2|, x_2, x', \eta) = x' \cdot \eta.$$

(For simplicity we shall study only the case with sign « - » and, in the last paragraph, we shall look at the dependence on η). Let us

remark that $B(x)$ is defined in $\bar{\Omega}$ where

$$\Omega = \{x \in \mathbf{R}^{2+n}; x_1 > |x_2|\}$$

and the hypotheses on A give: there exist positive constants M_j ($j = 0, 1, 2$) and a matrix B_0 such that

$$(2.3) \quad \text{if } \|x\| \geq M_0, \quad \text{then } B(x) = B_0,$$

and, for every $(x, \eta) \in \bar{\Omega} \times \mathbf{R}^n$,

$$(2.4) \quad -M_1 \|\eta\|^2 \geq \langle B(x)\eta, \eta \rangle \geq -M_2 \|\eta\|^2.$$

The symbol $p: \mathcal{T}^*(\bar{\Omega}) = \bar{\Omega} \times \mathbf{R}^{2+n} \rightarrow \mathbf{R}$ of equation (2.1) is

$$p(x, \xi) = \xi_1 - (\xi_2^2 - \langle B(x)\xi', \xi' \rangle)^{\frac{1}{2}}.$$

Since $(\partial/\partial \xi_1)p = 1$, we can choose x_1 as parameter of the null bicharacteristic curves, which are all the curves in $\mathcal{T}^*(\bar{\Omega})$ of the form

$$(2.5) \quad x = x(t), \quad \xi = \xi(t) \quad \text{with } x_1(t) = t$$

such that

$$(2.6) \quad \dot{x}_2 = -\xi_1^{-1} \xi_2, \quad \dot{x}' = \xi_1^{-1} B(x)\xi', \quad \dot{\xi}_j = -\frac{1}{2} \xi_1^{-1} \left\langle \frac{\partial}{\partial x_j} B(x)\xi', \xi' \right\rangle$$

$$(j = 1, \dots, 2+n).$$

2.1. PROPOSITION. Let $(x^0, \xi^0) \in p^{-1}(0)$. Then there exists $\bar{t} \geq 0$ such that the bicharacteristic curve, of the form (2.5), passing through (x^0, ξ^0) is defined on $[\bar{t}, +\infty)$ and $x(\bar{t}) \in \partial\Omega$. Moreover there are two cases:

(i) If $\xi^{0'} = 0$, then $\bar{t} = 0$ and

$$(x(t), \xi(t)) = ((t, \pm t, x^{0'}), (\xi_1, \mp \xi_1, 0)) \in \partial\mathcal{T}^*(\partial\Omega),$$

for every $t \geq 0$.

(ii) If $\xi^{0'} \neq 0$, then, for every $t > \bar{t}$, $x(t) \in \Omega$ and $\xi'(t)$ never vanishes.

In both the cases

$$(2.7) \quad |x_2(t) - x_2^0| \leq |t - x_1^0|$$

and, for a suitable constant $K > 0$ (independent from (x^0, ξ^0)),

$$(2.8) \quad \|\xi^{0'}\| \exp(-K|t - x_1^0|) \leq \|\xi'\| \leq \|\xi^{0'}\| \exp(K|t - x_1^0|).$$

PROOF. First we verify (i). From (2.4) and (2.6) we get

$$\frac{1}{2} \frac{d}{dt} \|\xi'\|^2 \leq \left| -\frac{1}{2} \sum_{j=3}^{n+2} \xi_1^{-1} \left\langle \frac{\partial}{\partial x_j} B(x) \xi', \xi' \right\rangle \xi_j \right| \leq \frac{1}{2} M_1^{-1} \left(\sup_{x,j} \left\| \frac{\partial}{\partial x_j} B(x) \right\| \right) \|\xi'\|^2$$

from which (2.8) follows. Thanks to (2.6) and $p(x(t), \xi(t)) = 0$, it is straightforward to prove (2.7) and that $(x(t), \xi(t))$ is bounded on every bounded interval. So we can extend the curve to an interval of the form $[\bar{t}, +\infty)$, with $x(\bar{t}) \in \partial\Omega$. Q.E.D.

3. In this section we state all the machinery which we need in order to prove our main theorem. Write

$$\partial\Omega = \partial_0\Omega \cup \partial_+\Omega \cup \partial_-\Omega$$

where

$$\partial_0\Omega = \{(0, 0)\} \times \mathbf{R}^n, \quad \partial_{\pm}\Omega = \{x \in \Omega; 0 < x_1 = \pm x_2\}.$$

Consider the $(n + 1)$ -dimensional submanifold $L_0(\eta) = \{(x^0, \xi^0) \in p^{-1}(0); x^0 \in \partial_0\Omega \text{ and } \xi^0 = \eta\}$ and the union $L(\eta)$ of the all null-bicharacteristic curves issued from $L_0(\eta)$; we point out that $\pi(L(\eta))$ (here π is the natural projection of $T^*(\bar{\Omega})$ onto Ω) is a family of curves issued from the wedge $\partial_0\Omega$, such that there are continuously many curves starting from each point of $\partial_0\Omega$ and as we shall see, on each of these curves the solution $\psi(x)$ must be constant. (So problem (2.1-2) is very different from the classical one for a Hamilton-Jacobi equation: in fact, in the latter, you usually give the solution ψ on a hypersurface L_0 and, for every $P \in L_0$, $\psi(P)$ propagates constant along a unique curve issued from P (see e.g. [8]). Unfortunately, by Proposition 2.1, $\pi(L(\eta)) \cap \partial_{\pm}\Omega = \emptyset$, so in order to study $\pi^{-1}(x) \cap L(\eta)$, for x near $\partial_{\pm}\Omega$, it is necessary to introduce the following machinery:

3.1 DEFINITION.

$$(3.1) \quad \tilde{L}_0(\eta) = \\ = \{(x^0, \xi^0); x_0 = (0, 0, y), \xi^0 = (1, \cos \theta, h(y) \sin \theta)\eta, y \in \mathbf{R}^n, 0 \leq \theta \leq \pi\}$$

where $h(y) = (-\langle B(x^0)\eta, \eta \rangle)^{-\frac{1}{2}}$

Remark that

$$L_0(\eta) = \{(x^0, \xi^0); (x_0, \xi^0) \in \tilde{L}_0(\eta), 0 < \theta < \pi, \xi^0 = (h(y) \sin \theta)^{-1} \xi^0\}.$$

We shall denote by $(x(t, \theta, y), \xi(t, \theta, y))$ and $(\tilde{x}(t, \theta, y), \tilde{\xi}(t, \theta, y))$ respectively the coordinates of the bicharacteristic of the form (2.5) issued from $(x^0, \xi^0) \in L_0(\eta)$ and $(x^0, \xi^0) \in \tilde{L}_0(\eta)$. From (2.6), by a homogeneity argument it follows

$$\begin{cases} \tilde{x}(t, \theta, y) = x(t, \theta, y), \\ \tilde{\xi}(t, \theta, y) = h(y) \sin \theta \xi(t, \theta, y), \end{cases} \quad 0 \leq t, 0 < \theta < \pi, y \in \mathbf{R}^n.$$

Now we choose positive constants ε, ω in a suitable way, as specified in Theorem 3.2 below, and let Σ be the union of the following subsets of $\mathbf{R}^+ \times (0, \pi) \times \mathbf{R}^n$:

$$(0, \varepsilon) \times (0, \pi) \times \mathbf{R}^n, \quad (0, M_0 + 1] \times (0, \omega) \times \mathbf{R}^n$$

and

$$\{(t, \theta); M_0 + 1 \leq t \text{ and } 0 < \theta < \omega(t - M_0)^{-1}\} \times \mathbf{R}^n;$$

(here M_0 is given from (2.3)); moreover let us define the following C^∞ functions: for every $(t, \theta, y) \in \mathbf{R}^+ \times [0, \pi] \times \mathbf{R}^n$,

$$F(t, \theta, y) := x(t, \theta, y) \quad \text{and} \quad u(t, \theta, y) := \frac{1}{2} [x_1(t, \theta, y) + x_2(t, \theta, y)].$$

So we can state a technical theorem which would be proved in next section:

3.2. THEOREM. There exist positive constants $C, \varepsilon, \delta, \omega$ such that $\omega < \pi, \varepsilon < M_0 + 1$ and

- (i) dF is non-singular on the closure of Σ in $\mathbf{R}^+ \times (0, \pi) \times \mathbf{R}^n$;

(ii) for every (t, θ, y) such that $0 \leq t, \theta \in [0, \pi], \|y\| \geq C,$

$$F(t, \theta, y) = (t, -t, h_0 t \sin \theta B_0 \eta + y)$$

(where $h_0 = (-\langle B_0 \eta, \eta \rangle)^{-1}$);

(iii) on $\bar{\mathbf{R}}^+ \times [0, \omega] \times \mathbf{R}^n, u(t, \theta, y) \geq \delta t \theta^2.$

4. In order to prove Theorem 3.2 we need some preliminary results.

If $\|y\| \gg 1,$ then we can explicitly compute $F(t, \theta, y):$

4.1. PROPOSITION. There exists a positive constant C such that

$$x_2(t, \theta, y) = -t \cos \theta, \quad x'(t, \theta, y) = h_0 t \sin \theta B_0 \eta + y$$

for every $t \geq 0, \theta \in [0, \pi], \|y\| \geq C.$

PROOF. It is easy to see from (2.3), (2.4) and (2.6) that there exists a suitable positive constant C such that $\|x(t, \theta, y)\| \geq M_0,$ for $t \geq 0, \theta \in [0, \pi], \|y\| \geq C;$ therefore (2.6) can be directly integrated. Q.E.D.

Since $x_1(t, \theta, y) = t,$ we get

4.2. COROLLARY. $dF(t, \theta, y)$ is non-singular for $t > 0, \theta \in (0, \pi), \|y\| \geq C.$

By definition

$$x_j, \xi_j \in C^\infty(\bar{\mathbf{R}}^+ \times [0, \pi] \times \mathbf{R}^n) \quad \text{and} \quad \xi_j \in C^\infty(\mathbf{R}^+ \times (0, \pi) \times \mathbf{R}^n),$$

$(j = 1, \dots, 2 + n);$ a straightforward computation gives the following.

4.3. LEMMA. There exist smooth functions r, R, s_1, s_2, S such that

$$x_2(t, \theta, y) = -t - \left[\frac{1}{2} h^2(y) \int_0^t \langle B(t', -t', y) \eta, \eta \rangle dt' \right] \theta^2 + r(t, \theta, y) \theta^3,$$

$$x'(t, \theta, y) = y + \left[h(y) \int_0^t B(t', -t', y) \eta dt' \right] \theta + R(t, \theta, y) \theta^2,$$

$$\xi_j(t, \theta, y) = 1 + s_j(t, \theta, y) \theta^2, \quad (j = 1, 2),$$

$$\xi_j'(t, \theta, y) = h(y) \eta \theta + S(t, \theta, y) \theta^2,$$

and

if $t = 0$, then $r = 0$, $R = 0$, $S = (\operatorname{sen} \theta - \theta)\theta^{-2}h(y)\eta$,

if $\|y\| \geq C$, then r , R do not depend on y .

Moreover $\xi'(t, \theta, y)$ can be smoothly extended to $\bar{R}^+ \times [0, \pi] \times \mathbf{R}^n$ by means of $\xi'(t, 0, y) = \xi'(t, \pi, y) = \eta$.

4.4. PROPOSITION. Let $t_0 > 0$. Then there exists a positive constant ω_1 such that $dF(t, \theta, y)$ is non singular on $(0, t_0] \times [0, \omega_1] \times \mathbf{R}^n$.

PROOF. Let

$$A(t, \theta, y) = \begin{pmatrix} \theta^{-1}t^{-1} \frac{\partial}{\partial \theta} x_2 & \theta^{-1}t^{-1} \frac{\partial}{\partial y} x_2 \\ \frac{\partial}{\partial \theta} x' & \frac{\partial}{\partial y} x' \end{pmatrix},$$

$$B(t, \theta, y) = \begin{pmatrix} t^{-1} \frac{\partial^2}{\partial \theta^2} x_2(t, 0, y) & 0 \\ \frac{\partial}{\partial \theta} x'(t, \theta, y) & I_n \end{pmatrix}.$$

From Lemma 4.3 and (2.4) it follows that $t^{-1}(\partial^2/\partial\theta^2)x_2(t, 0, y) \geq M_1/M_2$; therefore, for $(t, \theta, y) \in [0, t_0] \times [0, \pi] \times \mathbf{R}^n$, $B(t, \theta, y)$ is in a compact set of non singular matrices. Note that $\det((\partial x_2 \partial x')/(\partial \theta \partial y)) = \theta t \det A$. Now it is enough to show that for a suitable $M > 0$, $\|A - B\| \leq M\theta$ on $[0, t_0] \times [0, \pi] \times \mathbf{R}^n$.

Since $x_2 = 0$ for $t = 0$, from Lemma 4.3 we get

$$\begin{aligned} \theta^{-1} \frac{\partial}{\partial \theta} x_2(t, \theta, y) - \frac{\partial^2}{\partial \theta^2} x_2(t, 0, y) &= \\ &= \int_0^t \left[\theta^{-1} \frac{\partial}{\partial \theta} \dot{x}_2(t', \theta, y) - \frac{\partial^2}{\partial \theta^2} \dot{x}_2(t', 0, y) \right] dt' = \int_0^t \left(\theta \frac{\partial}{\partial \theta} \dot{r} + 3\dot{r} \right) dt' \theta. \end{aligned}$$

But r does not depend on y for $\|y\| \gg$, so

$$\left| \theta^{-1} \frac{\partial}{\partial \theta} x_2(t, \theta, y) - \frac{\partial^2}{\partial \theta^2} x_2(t, 0, y) \right| \leq M t \theta.$$

In a similar way one can show

$$\left| \theta^{-1} \frac{\partial}{\partial y} x_2(t, \theta, y) \right| \leq M t \theta \quad \text{and} \quad \left\| \frac{\partial}{\partial y} x'(t, \theta, y) - I_n \right\| \leq M \theta. \quad \text{Q.E.D.}$$

4.5. REMARK. Since an analogous of Lemma 4.3 holds for θ near π , we get: for every $t_0 > 0$ there exists ω_1 such that: $0 < \omega_1 < \pi$ and $dF(t, \theta, y)$ is non-singular on

$$(0, t_0] \times ((0, \omega_1] \cup [\pi - \omega_1, \pi)) \times \mathbf{R}^n.$$

4.6 PROPOSITION. There exists a positive constant ω_2 such that $dF(t, \theta, y)$ is not singular for $t \geq M_0 + 1$, $0 < \theta \leq \omega_2(t - M_0)^{-\frac{1}{2}}$, $y \in \mathbf{R}^n$.

In order to prove this Proposition we need some preparation. For $t \geq M_0$, all bicharacteristics are straight-line, so

$$\frac{\partial}{\partial \theta} x(t, \theta, y) = \frac{\partial}{\partial \theta} x(M_0, \theta, y) + \frac{\partial}{\partial \theta} \dot{x}(M_0, \theta, y)(t - M_0)$$

and an analogous formula holds for $(\partial/\partial y)x$. More precisely, thank to Lemma 4.3:

$$\frac{\partial}{\partial \theta} x_2(t, \theta, y) = c_{00}\theta + d_{00}\theta(t - M_0),$$

$$\frac{\partial}{\partial y_k} x_2(t, \theta, y) = c_{0k}\theta^2 + d_{0k}\theta^2(t - M_0),$$

$$\frac{\partial}{\partial \theta} x_{2+j}(t, \theta, y) = c_{j0} + d_{j0}(t - M_0),$$

$$\frac{\partial}{\partial y_k} x_{2+j}(t, \theta, y) = \delta_{jk} + c_{jk}\theta + d_{jk}\theta(t - M_0), \quad (j, k = 1, \dots, n).$$

here c_{jk} and d_{jk} are C^∞ functions of $\theta \in [0, \pi]$, $y \in \mathbf{R}^n$, ($j, k = 1, \dots, n$). Denote by J_k , C_k , D_k respectively the k -th column of matrices

$$J = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}, \quad (c_{jk})_{jk}, \quad (d_{jk})_{jk};$$

let

$$E_{j_k} = \begin{cases} J_k & \text{if } j = 0 \\ C_k & \text{if } j = 1 \\ D_k & \text{if } j = 2 \end{cases} \quad \text{and } E_{j_0 \dots j_n} = [E_{j_0 0}, \dots, E_{j_n n}] \\ \text{for } j_0, \dots, j_n \in \{0, 1, 2\}.$$

Finally let

$$\gamma = \gamma_{j_1 \dots j_n} = \# \{k \geq 1; j_k = 1\}, \quad \delta = \delta_{j_1 \dots j_n} = \# \{k \geq 1; j_k = 2\},$$

and

$$\alpha^* = \begin{cases} \alpha & \text{if } \alpha \geq 0, \\ 0 & \text{if } \alpha < 0. \end{cases}$$

PROOF OF PROPOSITION 4.6. Since $[\partial/\partial y_k \log h(y)] C_0(0, y) = C_k(0, y)$, $k = 1, \dots, n$ and the same formula holds for D_0, D_k , it follows that

$$\varphi_{1, j_1, \dots, j_n}(\theta, y) = \theta^{-\gamma - (\delta - 1)^*} \det E_{1, j_1, \dots, j_n}$$

and

$$\varphi_{2, j_1, \dots, j_n}(\theta, y) = \theta^{-(\gamma - 1)^* - \delta} \det E_{1, j_1, \dots, j_n}$$

are C^∞ functions. Therefore

$$\begin{aligned} \det \left(\frac{\partial x_2}{\partial \theta} \frac{\partial x'}{\partial y} \right) (t, \theta, y) &= \\ &= \sum_{j_1 \dots j_n} [\theta^{2\gamma + \delta + (\delta - 1)^* + 1} (t - M_0)^\delta \varphi_{1, j_1, \dots, j_n}(\theta, y) + \\ &+ \theta^{\gamma + (\gamma - 1)^* + 2\delta + 1} (t - M_0)^{\delta + 1} \varphi_{2, j_1, \dots, j_n}(\theta, y)] = \\ &= \theta [\varphi_{1, 0, \dots, 0}(\theta, y) + \sum_{\delta=0, \gamma>0} \theta^{2\gamma} \varphi_{1, j_1, \dots, j_n}(\theta, y)] + \\ &+ \theta(t - M_0) [\varphi_{2, 0, \dots, 0}(\theta, y) + \sum_{\gamma+\delta>0} \theta^{\gamma + (\gamma - 1)^* + 2\delta} (t - M_0)^\delta \varphi_{2, j_1, \dots, j_n}(\theta, y) + \\ &+ \sum_{\delta>0} \theta^{2\gamma + 2\delta - 1} (t - M_0)^{\delta - 1} \varphi_{1, j_1, \dots, j_n}(\theta, y)]. \end{aligned}$$

Now it is enough to make the following remarks:

$$\varphi_{1,0,\dots,0}(0, y) = c_{00}(0, y) = -h^2(y) \langle B_0 \eta, \eta, \eta \rangle \geq \frac{M_1}{M_2},$$

$$\varphi_{2,0,\dots,0}(0, y) = d_{00}(0, y) = -h^2(y) \int_0^{M_0} \langle B(t, -t, y) \eta, \eta \rangle dt \geq \frac{M_0 M_1}{M_2},$$

and $\varphi_{i_0, j_1, \dots, j_n}(\theta, y)$ are bounded on $[0, \pi] \times \mathbf{R}^n$. Q.E.D.

4.7. PROPOSITION. Let ω be such that $0 < \omega < \pi/2$. Then there exists $\varepsilon > 0$ such that $dF(t, \theta, y)$ is nonsingular on $(0, \varepsilon) \times [\omega, \pi - \omega] \times \mathbf{R}^n$.

PROOF. Initial data (3.1) give

$$\frac{\partial}{\partial \theta} x_2(0, \theta, y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y} x_2(0, \theta, y) = 0;$$

then from (2.6) and (3.1) it follows that

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-1} \det \left(\frac{\partial x_2}{\partial \theta} \frac{\partial x'}{\partial y} \right) (t, \theta, y) &= \\ &= \det \begin{pmatrix} \frac{\partial}{\partial \theta} \dot{x}_2 & \frac{\partial}{\partial y} \dot{x}_2 \\ \frac{\partial}{\partial \theta} x' & \frac{\partial}{\partial y} x' \end{pmatrix} (0, \theta, y) = \text{sen } \theta. \quad \text{Q.E.D.} \end{aligned}$$

The above results prove Theorem 3.2 (i) and (ii). For the last statement we show the following

4.8. PROPOSITION. There exist positive constants ω_s, δ such that $u(t, \theta, y) \geq \delta t^2$ on $\mathbf{R}^+ \times [0, \omega_s] \times \mathbf{R}^n$.

PROOF. Since

$$-\frac{1}{2} h^2(y) \int_0^t \langle B(t', -t', y) \eta, \eta \rangle dt' \geq \frac{t M_1}{2 M_2} \quad \text{and} \quad r(0, \theta, y) = 0$$

we get

$$u(t, \theta, y) \geq \frac{1}{2} \left[\frac{tM_1}{2M_2} \theta^2 + \int_0^t \dot{r}(t', \theta, y) dt' \theta^3 \right].$$

It is enough to remark that, for $t \geq M_0$, \dot{x}_2 (and so \dot{u} too) not depend on t , therefore \dot{u} is bounded on $\bar{\mathbf{R}}^+ \times [0, \pi] \times \mathbf{R}^n$. Q.E.D.

5. In this section we shall show that, for all fixed $\eta \in \bar{\mathbf{R}}^n$, there exists a (unique) solution $\psi(x)$ of (2.1-2), which is definite in a suitable domain.

5.1. LEMMA:

- (i) $F|_{\Sigma}: \Sigma \rightarrow F(\Sigma)$ is locally invertible and $F(\Sigma)$ is an open subset of Ω ;
- (ii) $F|_{\bar{\Sigma}}: \bar{\Sigma} \rightarrow \bar{\Omega}$ is a closed map;
- (iii) For every $x \in F(\Sigma)$, $F^{-1}(x) \cap \Sigma$ is finite.

PROOF. (i) follows from Theorem 3.2 (i) and Proposition 2.1.

(ii) Let $P_\nu = (t_\nu, \theta_\nu, y_\nu)$, $\nu \in N$, be a sequence in $\bar{\Sigma}$ with $\lim_{\nu \rightarrow +\infty} F(P_\nu) = \bar{x}$; then $t_\nu = x_1(P_\nu) \rightarrow \bar{x}_1$; therefore, if $\{P_\nu\}$ is not bounded, $\{y_\nu\}$ is not bounded too and one can apply Theorem 3.2 (ii) in order to see that $\{P_\nu\}$ has a limit point; hence, in every way, $\{P_\nu\}$ has a limit point.

(iii) Since F is injective for $\|y\| \geq C$, it follows that $F^{-1}(x) \cap \Sigma$ is bounded. Theorem 3.2 (i) says that $F^{-1}(x) \cap \Sigma$ has no accumulation points in $\bar{\Sigma} \cap (\bar{\mathbf{R}}^+ \times (0, \pi) \times \mathbf{R}^n)$; thanks to Proposition 2.1 it does not have any in $\bar{\Sigma}$ either. Q.E.D.

5.2. THEOREM. The map $F|_{\Sigma}: \Sigma \rightarrow F(\Sigma)$ is a diffeomorphism.

PROOF. By lemma 5.1, $F|_{\Sigma}$ is a covering map between arcwise connected spaces, so the cardinality of the fiber does not depend on the base-point.

Let $\Sigma_\varepsilon = (0, \varepsilon) \times (0, \pi) \times \mathbf{R}^n$, $\Omega^\varepsilon = \{x \in \Omega; x_1 < \varepsilon\}$. From lemma 5.1 (ii) it follows that $F(\bar{\Sigma}_\varepsilon) = \bar{F}(\bar{\Sigma}_\varepsilon)$, and from Proposition 2.1. $F(\partial\Sigma_\varepsilon) \cap \Omega^\varepsilon = \emptyset$; therefore $\bar{F}(\bar{\Sigma}_\varepsilon) \cap \Omega^\varepsilon = F(\Sigma_\varepsilon) \cap \Omega^\varepsilon$; hence $F(\Sigma_\varepsilon)$ is an open and closed subset of Ω^ε , i.e. $F(\Sigma_\varepsilon) = \Omega^\varepsilon$.

This proves that $F|_{\Sigma_\varepsilon}: \Sigma_\varepsilon \rightarrow \Omega^\varepsilon$ is a covering map between simply connected spaces, i.e. it is a homeomorphism. Q.E.D.

5.3. PROPOSITION. There exists $U > 0$ such that $\Omega_U \subseteq F(\Sigma)$.

PROOF. Choose $U < \min\{\varepsilon\delta\omega^2, \delta\omega^2\}$. Arguing as above it is enough to show $F(\partial\Sigma) \cap \Omega_U = \emptyset$.

From Theorem 3.2 (iii) it follows immediately that $u((\partial\Sigma - \partial\Sigma_\varepsilon) \cap (\mathbf{R}^+ \times (0, \pi) \times \mathbf{R}^n)) > U$. We shall see

$$(5.1) \quad u(\varepsilon, \theta, y) > U$$

for $\theta \geq \omega$, $y \in \mathbf{R}^n$. Both the hypersurface $\theta = \omega$ and $u(\varepsilon, \theta, y) = \varepsilon\delta\omega^2$ disconnect the strip $(0, \pi) \times \mathbf{R}^n$; then from $u(\varepsilon, 0, y) = 0$ and $u(\varepsilon, \omega, y) \geq \varepsilon\delta\omega^2$ it follows (5.1). Q.E.D.

Let now $t = t(x)$, $\theta = \theta(x)$, $y = y(x)$ be the components of the inverse map $\Omega_U \rightarrow F^{-1}(\Omega_U)$; they are smooth, moreover

5.4. LEMMA. (i) $t(x) = x$;

(ii) $\theta(x)$ can be continuously extended to $\bar{\Omega}_U - \partial_0\Omega$ by means of $\theta(\partial_\pm\Omega \cap \bar{\Omega}_U) = \arcsin(\pm 1)$;

(iii) $y(x)$ can be continuously extended to $\bar{\Omega}_U$ by means of $y(|x_2|, x_2, x') = x'$.

PROOF. Check the continuity of extensions (ii) and (iii) on a sequence $\{x_r\}$ in Ω_U converging to the boundary. Use Theorem 5.1 (ii) and Proposition 2.1.

Finally we can establish that

5.5. THEOREM. For every $\eta \in \mathbf{R}^n$ there exists a unique real solution $\psi \in C^\infty(\Omega_U)$ of (2.1-2) such that

$$(i) \quad \psi \in C(\bar{\Omega}_U),$$

$$(ii) \quad \frac{\partial}{\partial x_1} \psi \pm \frac{\partial}{\partial x_2} \psi \in C((\bar{\Omega}_U \cap \partial_\pm\Omega) \cup \Omega_U) \text{ and } \frac{\partial}{\partial x_j} \psi \in C(\bar{\Omega}_U),$$

$$(j = 3, \dots, 2 + n).$$

PROOF. *Existence.* Since the symplectic form $\sum_{j=1}^{2+n} d\xi_j \wedge dx_j$ vanishes identically on $L_0(\eta)$ we can apply Theorem 6.4.3 in [3] in order to see that the 1-form

$$\Omega_\sigma \in x \rightarrow (x, \xi(t(x), \theta(x), y(x))) \in L(\eta) \subseteq p^{-1}(0) \subseteq T^*(\bar{\Omega})$$

is closed. Thus there is a function $\psi \in C^\infty(\Omega_\sigma)$, unique up to a constant, such that

$$(5.2) \quad d\psi(x) = \xi(t(x), \theta(x), y(x)) ;$$

then ψ satisfies (2.1). Check that $(\partial/\partial t)[\psi(x(t, \theta, y))] = 0$, therefore

$$\frac{\partial}{\partial \theta} [\psi(x(t, \theta, y))] = \lim_{t \rightarrow 0^+} \frac{\partial}{\partial \theta} [\psi(x(t, \theta, y))] = \sum_{j=1}^{2+n} \xi_j(0, \theta, y) \partial_\theta x_j(0, \theta, y) = 0$$

and, in the same way,

$$\frac{\partial}{\partial y_k} [\psi(x(t, \theta, y))] = \eta_k \quad (k = 1, \dots, n).$$

Hence, up to a constant, $\psi(x(t, \theta, y)) = y \cdot \eta$. Next, for a suitable choice of the constant,

$$(5.3) \quad \psi(x) = y(x) \cdot \eta, \quad \text{for every } x \in \Omega_\sigma.$$

From Lemma 4.3.

$$\begin{aligned} \frac{\partial}{\partial x_1} \psi(x) - \frac{\partial}{\partial x_2} \psi(x) &= \\ &= [s_1(t(x), \theta(x), y(x)) - s_2(t(x), \theta(x), y(x))] \frac{\theta^2(x)}{h(y(x)) \sin \theta(x)}, \\ \frac{\partial}{\partial x'} \psi(x) &= \frac{[\eta + S(t(x), \theta(x), y(x)) \theta(x) h^{-1}(y(x))] \theta(x)}{\sin \theta(x)}. \end{aligned}$$

If $\bar{x} \in \bar{\Omega}_\sigma \cap \partial_- \Omega$ (resp. $\bar{\Omega}_\sigma \cap (\partial_0 \Omega \cup \partial_- \Omega)$) apply Lemma 5.4 to see that

$$\lim_{x \rightarrow \bar{x}} \frac{\partial}{\partial x_1} \psi(x) - \frac{\partial}{\partial x_2} \psi(x) = 0 \quad \left(\text{resp. } \lim_{x \rightarrow \bar{x}} \frac{\partial}{\partial x'} \psi(x) = \eta \right).$$

Uniqueness. Let ψ satisfy all the hypotheses. Certainly there exists $x^0 \in \Omega_\sigma$ with $d\psi(x^0) \neq 0$. Let $(x(t), \xi(t))$, with $\bar{t} \leq t < +\infty$ and $x(\bar{t}) \in \partial\Omega$, be the null bicharacteristic passing through $(x^0, d\psi(x^0))$. As is well known, $\xi(t) = d\psi(x(t))$.

Suppose for a moment that $x(\bar{t}) \in \partial_\pm \Omega$. Now $x(\bar{t}) = (\bar{t}, \pm \bar{t}, x'(\bar{t}))$ with $\bar{t} > 0$, therefore, from (2.2)

$$\xi'(\bar{t}) = \frac{\partial}{\partial x'} \psi(x(\bar{t})) = \frac{\partial}{\partial x'} (x' \cdot \eta)|_{x'=x'(\bar{t})} = \eta$$

and

$$\begin{aligned} \xi_1(\bar{t}) \pm \xi_2(\bar{t}) &= \left(\frac{\partial}{\partial x_1} \psi(x(\bar{t})) \pm \frac{\partial}{\partial x_2} \psi(x(\bar{t})) \right) = \\ &= \frac{d}{dt} [\psi(t, \pm t, x'(\bar{t}))]|_{t=\bar{t}} = \frac{d}{dt} x'(\bar{t}) \cdot \eta|_{t=\bar{t}} = 0 ; \end{aligned}$$

but this is in contradiction with $p(x(\bar{t}), \xi(\bar{t})) = 0$.

We conclude that $\bar{t} = 0$, then $(x(0), \xi(0)) \in L_0(\eta)$. This proves $(x^0, d\psi(x^0)) \in L(\eta)$. From (2.8) it follows that

$$\left\| \frac{\partial}{\partial x'} \psi(x^0) \right\| \geq \|\eta\| \exp(-kx_1^0),$$

so $d\psi$ never vanishes and the above argument can be repeated for every point of Ω_σ . Hence we obtain (5.2) for every $x \in \Omega_\sigma$; but, as already recalled, this determines ψ up to a constant. The thesis follows since $\psi = x \cdot \eta$ on $\partial\Omega$. Q.E.D.

6. Finally we have to consider the dependence on $\eta \in \mathbf{R}^n$, then we write $\psi_{(\eta)}$ instead of ψ , which is defined on $\bar{\Omega}_\sigma$ with $U = U(\eta)$.

It is easy to see, from (2.6), that $F(t, \theta, y)$ and $y(t, \theta, y)$ are homogeneous of degree 0 in η , therefore, by (5.3), $\psi_{(\eta)}$ is of degree 1. Thus, since $\{\eta \in \mathbf{R}^n; \|\eta\| = 1\}$ is compact, there exists $U > 0$ which works for every $\eta \neq 0$. So we have proved the following:

6.1. THEOREM. (i) There exists $U > 0$ such that (1.1) has two solutions

$$\varphi^\pm(u, v, z, \eta) \in C^\infty((0, U] \times \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n) \cap C([0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \mathbf{R}^n),$$

real and homogeneous of degree 1 in η . Moreover

$$\frac{\partial}{\partial u} \varphi^{\pm} \in C((0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \dot{\mathbf{R}}^n), \quad \frac{\partial}{\partial v} \varphi^{\pm} \in C([0, U] \times \dot{\mathbf{R}}^+ \times \mathbf{R}^n \times \dot{\mathbf{R}}^n),$$

$$\frac{\partial}{\partial z_j} \varphi^{\pm} \in C([0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \dot{\mathbf{R}}^n), \quad (j = 1, \dots, n).$$

(ii) If φ satisfies the above conditions, then φ coincides with φ^+ or φ^- in each connected component of $[0, U] \times \bar{\mathbf{R}}^+ \times \mathbf{R}^n \times \dot{\mathbf{R}}^n$.

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