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of modules and applications**

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Representable Equivalences between Categories of Modules and Applications.

CLAUDIA MENINI - ADALBERTO ORSATTI (*)

Dedicato a Giovanni Zacher.

1. Introduction.

All rings considered in this paper have a nonzero identity and all modules are unital. For every ring R , $\text{Mod-}R$ ($R\text{-Mod}$) denotes the category of all right (left) R -modules. The symbol M_R (${}_R M$) is used to emphasize that M is a right (left) R -module.

Categories and functors are understood to be additive. Any subcategory of a given category is full and closed under isomorphic objects.

1.1. Let A and R be two rings, \mathcal{D}_A and \mathcal{G}_R subcategories of $\text{Mod-}A$ and $\text{Mod-}R$ respectively.

Assume that a category equivalence (F, G) , $F: \mathcal{D}_A \rightarrow \mathcal{G}_R$ and $G: \mathcal{G}_R \rightarrow \mathcal{D}_A$ is given. We say that the equivalence (F, G) is *representable* if there exists a bimodule ${}_A P_R$, with $P_R \in \mathcal{G}_R$, such that the following natural equivalences of functors hold:

$$F \approx (- \otimes_A P) | \mathcal{D}_A, \quad G \approx \text{Hom}_R(P_R, -) | \mathcal{G}_R.$$

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In this case we say that the bimodule ${}_A P_R$ induces the equivalence (F, G) . Note that, if $A_A \in \mathcal{D}_A$, then A is canonically isomorphic to $\text{End}(P_R)$.

For example if $\mathcal{D}_A = \text{Mod-}A$ and $\mathcal{G}_R = \text{Mod-}R$, then a classical Morita's result [M] asserts that (F, G) is representable by a faithful balanced bimodule ${}_A P_R$ which is a progenerator on both sides and conversely any such a bimodule induces an equivalence between $\text{Mod-}A$ and $\text{Mod-}R$.

1.2. More recently Fuller [F] proved the following result: if $\mathcal{D}_A = \text{Mod-}A$ and if \mathcal{G}_R is closed under submodules, epimorphic images and arbitrary direct sums, then (F, G) is representable by a bimodule ${}_A P_R$ such that P_R is a *quasi-progenerator* i.e. P_R is quasi-projective, finitely generated (f.g.) and generates all its submodules. Conversely any quasi-progenerator P_R with $A = \text{End}(P_R)$ induces such an equivalence. If P_R is a progenerator then $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$ and R is dense in $\text{End}({}_A P)$ endowed with its finite topology. For unexplained terms see Section 2.

1.3. In this paper we prove the following representation theorem. Assume that

- a) $A_A \in \mathcal{D}_A$ and \mathcal{D}_A is closed under submodules.
- b) \mathcal{G}_R is closed under arbitrary direct sums and epimorphic images.
- c) A category equivalence (F, G) , $F: \mathcal{D}_A \rightarrow \mathcal{G}_R$, $G: \mathcal{G}_R \rightarrow \mathcal{D}_A$ is given

and let Q_R be a fixed, but arbitrary, injective cogenerator of $\text{Mod-}R$. Then there exists a bimodule ${}_A P_R$ with the following properties:

- 1) $P_R \in \mathcal{G}_R$, $A \cong \text{End}(P_R)$.
- 2) $\mathcal{G}_R = \text{Gen}(P_R)$, $\mathcal{D}_A = \mathcal{D}(K_A)$ where $K_A = \text{Hom}_R(P_R, Q_R)$ and $\mathcal{D}(K_A)$ is the subcategory of $\text{Mod-}A$ cogenerated by K_A .
- 3) The bimodule ${}_A P_R$ induces the equivalence (F, G) .

1.4. The categories \mathcal{D}_A and \mathcal{G}_R involved in 1.3 are the largest possible. Indeed, given any bimodule ${}_A P_R$ and setting $T = - \otimes_A P$, $H = \text{Hom}_R(P_R, -)$ we have $\text{Im}(T) \subseteq \text{Gen}(P_R)$ and $\text{Im}(H) \subseteq \mathcal{D}(K_A)$.

1.5. Under the assumptions *a*), *b*), *c*) in 1.3, suppose that, in addition, \mathfrak{G}_R is closed under submodules. Then we prove that $\mathfrak{D}(K_A) = \text{Mod-}A$ so that P_R is a quasi-progenerator.

Thus we obtain, in this way, a non trivial generalization of Fuller's Theorem on equivalences.

1.6. Under the assumptions *a*), *b*), *c*) in 1.3 it holds, in general, that $\mathfrak{D}_A \neq \text{Mod-}A$. This will be proved in Section 4 using tilting modules of Happel and Ringel [HR₂]. Nevertheless we are able to give, in Section 5, a number of conditions in order that $\mathfrak{D}_A = \text{Mod-}A$. In particular this is true if P_R is quasi-projective.

The following question is still open: characterize the modules $P_R \in \text{Mod-}R$ such that, setting $A = \text{End}(P_R)$, the bimodule ${}_A P_R$ induces an equivalence between $\mathfrak{D}(K_A)$ and $\text{Gen}(P_R)$.

1.7. Using Pontryagin duality on R , Theorem in 1.3 can be translated in a representation theorem for a given duality between the category \mathfrak{D}_A in 1.3 and a category ${}_R \mathfrak{C}$ of compact modules which is assumed to be closed under topological products and closed submodules.

This representation theorem leads us to solve an old question of ours [MO]: there exist dualities between \mathfrak{D}_A and ${}_R \mathfrak{C}$ which are not « good dualities ».

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2. Preliminaries.

2.1. Let A and R be two rings, ${}_A P_R$ any bimodule. Define the functors T and H by setting

$$T = - \otimes_A P: \text{Mod-}A \rightarrow \text{Mod-}R,$$

$$H = \text{Hom}_R(P_R, -): \text{Mod-}R \rightarrow \text{Mod-}A.$$

Let $\text{Gen}(P_R)$ be the full subcategory of $\text{Mod-}R$ generated by P_R . Recall that a module $M \in \text{Mod-}R$ belongs to $\text{Gen}(P_R)$ if there exists an epimorphism $P_R^{(X)} \rightarrow M \rightarrow 0$ where X is a suitable set. $\text{Gen}(P_R)$ is closed under taking epimorphic images and arbitrary direct sums.

Denote by $\overline{\text{Gen}}(P_R)$ the smallest subcategory of $\text{Mod-}R$ containing $\text{Gen}(P_R)$ and closed under taking submodules, epimorphic images and direct sums. Clearly $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$ if and only if $\text{Gen}(P_R)$ is closed under submodules.

Let $C(P_R)$ be the subcategory of $\text{Mod-}R$ consisting of all modules $M \in \text{Mod-}R$ having P_R -codominant dimension ≥ 2 i.e. for which there is an exact sequence of the form

$$P_R^{(X)} \rightarrow P_R^{(Y)} \rightarrow M \rightarrow 0.$$

Clearly

$$C(P_R) \subseteq \text{Gen}(P_R) \subseteq \overline{\text{Gen}}(P_R).$$

Let Q_R be a fixed, but arbitrary, injective cogenerator of $\text{Mod-}R$ and set $K_A = \text{Hom}_R(P_R, Q_R)$. Denote by $\mathcal{D}(K_A)$ the full subcategory of $\text{Mod-}A$ cogenerated by K_A and by $L(K_A)$ the subcategory of $\text{Mod-}A$ consisting of all modules $L \in \text{Mod-}A$ having K_A -dominant dimension ≥ 2 . This means that there exists an exact sequence

$$0 \rightarrow L \rightarrow K_A^X \rightarrow K_A^Y.$$

Clearly

$$L(K_A) \subseteq \mathcal{D}(K_A).$$

Finally, for every $M \in \text{Mod-}R$, set

$$t_P(M) = \sum \{\text{Im}(f) : f \in \text{Hom}_R(P_R, M)\}.$$

Then $t_P(M) \in \text{Gen}(P_R)$ and $\text{Hom}_R(P, M) \cong \text{Hom}_R(P, t_P(M))$ canonically.

2.2. PROPOSITION. *Let ${}_A P_R$ be any bimodule. Then:*

- a) $\text{Im}(T) \subseteq C(P_R) \subseteq \text{Gen}(P_R)$.
- b) $\text{Im}(H) \subseteq L(K_A) \subseteq \mathcal{D}(K_A)$.
- c) *For every $M \in \text{Mod-}R$, $H(M) \cong H(t_P(M))$ canonically.*

PROOF. a) Let $L \in \text{Mod-}A$. There is an exact sequence of the form

$$A^{(x)} \rightarrow A^{(y)} \rightarrow L \rightarrow 0 .$$

Tensoring by ${}_A P$ we get the exact sequence

$$P_R^{(x)} \rightarrow P_R^{(y)} \rightarrow T(L) \rightarrow 0 .$$

Hence $T(L) \in C(P_R)$.

b) Let $M \in \text{Mod-}R$. There exists an exact sequence

$$0 \rightarrow M \rightarrow Q_R^x \rightarrow Q_R^y .$$

Applying H we get the exact sequence

$$0 \rightarrow H(M) \rightarrow K_A^x \rightarrow K_A^y$$

so that $H(M) \in L(K_A)$.

c) is obvious.

2.3. Let ${}_A P_R$ be any bimodule. Recall that for every $M \in \text{Mod-}R$ there exists a natural morphism in $\text{Mod-}R$

$$\varrho_M: \text{Hom}_R(P_R, M) \otimes_A P \rightarrow M$$

given by $\varrho_M(f \otimes p) = f(p)$ ($f \in \text{Hom}_R(P_R, M)$, $p \in P$).

It is also well known that, for every $L \in \text{Mod-}A$ there is a natural morphism in $\text{Mod-}A$

$$\sigma_L: L \rightarrow \text{Hom}_R(P_R, L \otimes_A P)$$

given by

$$\sigma_L(l): p \mapsto l \otimes p \quad (l \in L, p \in P) .$$

The following remarks are useful:

- a) For every $M \in \text{Gen}(P_R)$, ϱ_M is surjective.
- b) For every $L \in \mathcal{D}(K_A)$, σ_L is injective.

Statement *a*) is obvious. Let us prove *b*). Let $L \in \mathcal{D}(K_A)$. Then there exists an inclusion $L \hookrightarrow \text{Hom}_R(P_R, Q_R)^X$. Thus for every $l \in L$, $l \neq 0$, there exists a $\xi \in \text{Hom}_A(L, \text{Hom}_R(P_R, Q_R))$ such that $\xi(l) \neq 0$. Hence there is a $p \in P$ such that $\xi(l)(p) \neq 0$. Let $\bar{\xi}: L \otimes^A P \rightarrow Q_R$ be the morphism defined by setting $\bar{\xi}(x \otimes y) = \xi(x)(y)$, $x \in L$, $y \in P$. Then $\bar{\xi}(l \otimes p) = \xi(l)(p) \neq 0$ and thus $l \otimes p \neq 0$ so that $l \notin \text{Ker}(\sigma_L)$. Hence $\text{Ker}(\sigma_L) = 0$.

2.4. PROPOSITION. *Let ${}_A P_R$ be a bimodule which induces an equivalence between a subcategory \mathcal{D}_A of $\text{Mod-}A$ and a subcategory \mathcal{S}_R of $\text{Mod-}R$. Then, if $A \in \mathcal{D}_A$, for every $M \in \mathcal{S}_R$ and for every $L \in \mathcal{D}_A$ the morphisms ϱ_M and σ_L are isomorphisms.*

PROOF. Let $M \in \mathcal{S}_R$. Then, by Proposition 2.2, $M \in \text{Gen}(P_R)$ and hence, by 2.3, ϱ_M is surjective. Set $N_A = \text{Hom}_R(P_R, M)$.

By assumption there is an isomorphism

$$\varrho: TH(M) = T(N) \rightarrow M.$$

Let $\theta: N_A \rightarrow \text{Hom}_R(P_R, M_R)$ be the morphism corresponding to ϱ because of the adjointness of T and H . Then for every $f \in N = \text{Hom}_R(P_R, M_R)$ and for every $p \in P$:

$$(1) \quad \theta(f)(p) = \varrho(f \otimes p).$$

Let $h \in \text{Hom}_R(M_R, M_R)$ such that $\theta = \text{Hom}_R(P_R, h)$.

Then, for every $f \in N$,

$$\theta(f) = h \circ f$$

and hence, for every $p \in P$,

$$(2) \quad \theta(f)(p) = h(f(p)) = (h \circ \varrho_M)(f \otimes p).$$

From (1) and (2) we get

$$\varrho(f \otimes p) = (h \circ \varrho_M)(f \otimes p)$$

for every $f \in N$, $p \in P$.

Thus $\varrho = h \circ \varrho_M$. Then ϱ_M is injective as ϱ is injective.

Let now $L \in \mathcal{D}_A$. Then, by Proposition 2.2, $L \in \mathcal{D}(K_A)$ and hence by 2.3, σ_L is injective. Let $\xi \in \text{Hom}_R(P, L \underset{A}{\otimes} P) = \text{Hom}_R(T(A), T(L))$. Then there is an $f \in \text{Hom}_A(A, L)$ such that $\xi = T(f)$. Let $x = f(1)$. Then, for every $p \in P$ we have

$$(\sigma_L(x))(p) = f(1) \otimes p = (T(f))(1 \otimes p) = \xi(p).$$

Thus $\sigma_L(x) = \xi$ and σ_L is surjective.

2.5. DEFINITION. A module ${}_A P \in \text{Mod-}A$ is called *weak generator* if, for every $L \in \text{Mod-}A$,

$$L \neq 0 \Rightarrow L \underset{A}{\otimes} P \neq 0.$$

2.6. LEMMA. Let $M \in \text{Gen}(P_R)$, $h: P_R^{(X)} \rightarrow M$ be an epimorphism. Let $h = (h_x)_{x \in X}$ with $h_x \in \text{Hom}_R(P_R, M)$. Then the right A -submodule $\sum_{x \in X} h_x A$ of $\text{Hom}_R(P_R, M)$ belongs to $\mathcal{D}(K_A)$. Moreover if

$$\sum_{x \in X} h_x A = \text{Hom}_R(P_R, M)$$

then for every $f: P_R \rightarrow M$ there exists a $g \in \text{Hom}_R(P_R, P_R^{(X)})$ such that $f = h \circ g$.

PROOF. The first assertion follows by Proposition 2.2. Assume now that $\sum_{x \in X} h_x A = \text{Hom}_R(P_R, M)$ and let $f: P_R \rightarrow M$ be an R -morphism. Then $f = \sum_{x \in X} h_x a_x$ where $a_x \in A$ and almost all a_x 's vanish. Let $g: P_R \rightarrow P_R^{(X)}$ be the diagonal morphism of the a_x 's, $x \in X$. Then for every $p \in P$ we have $(h \circ g)(p) = \sum h_x a_x(p) = f(p)$. Thus $f = h \circ g$.

2.7. DEFINITIONS. Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$. P_R is called *quasi-projective* if, for every diagram

$$\begin{array}{ccc} & P_R & \\ & \downarrow f & \\ P_R & \xrightarrow{h} & M \longrightarrow 0 \end{array}$$

with exact row, there exists an $a \in A$ such that $f = h \circ a$.

P_R is called Σ -quasi-projective if $P_R^{(X)}$ is quasi-projective for every set $X \neq \emptyset$. Clearly P_R is Σ -quasi-projective if and only if P_R is a projective object of $\text{Gen}(P_R)$.

2.8. LEMMA. Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$, $M \in \text{Gen}(P_R)$, $h = (h_x)_{x \in X}$ an epimorphism of $P_R^{(X)}$ onto M and assume that ϱ_M is injective. Then:

- a) The morphism $\varphi: \left(\sum_{x \in X} h_x A\right) \otimes_A P \rightarrow \text{Hom}_R(P_R, M) \otimes_A P$ given by the inclusion $\sum_{x \in X} h_x A \hookrightarrow \text{Hom}_R(P_R, M)$ is surjective.
- b) If ${}_A P$ is a weak generator, then $\sum_{x \in X} h_x A = \text{Hom}_R(P_R, M)$.

PROOF. See [A], Lemma 1 and Proposition 5.

2.9. LEMMA. Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$. If the functors $T = - \otimes_A P$ and $H = \text{Hom}(P_R, -)$ subordinate an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$ and if $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$, then:

- a) For every $M \in \text{Gen}(P_R)$ and for every epimorphism

$$h: P_R^{(X)} \rightarrow M, \quad h = (h_x)_{x \in X},$$

we have

$$\sum_{x \in X} h_x A = \text{Hom}_R(P_R, M).$$

- b) P_R is Σ -quasi-projective.

PROOF. Since (T, H) is an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$, by Proposition 2.4 for every $M \in \text{Gen}(P_R)$ and for every $L \in \mathcal{D}(K_A)$, ϱ_M and σ_L are isomorphisms. Moreover, to prove a), it is enough to prove that the morphism

$$\varphi: \left(\sum_{x \in X} h_x A\right) \otimes_A P \rightarrow \text{Hom}_R(P_R, M) \otimes_A P$$

defined in Lemma 2.8 is an isomorphism. Now, as ϱ_M is injective, by Lemma 2.8, φ is surjective.

Assume that φ is not injective. Set $L = \sum_{x \in X} h_x A$ and consider the inclusion $i: L \rightarrow \text{Hom}_R(P_R, M)$. Applying $-\otimes_A P$ we get the exact sequence in $\text{Mod-}R$

$$(1) \quad 0 \rightarrow Y \rightarrow T(L) \xrightarrow{T(i)} TH(M),$$

where $T(i) = \varphi$ and $Y \neq 0$. Since $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$, $Y \in \text{Gen}(P_R)$. Applying H to (1) and setting $\bar{M} = H(M)$ we have the exact sequence

$$(2) \quad 0 \rightarrow H(Y) \rightarrow HT(L) \xrightarrow{HT(i)} HT(\bar{M}).$$

On the other hand we have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{i} & \bar{M} \\ \sigma_L \downarrow & & \downarrow \sigma_{\bar{M}} \\ HT(L) & \xrightarrow{HT(i)} & HT(\bar{M}) \end{array}$$

Since $L \in \mathcal{D}(K_A)$, σ_L and $\sigma_{\bar{M}}$ are both isomorphisms. It follows that $HT(i)$ is injective so that, from (2), we get $H(Y) = 0$. Thus $Y = 0$ as $Y \in \text{Gen}(P_R)$. Contradiction.

b) By a) and Lemma 2.6, it follows that for every diagram with exact row

$$\begin{array}{ccccc} & & P_R & & \\ & & \downarrow g & & \\ P_R^{(H)} & \xrightarrow{h} & M & \longrightarrow & 0 \end{array}$$

there is a $g: P_R \rightarrow P_R^{(X)}$ such that $f = h \circ g$.

This means that P_R is Σ -quasi-injective.

3. The main result.

3.1. REPRESENTATION THEOREM. *Let A, R be two rings, $\mathcal{D}_A, \mathcal{S}_R$ full subcategories of $\text{Mod-}A$ and $\text{Mod-}R$ respectively, Q_R a fixed, but arbitrary, injective cogenerator of $\text{Mod-}R$.*

Assume that

- a) $A_A \in \mathfrak{D}_A$ and \mathfrak{D}_A is closed under taking submodules.
- b) \mathfrak{G}_R is closed under taking direct sums and epimorphic images
- c) A category equivalence $F: \mathfrak{D}_A \rightarrow \mathfrak{G}_R$, $G: \mathfrak{G}_R \rightarrow \mathfrak{D}_A$ is given with F, G additive functors.

Then there exists a bimodule ${}_A P_R$, unique up to isomorphisms, with the following properties:

- 1) $P_R \in \mathfrak{G}_R$, $A \cong \text{End}(P_R)$ canonically.
- 2) $\mathfrak{D}_A = \mathfrak{D}(K_A)$, where $K_A = \text{Hom}_R(P_R, Q_R)$ and $\mathfrak{G}_R = \text{Gen}(P_R)$
- 3) The functors F and G are naturally equivalent to the functors $T = - \otimes_A P$ and $H = \text{Hom}_R(P_R, -)$ respectively.
- 4) For every $L \in \mathfrak{D}(K_A)$ and for every $M \in \text{Gen}(P_R)$ the canonical morphisms σ_L and ϱ_M are isomorphisms

$$\varrho_M: \text{Hom}_R(P_R, M) \otimes_A P \rightarrow M \quad (\varrho_M(f \otimes p) = f(p)),$$

$$\sigma_L: L \rightarrow \text{Hom}_R(P_R, L \otimes_A P) \quad \sigma(l)(p \mapsto l \otimes p).$$

PROOF. Set $P_R = F(A)$. Then $A \cong \text{End}(P_R)$ canonically and we have the bimodule ${}_A P_R$. For every $M \in \text{Gen}(P_R)$ consider the canonical isomorphisms

$$G(M) \cong \text{Hom}_A(A, G(M)) \cong \text{Hom}_R(F(A), F(G(M))) \cong \text{Hom}_R(P_R, M).$$

Thus, looking at the closure properties of \mathfrak{G}_R , we deduce that

- i) G is naturally equivalent to the functor $H = \text{Hom}_R(P_R, -)$ and $\text{Gen}(P_R) \subseteq \mathfrak{G}_R$.

Consider now the functor $T: \mathfrak{D}_A \rightarrow \mathfrak{G}_R$ given by $T(L) = L \otimes_A P$ for every $L \in \mathfrak{D}_A$. T is well defined by i) and by Proposition 2.2. By well known facts, the functors T and H are adjoints. Since (F, H) is an equivalence, F and H are adjoints. Therefore F and T are equivalent. Thus

- ii) F is naturally equivalent to the functor $T = - \otimes_A P$.

Moreover, by i) and by Proposition 2.2 we get

iii) $\mathfrak{S}_R = \text{Gen}(P_R)$.

Set $K_A = \text{Hom}_R(P_R, Q_R)$. Then:

iv) $\mathfrak{D}_A = \mathfrak{D}(K_A)$.

The proof is due to E. Gregorio. First of all let us prove that \mathfrak{D}_A is closed under taking direct products. Let $(L_\lambda)_{\lambda \in A}$ be a family of modules in \mathfrak{D}_A . For every $\lambda \in A$ we have $L_\lambda = H(M_\lambda)$ where $M_\lambda \in \text{Gen}(P_R)$. Now in $\text{Mod-}R$ we have the following natural isomorphisms:

$$\begin{aligned} \text{Hom}_R\left(P_R, t_R\left(\prod_{\lambda \in A} M_\lambda\right)\right) &\cong \text{Hom}_R\left(P_R, \prod_{\lambda \in A} M_\lambda\right) \cong \\ &\cong \prod_{\lambda \in A} \text{Hom}_R(P_R, M_\lambda) \cong \prod_{\lambda \in A} L_\lambda . \end{aligned}$$

Since $t_R\left(\prod_{\lambda \in A} M_\lambda\right) \in \text{Gen}(P_R) = \mathfrak{S}_R$ and by a), it follows $\prod_{\lambda \in A} L_\lambda \in \mathfrak{D}_A$.

For similar reasons we have $K_A \in \mathfrak{D}_A$. Indeed:

$$K_A = \text{Hom}_R(P_R, Q_R) \cong \text{Hom}_R(P_R, t_P(Q_R)) \in \mathfrak{D}_A .$$

Therefore $\mathfrak{D}(K_A) \subseteq \mathfrak{D}_A$ by the closure properties of \mathfrak{D}_A . On the other hand, by a) and by Proposition 2.2, $\mathfrak{D}_A \subseteq \mathfrak{D}(K_A)$. Statement 4) follows from Proposition 2.4 in view of a) and 3). Finally, since $A \cong \text{End}(P_R)$ we have $A \cong H(P_R)$ so that $P_R \cong T(A)$ canonically. Thus P_R is unique up to isomorphisms.

From Theorem 3.1 we get the following important

3.2. PROPOSITION. *Suppose that the assumptions a), b), c) of Theorem 3.1 hold. Then the following conditions are equivalent:*

- (a) \mathfrak{S}_R is closed under taking submodules.
- (b) $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.
- (c) $\mathfrak{D}(K_A) = \text{Mod-}A$.
- (d) ${}_A P$ is a weak-generator.
- (e) For every $M \in \text{Gen}(P_R)$ and for every epimorphism

$$h: P_R^{(X)} \rightarrow M, \quad \text{Hom}_R(P_R, M) = \sum_{x \in X} h_x A .$$

- (f) P_R is Σ -quasi-projective.

PROOF. (a) \Leftrightarrow (b) is obvious in view of Theorem 3.1.

(b) \Rightarrow (f) by Theorem 3.1 and Lemma 2.9.

(f) \Rightarrow (b) Let $M \in \text{Gen}(P_R)$, U a submodule of M and consider the exact sequence

$$0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0.$$

As P_R is Σ -quasi-projective, it is a projective object of $\text{Gen}(P_R)$ so that we get the exact sequence

$$0 \rightarrow H(U) \rightarrow H(M) \rightarrow H(N) \rightarrow 0.$$

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc} TH(U) & \longrightarrow & TH(M) & \longrightarrow & TH(N) & \longrightarrow & 0 \\ \varrho_U \downarrow & & \downarrow \varrho_M & & \downarrow \varrho_N & & \\ 0 \longrightarrow & U & \longrightarrow & M & \longrightarrow & N & \longrightarrow 0 \end{array}$$

ϱ_M and ϱ_N are isomorphisms so that ϱ_U is surjective. Since $\text{Im}(T) \subseteq \text{Gen}(P_R)$ we get $U \in \text{Gen}(P_R)$. Therefore $\text{Gen}(P_R) = \overline{\text{Gen}(P_R)}$.

(c) \Rightarrow (d) is clear in view of Theorem 3.1.

(d) \Rightarrow (e) let $M \in \text{Gen}(P_R)$. By Theorem 3.1 ϱ_M is an isomorphism. Thus (e) follows from Lemma 2.8.

(e) \Rightarrow (f) by Lemma 2.6.

(f) \Rightarrow (e) Let $L \in \text{Mod-}A$. We have an exact sequence

$$(1) \quad A^{(X)} \rightarrow A^{(Y)} \rightarrow L \rightarrow 0.$$

Tensoring (1) by ${}_A P$ we get the exact sequence

$$(2) \quad A^{(X)} \otimes_A P \rightarrow A^{(Y)} \otimes_A P \rightarrow L \otimes_A P \rightarrow 0.$$

As P_R is Σ -quasi-projective applying $\text{Hom}_R(P_R, -)$ to (2) we get the exact sequence

$$HT(A^{(X)}) \rightarrow HT(A^{(Y)}) \rightarrow HT(L) \rightarrow 0.$$

Consider now the commutative diagram

$$\begin{array}{ccccccc}
 A^{(x)} & \longrightarrow & A^{(y)} & \longrightarrow & L & \longrightarrow & 0 \\
 \sigma_{A^{(x)}} \downarrow & & \sigma_{A^{(y)}} & & \sigma_L & & \\
 HT(A^{(x)}) & \longrightarrow & HT(A^{(y)}) & \longrightarrow & HT(L) & \longrightarrow & 0
 \end{array}$$

Since $\sigma_{A^{(x)}}$ and $\sigma_{A^{(y)}}$ are both isomorphisms, σ_L is an isomorphism too. Since $\text{Im}(H) \subseteq \mathcal{D}(K_A)$, we have $L \cong H\left(L \otimes_A P\right) \in \mathcal{D}(K_A)$. Thus $\mathcal{D}(K_A) = \text{Mod-}A$.

3.3. REMARK. The proposition above gives a non trivial generalization of Fuller's Theorem on equivalence (cf. [F], Theorem 1.1).

3.4. PROPOSITION. Let ${}_A P_R$ be a bimodule which induces an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$ and let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of modules in $\text{Gen}(P_R)$. Then

1) $\text{Hom}_R\left(P_R, \bigoplus_{\lambda} M_\lambda\right) \cong \bigoplus_{\lambda} \text{Hom}_R(P_R, M_\lambda)$ canonically.

In particular

- 2) $A \cong \text{End}(P_R)$.
- 3) For every set $X \neq \emptyset$, $\text{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)}$.

PROOF. 1) There exist the canonical isomorphisms:

$$\begin{aligned}
 \bigoplus_{\lambda} \text{Hom}_R(P_R, M_\lambda) &= \bigoplus_{\lambda} H(M_\lambda) \cong HT\left(\bigoplus_{\lambda} H(M_\lambda)\right) \cong \\
 &\cong H\left(\bigoplus_{\lambda} TH(M_\lambda)\right) \cong H\left(\bigoplus_{\lambda} M_\lambda\right) = \text{Hom}_R\left(P_R, \bigoplus_{\lambda} M_\lambda\right).
 \end{aligned}$$

2) and 3) are now obvious.

3.5. Theorem 1.3 suggests the following natural question:

(*) For a given ring R determine all modules $P_R \in \text{Mod-}R$ such that, setting $A = \text{End}(P_R)$, the bimodule ${}_A P_R$ induces an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$.

Suppose that ${}_A P_R$ is such a bimodule. Then the functors $T = - \otimes_A P$

and $H = \text{Hom}_R(P_R, -)$ subordinate an equivalence between $\text{Im}(H)$ and $\text{Im}(T)$ and moreover, in view of Proposition 2.2, the subcategories $\text{Im}(H) \subseteq \text{Mod-}A$ and $\text{Im}(T) \subseteq \text{Mod-}R$ are the *largest possible*. To answer question (*) without further assumptions seems to be quite difficult.

Let ${}_A P_R$ be a bimodule, Q_R an injective cogenerator of $\text{Mod-}R$, $K_A = \text{Hom}_R(P_R, Q_R)$ and the functors T, H have the usual meaning, M. Sato ([S], Theorem 1.3) has shown that the bimodule ${}_A P_R$ induces an equivalence between $\text{Im}(H)$ and $\text{Im}(T)$ if and only if $\text{Im}(H) = L(K_A)$, $\text{Im}(T) = C(P_R)$ and moreover ${}_A P_R$ induces an equivalence between $L(K_A)$ and $C(P_R)$ (see also the proof of Theorem 1.3 of [S]).

In this situation it could happen that $C(P_R) = \text{Gen}(P_R)$ while $L(K_A) \neq \mathcal{D}(K_A)$ as the following example shows.

3.6. AN EXAMPLE. Let p be a prime number, $\mathbb{Z}(p^\infty)$ the Prüfer group relative to p , J_p the ring of p -adic integers and consider the bimodule ${}_{J_p} \mathbb{Z}(p^\infty)_{J_p}$. Note that $\text{End}(\mathbb{Z}(p^\infty)_{J_p}) = J_p$.

In this case $C(\mathbb{Z}(p^\infty)_{J_p}) = \text{Gen}(\mathbb{Z}(p^\infty))$ = the category of all divisible p -primary abelian groups.

On the other hand, since $\mathbb{Z}(p^\infty)$ is an injective cogenerator of $\text{Mod-}J_p$, we have

$$K_{J_p} = \text{Hom}_{J_p}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) = J_p.$$

$\mathcal{D}(J_p)$ is the category of all reduced torsion-free J -modules, while $L(J_p)$ is the category of all cotorsion and torsion-free J_p -modules, which are exactly all the direct summands of direct products of copies of J_p .

By well known results of Harrison [H], the functors $T = - \otimes_{J_p} \mathbb{Z}(p^\infty)$ and $H = \text{Hom}_{J_p}(\mathbb{Z}(p^\infty), -)$ subordinate a category equivalence between

$$\text{Im}(H) = L(J_p) \quad \text{and} \quad \text{Im}(T) = \text{Gen}(\mathbb{Z}(p^\infty)_{J_p}) = C(\mathbb{Z}(p^\infty)_{J_p}).$$

Thus, in this case, $L(J_p) \subsetneq \mathcal{D}(J_p)$.

There exists a condition in order that a bimodule ${}_A P_R$ induces an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$ which involves the whole categories $\text{Mod-}A$ and $\text{Mod-}R$.

3.7. PROPOSITION. Let ${}_A P_R$ be a bimodule with $A = \text{End}(P_R)$. Then the following conditions are equivalent:

- (a) ${}_A P_R$ induces an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$.
- (b) For every $L \in \text{Mod-}A$ the canonical morphism σ_L is surjective and for every $M \in \text{Mod-}R$ the canonical morphism ϱ_M is injective.

PROOF. (a) \Rightarrow (b) Let $M \in \text{Mod-}R$ and let $i: t_P(M) \rightarrow M$ be the canonical inclusion. Then $H(i): H(t_P(M)) \rightarrow H(M)$ is an isomorphism and hence $TH(i): TH(t_P(M)) \rightarrow TH(M)$ is an isomorphism too.

Consider the commutative diagram:

$$\begin{array}{ccc}
 TH(t_P(M)) & \xrightarrow{TH(i)} & TH(M) \\
 \varrho_{t_P(M)} \downarrow & & \downarrow \varrho_M \\
 t_P(M) & \xrightarrow{i} & M
 \end{array}$$

As $t_P(M) \in \text{Gen}(P_R)$, by Theorem 3.1, $\varrho_{t_P(M)}$ is an isomorphism. Thus ϱ_M is injective.

Let now $L \in \text{Mod-}A$ and consider the exact sequence

$$0 \longrightarrow \text{Ker}(\sigma_L) \longrightarrow L \xrightarrow{\pi} L/\text{Ker}(\sigma_L) \longrightarrow 0$$

As $\text{Ker}(\sigma_L) = \{x \in L: x \otimes p = 0 \text{ for every } p \in P\}$ the map

$$T(\pi): T(L) \rightarrow T(L/\text{Ker}(\sigma_L))$$

is an isomorphism hence

$$HT(\pi): HT(L) \rightarrow HT(L/\text{Ker}(\sigma_L))$$

is an isomorphism too.

Now $L/\text{Ker}(\sigma_L)$ embeds into $\text{Hom}_R(P, L \otimes_A P)$ and hence belongs to $\mathcal{D}(K_A)$. Thus $\sigma_{L/\text{Ker}(\sigma_L)}$ is an isomorphism so that from the commutative diagram

$$\begin{array}{ccc}
 L & \xrightarrow{\pi} & L/\text{Ker}(\sigma_L) \\
 \sigma_L \downarrow & & \downarrow \sigma_{L/\text{Ker}(\sigma_L)} \\
 HT(L) & \xrightarrow{HT(\pi)} & HT(L/\text{Ker}(\sigma_L))
 \end{array}$$

we get that σ_L is surjective.

(b) \Rightarrow (a) follows in view of (a) and (b) of 2.3.

4. w -tilting modules.

4.1. In this section we will prove that under the assumptions a)' b), c) of Theorem 3.1 it holds, in general, that $\mathcal{D}(K_A) \neq \text{Mod-}A$.

4.2. Let R be a ring. Generalizing the concept of tilting module in the sense of Happel and Ringel [HR₂] we say that a right R -module P_R is a w -tilting module if the following conditions hold:

- 1) P_R is finitely presented.
- 2) P_R has projective dimension ≤ 1 .
- 3) $\text{Ext}_R^1(P, P) = 0$.
- 4) There exists an exact sequence in $\text{Mod-}R$ of the form

$$(1) \quad 0 \rightarrow R \rightarrow P' \rightarrow P'' \rightarrow 0$$

where P' and P'' are direct sums of direct summands of P_R .

Note that when R is a finite dimensional algebra over a field \mathcal{K} any tilting module in the sense of Happel and Ringel is a w -tilting module.

The following theorem is modelled on Brenner-Butler Theorem on tilting modules (see [HR₂]). As in their setting all modules are finitely generated, we shall give the proof for our more general case.

4.3. THEOREM. *Let P_R be a w -tilting module, $A = \text{End}(P_R)$ and let $\mathcal{D}_A = \{L \in \text{Mod-}A : \text{Tor}_1^R(L_A, {}_A P) = 0\}$. Then*

$$a) \text{ Gen}(P_R) = \{M \in \text{Mod-}R : \text{Ext}_R^1(P, M) = 0\}.$$

b) $A \in \mathcal{D}_A$ and \mathcal{D}_A is closed under submodules.

c) *For every $M \in \text{Gen}(P_R)$, $\text{Hom}_R(P_R, M) \in \mathcal{D}_A$ and the functors $H: \text{Gen}(P_R) \rightarrow \mathcal{D}_A$, $T: \mathcal{D}_A \rightarrow \text{Gen}(P_R)$ given by $H(M) = \text{Hom}_R(P_R, M)$ and $T(L) = L \otimes_A P$, for every $M \in \text{Gen}(P_R)$ and $L \in \mathcal{D}_A$, are an equivalence between $\text{Gen}(P_R)$ and \mathcal{D}_A . Therefore if Q_R is an arbitrary cogenerator of $\text{Mod-}R$, then, by setting $K_A = \text{Hom}_R(P_R, Q_R)$, we have $\mathcal{D}_A = \mathcal{D}(K_A)$.*

PROOF. First of all we show that for every set X , $\text{Ext}_R^1(P, P^{(X)}) = 0$. As P_R is finitely presented we have an exact sequence in $\text{Mod-}R$ of the form

$$(2) \quad 0 \rightarrow F_R \rightarrow R_R^n \rightarrow P_R \rightarrow 0$$

where $n \in N$ and F_R is a finitely generated right R -module.

Applying $\text{Hom}_R(-, P)$ we get the exact sequence

$$0 \rightarrow \text{Hom}_R(P, P) \rightarrow \text{Hom}_R(R^n, P) \rightarrow \text{Hom}_R(F, P) \rightarrow 0 = \text{Ext}_R^1(P, P),$$

hence every morphism $F_R \rightarrow P_R$ can be extended to a morphism $R^n \rightarrow P$. Consider now a morphism $f: F \rightarrow P^{(X)}$. As F is finitely generated, f is a diagonal morphism of a finite family of morphisms from F into P and hence f extends to a morphism from R^n into $P_R^{(X)}$. Thus the sequence

$$0 \rightarrow \text{Hom}_R(P, P^{(X)}) \rightarrow \text{Hom}_R(R^n, P^{(X)}) \rightarrow \text{Hom}_R(F, P^{(X)}) \rightarrow 0$$

is exact. Thus, as $\text{Ext}_R^1(R^n, P^{(X)}) = 0$ we get $\text{Ext}_R^1(P, P^{(X)}) = 0$. Now let $M \in \text{Gen}(P_R)$. Then there exists an exact sequence

$$0 \rightarrow M' \rightarrow P_R^{(X)} \rightarrow M \rightarrow 0.$$

Applying $\text{Ext}_R^1(P, -)$ we get the exact sequence

$$0 = \text{Ext}_R^1(P, P^{(X)}) \rightarrow \text{Ext}_R^1(P, M) \rightarrow \text{Ext}_R^2(P, M').$$

As $\text{Ext}_R^2(P_R, M') = 0$ we get $\text{Ext}_R^1(P, M) = 0$.

Conversely assume that $\text{Ext}_R^1(P, M) = 0$ and consider the exact sequence

$$0 \rightarrow t_P(M) \rightarrow M \rightarrow M/t_P(M) \rightarrow 0.$$

Applying $\text{Hom}_R(P, -)$ we get the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(P, t_P(M)) \xrightarrow{\alpha} \text{Hom}_R(P, M) \rightarrow \\ \rightarrow \text{Hom}_R(P, M/t_P(M)) \rightarrow \text{Ext}_R^1(P, t_P(M)) \rightarrow \text{Ext}_R^1(P, M) \rightarrow \\ \rightarrow \text{Ext}_R^1(P, M/t_P(M)) \rightarrow \text{Ext}_R^2(P, t_P(M)). \end{aligned}$$

As $t_P(M) \in \text{Gen}(P_R)$, we have $\text{Ext}_R^1(P, t_P(M)) = \text{Ext}_R^2(P, t_P(M)) = 0$.

On the other hand, α , by the definition of $t_P(M)$, is surjective so that $\text{Hom}_R(P, M/t_P(M)) = 0$.

Applying now the functor $\text{Hom}_R(-, M/t_P(M))$ to the exact sequence (1) we get the exact sequence

$$0 \rightarrow \text{Hom}_R(P'', M/t_P(M)) \rightarrow \text{Hom}_R(P', M/t_P(M)) \rightarrow \\ \rightarrow \text{Hom}_R(R, M/t_P(M)) \rightarrow \text{Ext}_R^1(P'', M/t_P(M)).$$

As $\text{Hom}_R(P, M/t_P(M)) = 0 = \text{Ext}_R^1(P, M/t_P(M))$ and as P' and P'' are direct sums of direct summands of P we get:

$$\text{Hom}_R(P', M/t_P(M)) = 0 = \text{Ext}_R^1(P'', M/t_P(M))$$

so that $\text{Hom}_R(R, M/t_P(M)) = 0$ and hence $M = t_P(M) \in \text{Gen}(P_R)$. Thus $a)$ is proved.

Let now $M \in \text{Gen}(P_R)$, $A = \text{Hom}_R(P, M)$ and $f: P^{(A)} \rightarrow M$ the codiagonal map of the morphisms λ 's.

Then $\text{Im}(f) = t_P(M) = M$ as $M \in \text{Gen}(P_R)$.

Consider now the exact sequence

$$(3) \quad 0 \rightarrow M' \rightarrow P^{(A)} \xrightarrow{f} M \rightarrow 0.$$

By applying $\text{Hom}_R(P_R, -)$ to it, we get the exact sequence

$$(4) \quad 0 \rightarrow \text{Hom}_R(P_R, M') \rightarrow \text{Hom}_R(P_R, P^{(A)}) \xrightarrow{f_*} \text{Hom}_R(P_R, M) \rightarrow \\ \rightarrow \text{Ext}_R^1(P, M') \rightarrow \text{Ext}_R^1(P, P^{(A)}) = 0$$

where $f_* = \text{Hom}_R(P, f)$. As f is the codiagonal map of the morphisms λ 's, $\lambda \in A$, f_* is surjective and hence $\text{Ext}_R^1(P, M') = 0$ so that $M' \in \text{Gen}(P_R)$. Thus (3) is a sequence in $\text{Gen}(P_R)$. Applying now the functor $-\otimes_A P$ to the exact sequence (4) we get:

$$0 \rightarrow \text{Tor}_1^A(H(M), P) \rightarrow H(M') \otimes_A P \rightarrow H(P^{(A)}) \otimes_A P \rightarrow H(M) \otimes_A P \rightarrow 0 \\ \begin{array}{ccccccc} & & \begin{array}{c} A \\ \downarrow \\ \varrho_{M'} \end{array} & & \begin{array}{c} A \\ \downarrow \\ \varrho_{P^{(A)}} \end{array} & & \begin{array}{c} A \\ \downarrow \\ \varrho_M \end{array} \\ 0 & \rightarrow & M' & \rightarrow & P^{(A)} & \rightarrow & M \rightarrow 0 \end{array}$$

where $Q_{M'}, Q_M$ are surjective and $Q_{P(A)}$ is an isomorphism. Thus Q_M is an isomorphism for every $M \in \text{Gen}(P_R)$. Note that as $M' \in \text{Gen}(P_R)_{Q_{M'}}$ is an isomorphism too and hence $\text{Tor}_1^A(H(M), P) = 0$ so that $H(M) \in \mathcal{D}_A$. On the other hand recall that $N \otimes_A P \in \text{Gen}(P_R)$ for every $N \in \text{Mod-}A$.

Applying now the functor $\text{Hom}_R(-, P)$ to the exact sequence (1) we get

$$0 \rightarrow \text{Hom}_R(P'', P) \rightarrow \text{Hom}_R(P', P) \rightarrow \text{Hom}_R(R, P) \rightarrow \text{Ext}_R^1(P'', P) = 0.$$

Thus ${}_A P \cong \text{Hom}_R(R, P)$ is a left A -module of projective dimension ≤ 1 as $\text{Hom}_R(P', P)$ and $\text{Hom}_R(P'', P)$ are direct summands of free modules. In particular $\text{Tor}_2^A(L_A, {}_A P) = 0$ for every $L_A \in \text{Mod-}A$. Let now $L \in \mathcal{D}_A$ and consider the injection

$$0 \rightarrow L' \rightarrow L$$

in $\text{Mod-}A$. By applying $\text{Tor}_1^A(-, P)$ we get the exact sequence

$$0 = \text{Tor}_2^A(L/L', P) \rightarrow \text{Tor}_1^A(L', P) \rightarrow \text{Tor}_1^A(L, P) = 0.$$

Thus $L' \in \mathcal{D}_A$ and therefore \mathcal{D}_A is closed under submodules. Let now $L \in \mathcal{D}_A$ and consider the exact sequence

$$(5) \quad 0 \rightarrow L' \rightarrow A^{(x)} \rightarrow L \rightarrow 0.$$

Clearly $A \in \mathcal{D}_A$ so that (5) is an exact sequence in \mathcal{D}_A .

Applying T we get the exact sequence

$$0 = \text{Tor}_1^A(P, L) \rightarrow T(L') \rightarrow T(A^{(x)}) \rightarrow T(L) \rightarrow 0.$$

Applying H to this sequence we get the exact sequence

$$0 \rightarrow HT(L') \rightarrow HT(A^{(x)}) \rightarrow HT(L) \rightarrow \text{Ext}_R^1(P, T(L')) = 0$$

as $T(L') \in \text{Gen}(P_R)$.

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & L' & \longrightarrow & A^{(x)} & \longrightarrow & L & \longrightarrow & 0 \\ & & \sigma_{L'} \downarrow & & \sigma_{A^{(x)}} \downarrow & & \sigma_L \downarrow & & \\ 0 & \longrightarrow & HT(L') & \longrightarrow & HT(A^{(x)}) & \longrightarrow & HT(L) & \longrightarrow & 0 \end{array}$$

Since $\sigma_{A(x)}$ is an isomorphism, σ_L is surjective. Thus, as $L' \in \mathcal{D}_A$, $\sigma_{L'}$ is surjective too. Therefore σ_L is an isomorphism and hence *b)*, and *c)* are proved.

The last assertion follows from Theorem 3.1.

4.4. PROPOSITION. *Let R be a ring, P_R a w -tilting module, $A = \text{End}(P_R)$, Q_R an arbitrary cogenerator of $\text{Mod-}R$, $K_A = \text{Hom}_R(P_R, Q_R)$. Then the following statements are equivalent:*

- (a) $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.
- (b) $\text{Gen}(P_R) = \text{Mod-}R$.
- (c) P_R is Σ -quasi-projective.
- (d) P_R is projective.
- (e) $\mathcal{D}(K_A) = \text{Mod-}A$.

PROOF. (a) \Leftrightarrow (c) \Leftrightarrow (e). Follow by Theorem 4.3 and Proposition 3.2.

(a) \Leftrightarrow (b). Follows in view of 4) of the definition of w -tilting module.

(c) \Rightarrow (d) As P_R is Σ -quasi-projective it is a projective object of $\text{Gen}(P_R)$. Since (c) \Rightarrow (a) \Rightarrow (b), $\text{Gen}(P_R) = \text{Mod-}R$ so that P_R is projective.

(d) \Rightarrow (c) Is trivial.

4.5. CONCLUSION. Let P_R be a w -tilting non projective module (for an example see [HR₁], pp. 126-127), $A = \text{End}(P_R)$, Q_R an arbitrary cogenerator of $\text{Mod-}R$, $K_A = \text{Hom}_R(P_R, Q_R)$. Then by Theorem 4.3 P_R gives rise to an equivalence between $\text{Gen}(P_R)$ and \mathcal{D}_A which fulfils assumptions *a)*, *b)*, *c)* of Theorem 3.1 but, in view of Proposition 4.4, $\mathcal{D}(K_A) = \mathcal{D}_A \neq \text{Mod-}A$.

5. Quasi-progenerators.

5.1. In this section, under the assumptions of Theorem 3.1, we determine a number of sufficient conditions in order that $\mathcal{D}(K_A) = \text{Mod-}A$.

5.2. PROPOSITION. *Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$. The following conditions are equivalent:*

- (a) *For every $n \in \mathbf{N}$, P_R generates all submodules of P_R^n .*
- (b) *$\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.*
- (c) *For every $M \in \overline{\text{Gen}}(P_R)$, $M \cong \text{Hom}_R(P_R, M) \otimes_A P$ canonically.*
- (d) *${}_A P$ is flat and for every $M \in \text{Gen}(P_R)$, $M \cong \text{Hom}_R(P_R, M) \otimes_A P$ canonically.*

If these conditions are fulfilled, then the canonical image of R in $\text{End}(P_R)$ is dense in $\text{End}({}_A P)$ endowed with its finite topology.

PROOF. The equivalences (a) \Leftrightarrow (b) \Leftrightarrow (d) are due to Zimmermann-Huisgen (cf. [ZH], Lemma 1.4).

The equivalence between (b) and (c) has been noted by Sato (cf. [S], Lemma 2.2). The last statement is due to Fuller (cf. [F], Lemma 1.3).

For another proof of this proposition see [MO], Proposition 5.5

5.3. REMARK. It could happen that for every $M \in \text{Gen}(P_R)$ $M \cong \text{Hom}_R(P_R, M) \otimes_A P$ canonically, but this is not true for every $M \in \overline{\text{Gen}}(P_R)$.

In fact looking at Example 3.6 we have, for every $M \in \text{Gen}(\mathbf{Z}(p^\infty)_{\mathbf{Z}})$ the required canonical isomorphism. On the other hand the cyclic group $\mathbf{Z}(p)$ of order p belongs to $\overline{\text{Gen}}(\mathbf{Z}(p^\infty)_{\mathbf{Z}})$, but $\text{Hom}_{\mathbf{Z}}(\mathbf{Z}(p^\infty), \mathbf{Z}(p)) = 0$.

5.4. THEOREM. *Let ${}_A P_R$ be a bimodule and assume that ${}_A P_R$ induces an equivalence between $\mathfrak{D}(K_A)$ and $\text{Gen}(P_R)$. Then $A = \text{End}(P_R)$ and the following conditions are equivalent:*

- (a) $\mathfrak{D}(K_A) = \text{Mod-}A$.
- (b) $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.
- (c) P_R is Σ -quasi-projective.
- (d) P_R is quasi-projective and finitely generated.
- (e) P_R is quasi-projective.
- (f) ${}_A P$ is flat ($\Leftrightarrow {}_A K$ is injective).
- (g) ${}_A P$ is a weak generator.

PROOF. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (g)$ follow from Proposition 3.2.

$(b) \Leftrightarrow (f)$ Follows from Proposition 5.2 in view of Proposition 2.4.

$(c) \Leftrightarrow (d)$ Follows by Proposition 3.4 from Proposition 8 in [A].

$(d) \Rightarrow (e)$ Is trivial.

$(e) \Rightarrow (f)$ Follows from Proposition 4 in [A] in view of Proposition 3.7.

5.5. DEFINITION (Fuller [F]). A module $P_R \in \text{Mod-}R$ is called a *quasi-progenerator* if P_R is quasi-projective, finitely generated and generates all its submodules.

We are now ready to prove the concluding theorem of this section.

5.6. THEOREM. *Let ${}_A P_R$ be a bimodule. The following conditions are equivalent:*

- (a) ${}_A P_R$ induces an equivalence between $\mathcal{D}(K_A)$ and $\overline{\text{Gen}}(P_R)$.
- (b) ${}_A P_R$ induces an equivalence between $\text{Mod-}A$ and $\text{Gen}(P_R)$.
- (c) P_R is a quasi-progenerator and $A = \text{End}(P_R)$.
- (d) $A = \text{End}(P_R)$, P_R is quasi-projective and generates all its submodules, ${}_A P$ is faithfully flat.

If these conditions hold then:

- 1) $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$ and K_A is an injective cogenerator of $\text{Mod-}A$.
- 2) The canonical image of R in $\text{End}({}_A P)$ is dense whenever $\text{End}({}_A P)$ is endowed with its finite topology.

PROOF. $(a) \Leftrightarrow (b)$ By Theorem 5.4 and Proposition 2.2.

$(b) \Rightarrow (c)$ By Theorem 5.4 P_R is quasi-projective and finitely generated and $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$. Then P_R generates all its submodules.

$(b) \Rightarrow (d)$ Since $A \in \text{Im}(H)$, $A = \text{End}(P_R)$. By Theorem 5.4 ${}_A P$ is flat and a weak generator. Thus ${}_A P$ is faithfully flat.

(d) \Rightarrow (a) By Lemma 2.2 of [F] $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$. Hence by Proposition 5.2, for every $M \in \text{Gen}(P_R)$, $M \cong TH(M)$ canonically. Let $L \in \text{Mod-}A$. Since $T(L) \in \text{Gen}(P_R)$ we have $TH(T(L)) \cong T(L)$. It follows $T(HT(L)) \cong T(L)$ hence $HT(L) \cong L$ as ${}_A P$ is faithfully flat.

(e) \Rightarrow (d) By Lemma 2.2 of [F] $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$ so that, by Proposition 5.2, ${}_A P$ is flat. It is well known that ${}_A P$ is faithfully flat if and only if ${}_A P$ is flat and moreover for every right ideal I of A such that $IP = P$ we have $I = A$. Thus assume $IP = P$. As P_R is finitely generated there exist finitely many endomorphisms $a_1, \dots, a_n \in I$ such that $\sum_{i=1}^n a_i P = P_R$. This yields an epimorphism $P_R^n \xrightarrow{\sum a_i} P_R \rightarrow 0$. Since P_R is quasi-projective this epimorphism splits so that there exists a morphism $\beta: P_R \rightarrow P_R^n$ such that $\beta \circ \sum_{i=1}^n a_i = 1$. We have $\beta = (b_1, \dots, b_n)$ where $b_i \in A$. Thus $1 = \sum_{i=1}^n b_i a_i \in I$ so that $I = A$.

Suppose now that the conditions above hold.

1) We know that $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.

Moreover K_A is injective as ${}_A P$ is flat (recall that $K_A = \text{Hom}_R(P_R, Q_R)$ where Q_R is an injective cogenerator of $\text{Mod-}R$) and K_A is a cogenerator as $\text{Mod-}A = \text{Im}(H) \subseteq \mathcal{D}(K_A)$.

2) Follows from Proposition 5.2.

5.7. REMARK. The equivalences (b) \Leftrightarrow (e) \Leftrightarrow (d) and statements $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$ in 1) and 2) of 5.6 are due to Fuller (see [F] Theorem 2.6).

The equivalence (a) \Leftrightarrow (b) is due to E. Gregorio ([G], Theorem 1.11).

5.8. REMARK. Let P_R be a quasi-progenerator and $A = \text{End}(P_R)$. Then ${}_A P_R$ induces an equivalence between $\mathcal{D}(K_A)$ and $\text{Gen}(P_R)$. While $\mathcal{D}(K_A) = \text{Mod-}A$ in general $\text{Gen}(P_R) \neq \text{Mod-}R$ as the following example shows.

Let R be a right primitive ring, P_R a faithful simple right R -module, $A = \text{End}(P_R)$. Since P_R is a quasi-progenerator, ${}_A P_R$ induces an equivalence between $\text{Mod-}A$ and $\text{Gen}(P_R)$.

$\text{Mod-}A$ is the category of right vector spaces over the division ring A and $\text{Gen}(P_R)$ is the category of all semisimple modules of the form $P_R^{(X)}$. Thus, if R is not right artinian, $\text{Gen}(P_R) \neq \text{Mod-}R$.

6. Equivalences and dualities.

6.1. Denote by $R\text{-CM}$ the category of all compact Hausdorff left modules over the discrete ring R and let \mathbf{T} be the topological quotient of the real field \mathbf{R} , endowed with the usual topology, modulo the group \mathbf{Z} of rational integers. For every $N \in R\text{-CM}$, let $\Gamma_1(N) = \text{Chom}_{\mathbf{Z}}(N, \mathbf{T})$ be the set of all continuous morphisms of abelian groups from N into \mathbf{T} .

$\Gamma_1(N)$ is an abelian group which has a natural structure of right R -module defined by setting

$$(\xi r)(x) = \xi(rx) \quad (\xi \in \Gamma_1(N), r \in R, x \in N).$$

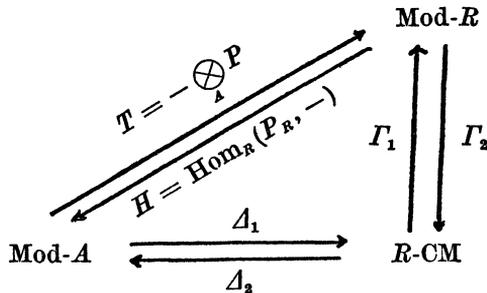
For every $M \in \text{Mod-}R$ let $\Gamma_2(M)$ be the left R -module $\text{Hom}_{\mathbf{Z}}(M, \mathbf{T})$ endowed with the topology of pointwise convergence.

It is well known that $\Gamma_2(M)$ is a compact group so that $\Gamma_2(M) \in R\text{-CM}$. In this way we obtain the contravariant functors

$$\Gamma_1: R\text{-CM} \rightarrow \text{Mod-}R \quad \text{and} \quad \Gamma_2: \text{Mod-}R \rightarrow R\text{-CM}$$

which coincide with the usual Pontryagin Duality functors when $R = \mathbf{Z}$. Clearly, from Pontryagin's classical results, we have that $\Gamma_2 \circ \Gamma_1 \approx 1_{R\text{-CM}}$ and $\Gamma_1 \circ \Gamma_2 \approx 1_{\text{Mod-}R}$: (Γ_1, Γ_2) is called the Pontryagin Duality over R . For more details about this duality see [MO].

6.2. Let ${}_A P_R$ be a bimodule with $A = \text{End}(P_R)$. Consider the following commutative diagram of categories and functors:



where $\Delta_1 = \Gamma_2 \circ T$ and $\Delta_2 = H \circ \Gamma_1$. Set ${}_R K_A = \Gamma_2({}_A P_R)$ and for every

$N, N' \in R\text{-CM}$ let $\text{Chom}_R(N, N')$ be the group of all continuous R -morphisms from N into N' .

Clearly $A \cong \text{Chom}_R({}_R K, {}_R K)$ canonically.

Denote by $\mathcal{C}({}_R K)$ the category of all topological modules M over the discrete ring R which are topologically isomorphic to closed submodules of topological powers of ${}_R K$.

As ${}_R K = \Gamma_2(P_R) \in R\text{-CM}$ we have that $\mathcal{C}({}_R K) \subseteq R\text{-CM}$.

Let $L \in \text{Mod-}A$. $N \in R\text{-CM}$. We have the canonical isomorphisms in $R\text{-CM}$ and $\text{Mod-}A$ respectively,

$$\begin{aligned} \Delta_1(L) = \Gamma_2(L \otimes_A P) &= \text{Hom}_{\mathbf{Z}}(L \otimes P, \mathbf{T}) \cong \\ &\cong \text{Hom}_A(L, \text{Hom}_{\mathbf{Z}}(P, \mathbf{T})) \cong \text{Hom}_A(L, K_A) \end{aligned}$$

where $\text{Hom}_A(L, K_A)$ is regarded as a topological submodule of the topological product ${}_R K^{\mathbf{Z}}$. Note that, since ${}_R K$ is compact and $\text{Hom}_A(L, K_A)$ is closed in ${}_R K^{\mathbf{Z}}$, it follows that $\text{Hom}_A(L, K_A)$ is compact. Clearly the topologically isomorphic modules $\Gamma_2(L \otimes P)$ and $\text{Hom}_A(L, K_A)$ have the same characters.

$$\begin{aligned} \Delta_2(N) = H(\Gamma_1(N)) = \text{Hom}_R(P_R, \Gamma_1(N)) &\cong \\ &\cong \text{Chom}_R(\Gamma_2 \Gamma_1(N), \Gamma_2(P_R)) \cong \text{Chom}_R(N, {}_R K). \end{aligned}$$

Thus $\Delta_1(L) \in \mathcal{C}({}_R K)$ while $\Delta_2(N) \in \mathcal{D}(K_A)$.

Let

$$\omega_L: L \rightarrow \text{Chom}_R(\text{Hom}_A(L, K_A), {}_R K)$$

and

$$\omega_N: N \rightarrow \text{Hom}_A(\text{Chom}_R(N, {}_R K), K_A)$$

be the canonical morphisms:

$$(\xi)\omega_L(x) = \xi(x), \quad \xi \in \text{Hom}_A(L, K_A), \quad x \in L$$

and

$$((y)\omega_N)(\eta) = (y)\eta, \quad \eta \in \text{Chom}_R(N, {}_R K), \quad y \in N.$$

Then, by means of the Pontryagin duality over R , ω_L and ω_N correspond respectively to the canonical morphisms σ_L and $\varrho_{\Gamma_1(N)}$.

Indeed let $L \in \text{Mod-}A$ and consider the diagram:

$$L \xrightarrow{\sigma_L} \text{Hom}_R(P, L \otimes_A P) \xrightarrow{\varrho_L} \text{Chom}_R(\Gamma_2(L \otimes P), \Gamma_2(P)) \xrightarrow{\varrho_L} \Delta_2 \Delta_1(L)$$

where φ_L associates to every $\xi \in \text{Hom}_R(P, L \otimes_A P)$ its transposed by Γ_2 and $\psi_L = \text{Chom}_R(\xi_L^{-1}, 1_{\Gamma_2(P)})$ where

$$\begin{aligned} \xi_L: \Gamma_2(L \otimes_A P) &= \\ &= \text{Hom}(L \otimes_A P, \mathbf{T}) \xrightarrow{\sim} \text{Hom}_A(L, \text{Hom}_{\mathbf{Z}}({}_A P, \mathbf{T})) = \text{Hom}_A(L, K_A) \end{aligned}$$

is the natural isomorphism. Set $\gamma_L = \psi_L \circ \varphi_L \circ \sigma_L$. We want to show that for every $\chi: L \rightarrow K_A$ we have

$$(1) \quad (\chi)(\gamma_L(l)) = \chi(l) \quad \text{for every } l \in L.$$

Let $\tilde{\chi} = (\chi)\xi_L^{-1} \in \Gamma_2(L \otimes_A P)$. We have

$$\tilde{\chi}(l \otimes p) = \chi(l)(p) \quad \text{for every } l \in L, p \in P.$$

Therefore

$$(\tilde{\chi} \circ \sigma_L)(l)(p) = \tilde{\chi}(l \otimes p) = \chi(l)(p) \quad l \in L, p \in P$$

and we have $\tilde{\chi} \circ \sigma_L(l) = \chi(l)$ for every $l \in L$ so that

$$(\chi)(\gamma_L(l)) = (\chi)[(\psi_L \circ \varphi_L \circ \sigma_L)(l)] = (\tilde{\chi})[(\varphi_L \circ \sigma_L)(l)] = \tilde{\chi} \circ \sigma_L(l) = \chi(l).$$

Thus (1) is proved.

For $N \in R\text{-CM}$ the correspondence between $\varrho_{\Gamma_1(N)}$ and ω_N is proved by an adjointness argument.

6.3. COROLLARY. *Let $L \in \text{Mod-}A$ ($N \in R_L\text{CM}$). Then $\omega: (\omega_N)$ is an isomorphism (a topological isomorphism) if and only if $\sigma_L(\varrho_{\Gamma_1(N)})$ is an isomorphism.*

6.4. From Theorem 3.1 and from 6.2 we easily obtain a theorem of representations for dualities.

6.5. THEOREM. *Let R, A be two rings, \mathfrak{D}_A a subcategory of $\text{Mod-}A$, ${}_R\mathfrak{C}$ a subcategory of $R\text{-CM}$. Assume that*

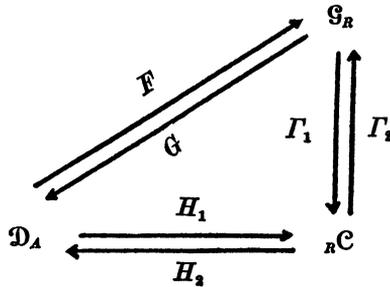
- a) $A_A \in \mathfrak{D}_A$ and \mathfrak{D}_A is closed under taking submodules.

- b) ${}_R\mathcal{C}$ is closed under taking closed submodules and topological products.
- c) A duality $H_1: \mathcal{D}_A \rightarrow {}_R\mathcal{C}$, $H_2: {}_R\mathcal{C} \rightarrow \mathcal{D}_A$ is given with H_1, H_2 additive functors.

Then there exists a bimodule ${}_R K_A$ with the following properties:

- 1) ${}_R K \in {}_R\mathcal{C}$ and $A \cong \text{Chom}_R(K, K)$ canonically.
- 2) $\mathcal{D}_A = \mathcal{D}(K_A)$, ${}_R\mathcal{C} = \mathcal{C}({}_L K)$ where $\mathcal{C}({}_R K)$ consists of all compact modules which are closed submodules of topological powers of ${}_R K$.
- 3) $H_1 \approx \Delta_1$, $H_2 \approx \Delta_2$.
- 4) For every $L \in \mathcal{D}_A$, ω_L is an isomorphism and for every $N \in {}_R\mathcal{C}$, ω_N is a topological isomorphism.

PROOF. Consider the commutative diagram



where $F = \Gamma_2 \circ H_1$, $G = H_2 \circ \Gamma_1$ and $\mathcal{G}_R = \Gamma_2({}_R\mathcal{C})$.

Then \mathcal{G}_R is closed under taking homomorphic images and arbitrary direct sums. Set ${}_A P_R = \Gamma_1({}_R K_A)$. Clearly $A \cong \text{End}({}_R P_R)$ canonically. Set $Q_R = \text{Hom}_Z(R, T)$. Then Q_R is an injective cogenerator of $\text{Mod-}R$. We have the canonical isomorphisms:

$$\begin{aligned} \text{Hom}_R(P_R, Q_R) &= \text{Hom}_R(P_R, \text{Hom}_Z(R, T)) \cong \text{Hom}_Z(P_R \otimes_R R, T) = \\ &= \text{Hom}_Z(P_R, T) = \Gamma_2({}_A P_R) = {}_R K_A. \end{aligned}$$

At this point we can apply Theorem 3.1 getting:

$$\mathcal{D}_A = \mathcal{D}(K_A), \quad \mathcal{G}_R = \text{Gen}({}_R P_R),$$

hence

$${}_R\mathcal{C} = \mathcal{C}({}_R K), \quad F \approx \left(- \otimes_A P\right)|_{\mathcal{D}_A}, \quad G \approx \text{Hom}_R(P_R, -)|_{\mathcal{S}_L}$$

and for every $L \in \mathcal{D}_A$, $M \in \mathcal{S}_R$, σ_L and ϱ_M are isomorphisms. Then $H_1 \approx \Delta_1$, $H_2 \approx \Delta_2$ and statement 4) follows by Corollary 6.3.

6.6. Let ${}_R K \in R\text{-CM}$, $A = \text{Chom}_R(K, K)$. Assume that the couple of functors (Δ_1, Δ_2) induces a duality between $\mathcal{D}(K_A)$ and $\mathcal{C}({}_R K)$. We say that this duality is *good* if $\mathcal{C}({}_R K)$ has the extension property of K -characters. This means that for every $N' \in \mathcal{C}({}_R K)$ and for every (closed) submodule $N' \subseteq N$ every continuous morphism of N' in ${}_R K$ extends to a continuous morphism of N in ${}_R K$.

In [MO] it was proved that the considered duality is a good duality if and only if $\mathcal{D}(K_A) = \text{Mod-}A$.

The results of Section 4 solve an old problem of us: namely there exists dualities between $\mathcal{D}(K_A)$ and $\mathcal{C}({}_R K)$ which are not good dualities.

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