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Solving a Collection of Free Coexistence-Like Problems in Stability.

GAETANO ZAMPIERI (*)

1. Introduction (1).

In the paper [2] the author determined and constructed the collection of the real continuous maps f such that the origin is a stable equilibrium for the system

$$\ddot{x} + xf(x) = 0 \,, \quad \ddot{y} + yf(x) = 0 \,, \quad (x, y) \in \mathbb{R}^2 \,, \tag{1.1}$$

(where f is defined in some neighbourhood of 0 in $\mathbb{R}$).

From the mechanical point of view we have a purely positional force which is central and nonconservative (unless $f = \text{const}$). The system (1.1) admits two first integrals. One is energy for the first equation $(1.1)_1$, and the other is

$$\dot{y}x - \dot{x}y \,. \tag{1.2}$$

If $f(0) \leq 0$, we trivially have instability as can be seen from the solutions along the y axis. Let $f(0) > 0$, then the origin is a center

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(1) This Introduction chiefly summarizes the definitions and the main results contained in ref. (3). These provide the right frame for this paper but they are not strictly necessary. In this Sect. 1, and only here, we avoid the details.

for $(1.1)_1$, and $(1.1)_2$ leads to a family of Hill's equations. By plugging any given periodic solution $x(\cdot)$ of $(1.1)_1$ into $(1.1)_2$, we have a Hill's equation which has the solution $x(\cdot)$. Either all its other solutions are periodic or they are all unbounded, except the multiples of $x(\cdot)$—see e.g. Chapter 1 in ref. [1]—. In the literature the former case is called an instance of *coexistence*—see [1]—.

The origin is a stable equilibrium for (1.1) iff, for some $\varepsilon > 0$, we have coexistence of periodic solutions to each aforementioned Hill's equation defined by $x(\cdot)$ such that $x(0)^2 + \dot{x}(0)^2 < \varepsilon$. Thus we say that such problems in stability are *related to coexistence*. Let us see the general definition.

The problem in stability of the origin for

$$(1.3) \qquad \ddot{x} + xf(x) = 0\,, \quad \ddot{y} + yw(x) = 0\,, \quad f(0) > 0\,, \quad f \in C^1\,, \quad w \in C^0\,,$$

is said to be *related to coexistence* if (1.3) has a first integral like

$$(1.4) \qquad \dot{y}\, s(x, \dot{x}) - y\, \dot{s}(x, \dot{x})\,.$$

In this case we say that (f, w, s) is *coexistence-like*. This is essentially Def. 1 in [3] (the condition in Def. 1 of [3] is actually slightly more restrictive).

Let (f, w_1, s) and (f, w_2, s) be coexistence-like. Then $w_1 = w_2$ (in some neighbourhood of 0)—see Prop. 2 in [3]—and we say that (f, s) is coexistence-like.

We say that s *yields a free coexistence-like problem* iff (f, s) is coexistence-like for every f with $f(0) > 0$—see Def. 2 in [3]—. For instance $s(x, \dot{x}) = x$ agrees with this definition, and the map w, determined by (f, s), coincide with f. For such maps s, we say that the problem of finding the family of all the maps f such that the origin is stable for (1.3) (with w determined by (f, s)), is a *free coexistence-like problem*.

In [3] we determine all these problems. They are only those yielded by

$$(1.5) \qquad s(x, \dot{x}) = ax + b\,, \qquad \text{with } a, b \in \mathbf{R},\ a \neq 0\,,$$

and by

$$(1.6) \qquad s(x, \dot{x}) = \dot{x}(cx + d)\,, \qquad \text{with } c, d \in \mathbf{R},\ c^2 + d^2 \neq 0\,.$$

The map w determined by f and s in (1.5) is $x \mapsto axf(x)/(ax+b)$. For $b=0$ we have $w(0)=f(0)$, otherwise $w(0)=0$. By this discontinuity in the parameter b, we do not expect the solutions to the problems in stability given by (1.1)—which are obtained for $b=0$—to have some relationship with those given by

$$(1.7) \qquad \ddot{x}+xf(x)=0\,, \quad \ddot{y}+y\,\frac{xf(x)}{x+\delta}=0\,, \qquad \delta\neq 0\,.$$

In fact these last equilibria are always unstable—consider again the solutions along the y axis—while a large class of maps f, with $f(0)>0$, yield stability for (1.1) (but instability is generic)—see [2].

Therefore it is natural to split (1.5) into the following two possibilities (where we also eliminate a trivial nonvanishing factor):

$$(1.8) \qquad s(x,\dot{x})=x+\delta\,, \quad \text{with } \delta\in\mathbf{R}^* \text{ (which yield (1.7))}\,,$$

and

$$(1.9) \qquad s(x,\dot{x})=x\,, \quad \text{(which yields (1.1))}\,.$$

A similar argument of discontinuity does hold for (1.6) too. So we arrive at the two cases

$$(1.10) \qquad s(x,\dot{x})=\dot{x}(1+\alpha x)\,, \quad \alpha\in\mathbf{R} \text{ (which yield (3.1) of Sec. 3)}\,,$$

and

$$(1.11) \qquad s(x,\dot{x})=x\dot{x}$$

$\big($yielding $\ddot{x}+xf(x)=0$, $\ddot{y}+(4f(x)+xf'(x))=0\big)$.

This paper solves the problems given by (1.10), that is we determine and construct—in Sec. 4 and Sec. 5 respectively—all the maps f such that the origin is a stable equilibrium for the system in (3.1).

Remark. There are there sharply distinct proofs. The first, related to (1.9), is in [2]. The second, related to (1.10), is in this paper. The third (which is perhaps the most difficult) will appear in a future paper. That paper will solve the problem given by (1.11).

2. Preliminaries.

Consider the scalar equation

$$\ddot{x} + xf(x) = 0 \tag{2.1}$$

where f is C^1 in some open interval of $\mathbb{R}$ which contains 0. This equation has the first integral of energy

$$\dot{x}^2 + 2v(x) \quad \text{with } v(x) = \int_0^x \xi f(\xi)\, d\xi . \tag{2.2}$$

The potential energy v is C^2 and admits $v'''(0)$.

We shall use the following maps u and h defined if $f(0) > 0$. The map u is the C^2-diffeomorphism of some (maximal) interval J, with $0 \in J = \mathring{J}$ onto some symmetric interval I, such that

$$u(x)^2 = 2v(x), \quad u'(0) > 0, \tag{2.3}$$

—see the Appendix I where we give the proof. We denote by X its inverse, so

$$X = u^{-1}, \quad X'(0) = [f(0)]^{-\frac{1}{2}} . \tag{2.4}$$

Futhemore we set

$$h\colon J \to J,\ x \mapsto X(-u(x)), \quad \text{thus } v(h(x)) = v(x) .$$

We have

$$h \in \mathrm{Diff}^2(J), \quad h^{-1} = h, \quad h(0) = 0, \quad h'(0) = -1 . \tag{2.5}$$

Let $f(0) > 0$, then $v''(0) = f(0) > 0$. Thus v has a strict minimum at 0 which is a stable isolated equilibrium for (2.1). The origin of the $x, \dot{x}$-plane is a center for (2.1). We denote by $x(\cdot, x_0, 0)$ the solution of (2.1) with $(x_0, 0)$ as initial condition. For $x_0 \in J \setminus \{0\}$ it is periodic and $\ell(x_0)$ will denote its period. Of course

$$x(\ell(x_0)/2, x_0, 0) = h(x_0) . \tag{2.6}$$

3. A condition for stability.

Now, let $\alpha \in \mathbf{R}$ and let us consider the system

$$(3.1) \qquad \begin{cases} \ddot{x} + xf(x) = 0\,, \qquad \ddot{y} + yw(x) = 0 \\ \text{with} \\ w(x) = \dfrac{f(x) + xf'(x) + \alpha x[4f(x) + xf'(x)]}{1 + \alpha x}\,. \end{cases}$$

It admits the first integral in (2.2) and the first integral

$$(3.2) \qquad \dot{y}s(x, \dot{x}) - y\,\dot{s}(x, \dot{x}) \qquad \text{with } s(x, \dot{x}) = \dot{x}(1 + \alpha x)\,.$$

If $f(0) \leq 0$ then the origin is unstable for (3.1) (consider the solutions with $x(0) = 0 = \dot{x}(0)$). Thus in the sequel $f(0) > 0$. Better, we consider the nonrestrictive assumption $f(0) = 1$ (otherwise use the transformation $t \mapsto [f(0)]^{\frac{1}{2}} t$).

Let $x_0 \in J \cap \mathbf{R}_+^*$ and let us consider the Hill's equation

$$(3.3) \qquad \ddot{y} + yw(x(t, x_0, 0)) = 0$$

whose coefficient is even and periodic as well as $x(\cdot\,, x_0, 0)$. The map

$$(3.4) \qquad \psi\colon t \mapsto -\frac{s(x(t, x_0, 0), \dot{x}(t, x_0, 0))}{(1 + \alpha x_0)x_0 f(x_0)}$$

—see (3.2)—is an odd solution of (3.3) with $\psi(0) = 0$ and $\dot{\psi}(0) = 1$. Moreover ψ has the period $\ell(x_0)$. Let φ be the solution of (3.3) with $\varphi(0) = 1$ and $\dot{\varphi}(0) = 0$. By (3.2)

$$(3.5) \qquad \varphi\dot{\psi} - \dot{\varphi}\psi = 1\,.$$

As is well known from the theory of the Hill's equation, either φ has period $\ell(x_0)$ too (*coexistence*) or any solution, which is not $a\psi$ for some $a \in \mathbf{R}$, is unbounded; a necessary and sufficient condition for the former case is

$$(3.6) \qquad \dot{\varphi}(\ell(x_0)/2) = 0$$

—see the Appendix II or deduce from the results in ref. [1], Chapter 1.

Now we arrived at the following condition for the stability of the origin for (3.1): *the origin is stable iff there exists some* $a \in \mathbf{R}_+^*$ *such that* (3.6) (where φ depends on x_0) *is satisfied for every* $x_0 \in]0, a[$.

This condition does not display the functions f which give stability. However it is our starting point in the next Section where we find such maps.

4. The stable cases.

Let us work on condition (3.6). By (3.5)

$$\varphi(t) = (\varphi(\bar{t})/\psi(\bar{t}))\,\psi(t) - \psi(t)\int_{\bar{t}}^{t}\psi(\xi)^{-2}\,d\xi \qquad \text{for } t, \bar{t} \in]0, \ell(x_0)/2[\,.$$

$$\dot{\varphi}(t) = (\varphi(\bar{t})/\psi(\bar{t}))\,\dot{\psi}(t) - \dot{\psi}(t)\int_{\bar{t}}^{t}\psi(\xi)^{-2}\,d\xi - \psi(t)^{-1}\,. \tag{4.1}$$

Since $\dot{\varphi}(0) = 0$ and $\dot{\psi}(0) = 1$, then

$$(\varphi(\bar{t})/\psi(\bar{t})) = \lim_{\tilde{t}\to 0}\left[\int_{\tilde{t}}^{\bar{t}}\psi(\xi)^{-2}\,d\xi + \psi(\tilde{t})^{-1}\,\dot{\psi}(\tilde{t})^{-1}\right].$$

The condition (3.6) can be written as

$$\lim_{t\to \ell(x_0)/2}(\dot{\varphi}(t)/\dot{\psi}(t)) = 0\,.$$

By (4.1), this is equivalent to

$$\lim_{t\to \ell(x_0)/2}\;\lim_{\tilde{t}\to 0}\left\{\int_{t}^{\tilde{t}}\psi(\xi)^{-2}\,d\xi + [\psi(\xi)^{-1}\,\dot{\psi}(\xi)^{-1}]_{\xi=t}^{\xi=\tilde{t}}\right\} = 0\,.$$

Now, we pass from the variable t to the variable x by using the first integral in (2.2) and the definition (3.4). We can also perform a single integral in the following way (remark that the limit operations above have a finite result in any case though, in general, this is not 0):

$$0 = \lim_{x\to x_0-}\left\{-\int_{h(x)}^{x}(1+\alpha\xi)^{-2}\big(2v(x_0)-2v(\xi)\big)^{-\frac{3}{2}}\,d\xi\, + \right.$$
$$\left. + \left[(1+\alpha\xi)^{-1}\big(2v(x_0)-2v(\xi)\big)^{-\frac{1}{2}}\Big((1+\alpha\xi)\,\xi f(\xi)-\alpha\big(2v(x_0)-2v(\xi)\big)\Big)^{-1}\right]_{\xi=h(x)}^{\xi=x}\right\}.$$

Let us perform a second transformation of variable by considering $z = u(x)$ where u is the map in (2.3) (and $u^{-1} = X$). We also briefly write u_0 for $u(x_0)$.

$$0 = \lim_{z\to u_0-}\left\{-\int_{-z}^{z}X'(\eta)(1+\alpha X(\eta))^{-2}(u_0^2-\eta^2)^{-\frac{3}{2}}\,d\eta\, + \right.$$
$$\left. + \left[(1+\alpha X(\eta))^{-1}(u_0^2-\eta^2)^{-\frac{1}{2}}\Big((1+\alpha X(\eta))\,X(\eta)\,f(X(\eta))-\alpha\underline{(u_0^2-\eta^2)}\Big)^{-1}\right]_{\eta=-z}^{\eta=z}\right\}.$$

We can eliminate the underlined term $(u_0^2-\eta^2)$. In fact an easy calculation yields

$$\lim_{z\to u_0-}\Big\{(u_0^2-z^2)^{-\frac{1}{2}}\big[(1+\alpha X(\eta))\,X(\eta)\,f(X(\eta))-\alpha(u_0^2-\eta^2)\big]^{-1} +$$
$$- (u_0^2-z^2)^{-\frac{1}{2}}\big[(1+\alpha X(\eta))\,X(\eta)\,f(X(\eta))\big]^{-1}\Big\} = 0\,.$$

Now (2.3) yields $u(x)\,u'(x) = xf(x)$; therefore

$$(z/X'(z)) = X(z)\,f(X(z))\,.$$

By this relation our condition becomes

$$0 = \lim_{z\to u_0-}\left\{-z\int_{-z}^{z}X'(\eta)(1+\alpha X(\eta))^{-2}(u_0^2-\eta^2)^{-\frac{3}{2}}\,d\eta\, + \right.$$
$$+ (u_0^2-z^2)^{-\frac{1}{2}}\big[X'(z)(1+\alpha X(z))^{-2}+X'(-z)(1+\alpha X(-z))^{-2}\big]\,.$$

Since

$$z\int_{-z}^{z}(u_0^2-\eta^2)^{-\frac{3}{2}}\,\mathrm{d}\eta - 2z^2u_0^{-2}(u_0^2-z^2)^{-\frac{1}{2}} = 0\,,$$

then we can add the left hand side to the map in our condition. By this, and by adding $2(u_0^2 - z^2)^{-\frac{1}{2}} - 2(u_0^2 - z^2)^{-\frac{1}{2}}$, we have

$$0 = \lim_{z \to u_0 -} \left\{ -z \int_{-z}^{z} [X'(\eta)(1 + \alpha X(\eta))^{-2} - 1](u_0^2 - \eta^2)^{-\frac{3}{2}} \, d\eta + \right.$$
$$+ (u_0^2 - z^2)^{-\frac{1}{2}} [X'(z)(1 + \alpha X(z))^{-2} + X'(-z)(1 + \alpha X(-z))^{-2} - 2] +$$
$$\left. + 2(u_0^2 - z^2)^{-\frac{1}{2}} - 2z^2 u_0^{-2}(u_0^2 - z^2)^{-\frac{1}{2}} \right\}.$$

Now, let us remark that

$$\lim_{z \to u_0 -} [2(u_0^2 - z^2)^{-\frac{1}{2}} - 2z^2 u_0^{-2}(u_0^2 - z^2)^{-\frac{1}{2}}] = 0 ,$$

and let us introduce the odd map Q given by

$$(4.2) \qquad Q(0) = 0 \quad \text{and} \quad Q'(z) = $$
$$= 2 - X'(z)(1 + \alpha X(z))^{-2} - X'(-z)(1 + \alpha X(-z))^{-2}$$

(remark that $Q'(0) = 0$). By this we see at once that our condition is equivalent to

$$0 = \lim_{z \to u_0 -} \left\{ z \int_0^z Q'(\eta)(u_0^2 - \eta^2)^{-\frac{3}{2}} \, d\eta - Q'(z)(u_0^2 - z^2)^{-\frac{1}{2}} \right\}.$$

An integration by parts, and a trivial vanishing limit, finally yield

$$(4.3) \qquad \int_0^{u_0} \eta Q''(\eta)(u_0^2 - \eta^2)^{-\frac{1}{2}} \, d\eta = 0 .$$

Thus *the origin is a stable equilibrium for the system* (3.1) *iff there exists* $b \in \mathbf{R}_+^*$ *such that* (4.3) *holds for every* $u_0 \in]0, b[$.

Let us prove that *this last condition is equivalent to*

$$(4.4) \qquad Q(z) = 0 \quad \textit{for every } z \in]0, b[.$$

The condition in (4.4) obviously implies (4.3) for any $u_0 \in]0, b[$. Let us assume that (4.3) holds for every $u_0 \in]0, b[$ and let $z \in]0, b[$. Then

$$0 = 2\int_0^z \theta(z^2 - \theta^2)^{-\frac{1}{2}}\left(\int_0^\theta \eta Q''(\eta)(\theta^2 - \eta^2)^{-\frac{1}{2}}\, d\eta\right) d\theta =$$

$$= 2\int_0^z \left(\int_\eta^z \theta(z^2 - \theta^2)^{-\frac{1}{2}}(\theta^2 - \eta^2)^{-\frac{1}{2}}\, d\theta\right)\eta Q''(\eta)\, d\eta = \pi\int_0^z \eta Q''(\eta)\, d\eta$$

(the last equality can be obtained by the change of variables $\theta \mapsto \mapsto (\theta^2 - \eta^2)/(z^2 - \eta^2)$). Therefore

(4.5) $$A(z) := \int_0^z \eta Q''(\eta)\, dn = 0 \quad \text{for every } z \in]0, b[\,.$$

By (4.5) we have that the continuous map $Q''|]0, b[$ cannot have non-vanishing values (otherwise e.g. $Q''(\bar{z}) > 0$ and $A(\bar{z}) = 0$ should imply $A(\bar{z} + \varepsilon) > 0$ for any $\varepsilon > 0$ small enough). This result, $Q'(0) = 0$, and $Q(0) = 0$, give (4.4). □

By (4.2) and by considering $Q \circ u$—see Sec. 2—we finally have

PROPOSITION. *The origin is a stable equilibrium for the system* (3.1) *with* $f(0) = 1$ *iff there exists some* $a \in \mathbf{R}_+^*$ *such that*

(4.6) $$2u(x) = \frac{x - h(x)}{(1 + \alpha x)(1 + \alpha h(x))}$$

for every $x \in]0, a[$—see Sec. 2 for the definition of h.

5. Constructing the stable cases.

We can construct the collection of all the maps f which yield stability of the origin for (3.1) in the following two ways which are similar to those considered in [2] (after the Corollary of Sec. 4) for the system (1.1) (but f was only continuous in [2]). We consider $f(0) = 1$ only (the general case $f(0) > 0$ is trivially obtained).

Firstly, let us remark that (4.6) is equivalent to

$$h(x) = \frac{x - 2u(x) - 2\alpha x u(x)}{1 + 2\alpha u(x) + 2\alpha^2 x u(x)}\,. \tag{5.1}$$

Thus we can start our construction (arbitrarily) giving a C^1 map f in a right neighbourhood of 0 (with $f(0) = 1$). Then we define $v(x)$ as in (2.2) for $x \geqq 0$. Now, (5.1), with $u(x) = (2v(x))^{\frac{1}{2}}$, defines h in some interval $[0, a[$, and we can choose $a > 0$ such that the condition $h \circ h = \mathrm{id}$ extends h to a C^2 diffeomorphism of some open interval J onto itself (the proof is just a very boring calculation).

Finally, we extend v and f to a neighbourhood of 0 by setting $v(h(x)) = v(x)$, and $f(x) = v'(x)/x$. The f so obtained is a C^1 extension of the starting map.

Therefore, *for any C^1 map defined in a right neighbourhood of* 0, *and equals to* 1 *at* 0, *there exists a C^1 extension to some neighbourhood of* 0 *such that the origin is a stable equilibrium for the system* (3.1). *Moreover, any two of these extensions coincide in some neighbourhood of* 0. In other words, we have uniqueness of the extension yielding stability up to the equivalence which defines the C^1 germs at 0.

Also the other construction in [2], Sec. 4, can be followed. We just have to consider (2.5), instead of (3.3) of [2] (the difference is just $h \in \mathrm{Diff}^2$ instead of $h \in \mathrm{Diff}^1$), and $f(x) = u(x)\,u'(x)/x$, with $u(x)$ given by (4.6), instead of (4.4) of [2]—see the remark in Sec. 6 after the three stars.

6. Appendix I.

Let f be continuous in some open interval of $\mathbf{R}$ containing 0 and let $f(0) > 0$. The map

$$v\colon x \mapsto \int_0^x \xi f(\xi)\, d\xi$$

is C^1 and admits $v''(0) = f(0) > 0$. Consider the map $\bar{u}\colon x \mapsto (\operatorname{sgn} x)\cdot \cdot (2v(x))^{\frac{1}{2}}$. Since $(2v(x)/x^2) \to f(0) > 0$ as $x \to 0$, then $\bar{u}'(0) = (f(0))^{\frac{1}{2}} > 0$. Furthermore $\bar{u}'(x) = (xf(x)/\bar{u}(x)) \to \bar{u}'(0)$ as $x \to 0$. Therefore there exists a maximal open interval J, with $0 \in J$, such that $u = \bar{u}|J$ is a C^1

diffeomorphism onto a symmetric interval I:

$$(6.1)\qquad u \in \mathrm{Diff}^1(J; I)\,, \qquad (u(x))^2 = 2v(x)\,, \qquad u'(0) = (f(0))^{\frac{1}{2}}\,.$$

Remark that

$$(6.2)\qquad u(x)\,u'(x) = x\,f(x)\,.$$

This was already written in the paper [2] by the author.

In this paper we consider $f \in C^1$. Let us prove that

$$(6.3)\qquad f \in C^1 \quad \text{implies} \quad u \in \mathrm{Diff}^2(J; I)\,.$$

By (6.2)

$$u''(x) = \frac{f(x)}{u(x)} + \frac{x}{u(x)} f'(x) - \frac{x^2 f(x)^2}{u(x)^3} \qquad \text{for } x \neq 0\,.$$

So $u''(x)$ has a finite limit as $x \to 0$ if $1/u(x) - x^2 f(x)/u(x)^3$ has a finite limit, and this happens if $2v(x)/x^3 - f(x)/x$ has a finite limit—see $(6.1)_2$.

In order to calculate this last limit, let us remark that v is (only) C^2 for $f \in C^1$, but there exists $v'''(0)$:

$$v'''(0) = \lim_{x \to 0} \frac{f(x) + xf'(x) - f(0)}{x} = 2f'(0)\,.$$

Therefore Taylor's theorem gives

$$v(x) = (f(0)/2)\,x^2 + (f'(0)/3)\,x^3 + r(x)\,x^3\,, \qquad \text{with } r(x) \to 0 \text{ as } x \to 0\,.$$

By this expression, the aforementioned last limit is equal to $-f'(0)/3$. Thus u is C^2, and finally we have (6.3).

* * *

Let us remark that (6.2), with $u \in \mathrm{Diff}^2(J; I)$ as before, and $f(0) = u'(0)^2$, define a C^1 map $f\colon J \to \mathbb{R}$. In fact we easily see that $f'(x) \to 3u'(0)\,u''(0)/2$ as $x \to 0$.

7. Appendix II.

Let us prove the sentence which includes (3.6) in Sec. 3.

For the sake of simplicity let us set $T = \ell(x_0)$.

The map $t \mapsto \varphi(t+T)$ is a solution to (3.3). Since the typical solution to this equation can be written as

$$y(t) = y(0)\,\varphi(t) + \dot{y}(0)\,\psi(t)\,, \tag{7.1}$$

then, in particular

$$\varphi(t+T) = \varphi(t) + \dot{\varphi}(T)\,\psi(t)\,.$$

So

$$\dot{\varphi}(t+T) = \dot{\varphi}(t) + \dot{\varphi}(T)\,\dot{\psi}(t)\,. \tag{7.2}$$

Let us consider $t = nT$ for any $n \in Z$. Then (7.2) easily yields $\dot{\varphi}(nT) = n\dot{\varphi}(T)$. This result, and the expression obtained from (7.1) by taking the derivatives, imply

$$\dot{y}(nT) = ny(0)\,\dot{\varphi}(T) + \dot{y}(0)\,, \quad \text{for every } n \in Z\,. \tag{7.3}$$

To complete the proof we just have to show that $\dot{\varphi}(T) = 0$ if and only if $\dot{\varphi}(T/2) = 0$. Consider (7.2) with $t = -T/2$:

$$\dot{\varphi}(T/2) = \dot{\varphi}(-T/2) + \dot{\varphi}(T)\,\dot{\psi}(-T/2)\,.$$

This, $\dot{\varphi}(-T/2) = -\dot{\varphi}(T/2)$, and $\dot{\psi}(-T/2) \neq 0$, give the result.

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