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The p^*p-Injectors of a Finite Group.

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1. Introduction and notation.

All groups considered in this note are finite. In ([2], 5.4 Satz) Blessenohl and Laue gave the description of the $\mathfrak{N}^*$-injectors of a grou G, where $\mathfrak{N}^*$ is the class of the generalized nilpotent groups; these injectors are the subgroups $VF^*(G)$, where $F^*(G)$ is the generalized Fitting subgroup of G and V describes the conjugacy class of the nilpotent injectors of the subgroup $C_G(F^*(G)/F(G)) = C_G(E(G))$ where $E(G)$ is the semisimple radical of G. In ([4], Proposition 8) it is shown that if $\mathfrak{F}$ is a Fitting class between $\mathfrak{N}$ (i.e. the class of nilpotent groups) and $\mathfrak{N}^*$, then in a group G such that $F^*(G) \in \mathfrak{F}$ there is a unique conjugacy class of $\mathfrak{F}$-injectors that are precisely the $\mathfrak{N}^*$-injectors of G, and conversely.

The aim of this note is to give a description of the p^*-by-p injectors and the p^*p-injectors of a group, by following the analogies between the class of semisimple groups and its radical $E(G)$ and the class of p^*-groups and its radical $O_{p^*}(G)$ (i.e. the generalized p'-core of G) and those between the class of generalized nilpotent groups

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and its radical $F^*(G)$ and the class of the p^*p-groups and its corresponding radical $O_{p^*p}(G)$ as was developed by Bender in [1].

Given a fixed prime p we shall denote by $\mathfrak{S}_p$ the class of all p-groups, $\mathfrak{E}_{p'}$ the class of all p'-groups, $\mathfrak{E}_{p^*}$ that of all p^*-groups and $\mathfrak{E}_{p^*p}$ that of the p^*p-groups; the corresponding radicals in a group G are denoted, as usual, by $O_p(G)$, $O_{p'}(G)$, $O_{p^*}(G)$ and $O_{p^*p}(G)$. $O^p(G)$ is the p-residual of G. Finally if H is a subgroup of G, $C^*_G(H)$ is the generalized centralizer of H in G.

For all definitions we refer to Bender [1].

2. The injectors.

Lemma 1. The class $\mathfrak{E}_{p^*}\mathfrak{S}_p = (G\colon O^p(G) \in \mathfrak{E}_{p^*})$ is a Fitting class containing the Fitting class $\mathfrak{E}_{p^*p}$. The corresponding radical is $O_{p^*,p}(G)$.

Proof. The fact that $\mathfrak{E}_{p^*}\mathfrak{S}_p$ is closed for normal subgroups is precisely ([3], Ch. X, 14.11). If $G = N_1N_2$ such that $N_i \trianglelefteq G$ and $N_i \in \mathfrak{E}_{p^*}\mathfrak{S}_p$, $i = 1, 2$, we have $O_{p^*}(N_1)O_{p^*}(N_2) \leqslant O_{p^*}(G)$ and $P \cap N_i$ is a Sylow p-subgroup of N_i, $i = 1, 2$, if P is such a subgroup of G, then $N_i = (P \cap N_i)O_{p^*}(N_i)$, $i = 1, 2$, and finally we have $G = PO_{p^*}(G)$. For the remaining statements one can see ([1], 4.13) and ([1], 6.1 (iii), (iv)).

Remarks. 1) The inclusion in the lemma is strict for every prime p, take for instance E a simple group with selfcentralizing Sylow p-subgroup P and C_p the cyclic group of order p, then the regular wreath product $G = E \operatorname{wr} C_p$ does not belong to $\mathfrak{E}_{p^*p}$ because $C^*_G(P^p) \leqslant E^p$ (here E^p is the base group of G and P^p is a Sylow p-subgroup of E^p).

2) The class of all groups that are a central product of a p^*-group and a p-group is not a Fitting class.

Lemma 2. If $\mathfrak{F}$ is any Fitting class such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$, then

i) The $\mathfrak{F}$-maximal subgroups of G containing $G_{\mathfrak{F}}$ are the subgroups $(O_{p^*}(G)P)_{\mathfrak{F}}$ where P describes the Sylow p-subgroups of G.

ii) If $(O_{p^*}(G)P)_{\mathfrak{F}} \leqslant H \leqslant G$ then there is a Sylow p-subgroup P_0 of H such that $(O_{p^*}(G)P)_{\mathfrak{F}} = (O_{p^*}(H)P_0)_{\mathfrak{F}}$.

PROOF. i) Consider $G_{\mathfrak{F}} \leqslant S \leqslant G$ then $O_{p^*p}(G) \leqslant S$ allows us to use ([1], 4.22) and deduce that $O_{p^*}(S) = O_{p^*}(G)$; but if, moreover, S belongs to $\mathfrak{F}$, we have $S = O_{p^*}(G)Q$ where Q is a Sylow p-subgroup of S. By taking a Sylow p-subgroup P of G such that $Q \leqslant P$, we have $S = O_{p^*}(G)Q \trianglelefteq\trianglelefteq O_{p^*}(G)P$, so if S is $\mathfrak{F}$-maximal in G, clearly $S = (O_{p^*}(G)P)_{\mathfrak{F}}$.

ii) As before $O_{p^*}(H) = O_{p^*}(G)$, so if T is a Sylow p-subgroup of $(O_{p^*}(G)P)_{\mathfrak{F}}$ and P_0 is one of H containing T we can write

$$(O_{p^*}(G)P)_{\mathfrak{F}} = O_{p^*}(G)T \trianglelefteq\trianglelefteq O_{p^*}(H)P_0$$

therefore

$$(O_{p^*}(G)P)_{\mathfrak{F}} \leqslant (O_{p^*}(H)P_0)_{\mathfrak{F}}$$

both $\mathfrak{F}$-subgroups of G, so by (i), the equality is valid.

The first statement of this lemma means that the Fitting classes between $\mathfrak{E}_{p^*p}$ and $\mathfrak{E}_{p^*}\mathfrak{S}_p$ are dominant in the class of all finite groups, so it describes the injectors of every group (see [2], 5.1 Satz). On the other hand, by the second part of the lemma every injector of a group G is an injector of each subgroup containing it.

REMARK. A group G is p-constrained if and only if $O_{p^*}(G) = O_{p'}(G)$ (see [1], 6.12 or [3], Ch. X, 14.12 Theorem b)); so if we restrict ourselves to the Fitting class of all p-constrained groups we can deduce from Lemma 2 that the class $\mathfrak{E}_{p'}\mathfrak{S}_p$ of p-nilpotent groups is dominant in the class of p-constrained groups by obtaining a description for the p-nilpotent injectors of a p-constrained group G they are the subgroups of the form $O_{p'}(G)P$ where P describes the set of Sylow p-subgroups of G and such a subgroup is a p-nilpotent injector of every subgroup of G containing it.

But conversely, let us assume that G is a group having a unique conjugacy class of p-nilpotent injectors, then the same is true for $\bar{G} = G/O_{p'}(G)$, therefore by [5] $E(\bar{G})$ is p-nilpotent and due to the fact that $O_{p'}(\bar{G})$ is trivial, $E(\bar{G})$ must be trivial too i.e. $\bar{G}$ is p-constrained, so G itself is p-constrained.

LEMMA 3. Let G be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-group and Q a Sylow p-subgroup of $O_{p^*}(G)$, then $O_{p^*p}(G) = O_{p^*}(G)T$ where T is a Sylow p-subgroup of $C^*_G(Q)$.

PROOF. Let us write $X = O_{p^*p}(G)$, $Y = O_{p^*}(G)$, Z a Sylow p-subgroup of $O_{p^*p}(C^*_G(Q))$, and T a Sylow p-subgroup of $C^*_G(Q)$ such that $Z \leqslant T$. Put $S = YT$ and observe that Y and S/Y are p^*p-groups; on the other hand $S \leqslant YC^*_G(Q)$ implies

$$S = Y(C^*_G(Q) \cap S) \leqslant YC^*_S(Q) \leqslant S$$

so $S = YC^*_S(Q)$ and by ([1], 4.10) we have that S is a p^*p-group. But on the other hand by using ([1], 4.18) and ([1], 6.10) or ([3], Ch. X, 14.13) we can write

$$X = YO_{p^*p}(C^*_G(Q)) = YO_{p^*}(C^*_G(Q))Z = YZ \leqslant YT = S .$$

But G is an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-group, so $S = X$.

From the last two lemmas we can assure

THEOREM. The $\mathfrak{E}_{p^*p}$-injectors of a group G are the subgroups of the conjugacy class $O_{p^*}(G)T$ where T describes the set of all Sylow p-sybgroups of $C^*_{O_{p^*}(G)P}(Q)$ where P describes the set of such subgroups of G and Q those of $O_{p^*}(G)$.

If one follows the notation and the unified approach given by Bender ([1], § 4), namely taking for π either the set of all primes or a set consisting of a single one we can give a unified formula for the $\mathfrak{N}^*$-injectors and the $\mathfrak{E}_{p^*p}$-injectors of G; in an abridged form

$$\mathrm{Inj}_{\mathfrak{E}_{\pi^*\pi}}(G) = \left\{O_{\pi^*}(G)T \colon T \in \mathrm{Inj}_{\mathfrak{N}_\pi}\left(C^*_{O_{\pi^*}(G)G_\pi}(O_{\pi^*}(G)_\pi)\right)\right\}$$

where if X is any group X_π stands for a Sylow π-subgroup of X in the above explained sense.

The p^*p-counterpart of ([4], Proposition 8) is the following

PROPOSITION. Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$ and $\mathfrak{F}\mathfrak{S}_p = \mathfrak{F}$, then if G is a group such that $O_{p^*,p}(G)$ belongs to $\mathfrak{F}$, then G has a unique conjugacy class of $\mathfrak{F}$-injectors, namely the $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injectors. Conversely, if G is a group whose $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injectors belong to $\mathfrak{F}$, then $O_{p^*,p}(G) \in \mathfrak{F}$.

PROOF. It is easy to prove that each $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G belongs to $\mathfrak{F}$; on the other hand given any $\mathfrak{F}$-injector H of G, then H contains $G_{\mathfrak{F}} = O_{p^*,p}(G)$ so there is an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector V of G containing H, therefore $H = V$; finally use the dominance of $\mathfrak{E}_{p^*}\mathfrak{S}_p$ to deduce the existence of such injectors. The converse is trivial.

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