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## Partial Differential Equations in Domains with Self-Contact.

A. VISINTIN (\*)

ABSTRACT - Let  $\Omega$  be a Euclidean domain,  $\Gamma_1, \Gamma_2 \subset \Omega$  be « smooth » surfaces of codimension 1, and  $\alpha: \Gamma_1 \rightarrow \Gamma_2$  be a « smooth » application. Here second and fourth order partial differential equations are studied in  $\Omega$  under the constraints

$$(1) \quad v(\sigma) = v(\alpha(\sigma)) \quad \text{on } \Gamma_1$$

and, for fourth order equations,

$$(2) \quad \nabla v(\sigma) = \nabla v(\alpha(\sigma)) \quad \text{on } \Gamma_1;$$

these yield special discontinuity conditions on  $\Gamma_1$  and  $\Gamma_2$ : These constraints can correspond to non-standard geometrical structures, which have natural applications in engineering.

### 1. Introduction.

Let us consider a body occupying a domain  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ); assume that its state is characterized by a field  $v: \Omega \rightarrow \mathbb{R}$  and that its equilibrium is governed by a potential of the form

$$(1.1) \quad E(v) := \int_{\Omega} [\Psi_1(v, \nabla v) - fv] dx \quad \forall v \in H^1(\Omega),$$

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where  $\Psi_1$  is a convex function and  $f \in L^2(\Omega)$  is a datum. This is a typical situation; for instance, it occurs in thermostatics, in electrostatics, and so on.

Now we consider a «smooth»  $(N-1)$ -dimensional manifold  $\Gamma_1 \subset \Omega$  and a «smooth» injective application  $\alpha: \Gamma_1 \rightarrow \Omega$ ; then we introduce the condition

$$(1.2) \quad v(x) = v(\alpha(x)) \quad \text{a.e. on } \Gamma_1.$$

This constraint corresponds to two basic situations in applications; in both cases  $\Omega$  will be regarded as imbedded in some  $\mathbb{R}^M$ , with  $M > N$ . In the first example each  $\alpha \in \Gamma_1$  is connected with  $\alpha(\sigma) \in \alpha(\Gamma_1)$  by means of a highly conducting wire; in electrostatics this corresponds to a short circuit. The second example corresponds to deforming  $\Omega$

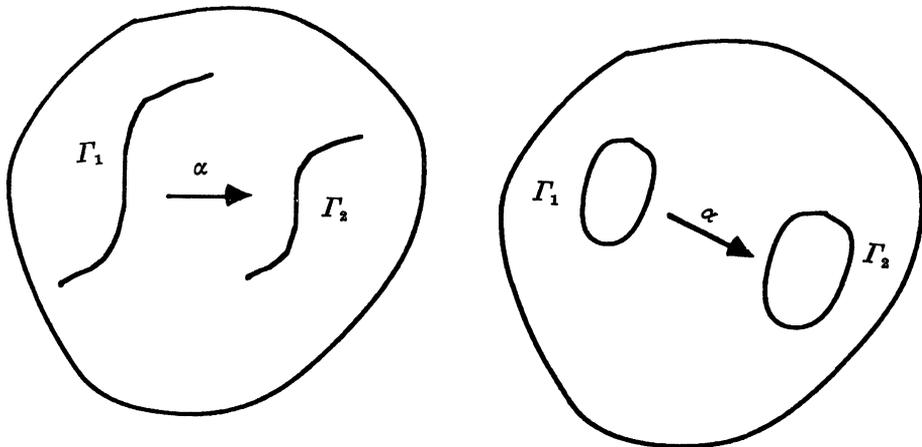


Fig. 1. Two examples of 1-dimensional manifolds in  $\Omega \subset \mathbb{R}^2$ , with and without boundaries, respectively. In each case  $\Gamma_1$  and  $\Gamma_2$  are «identified» by means of a map  $\alpha: \Gamma_1 \rightarrow \Gamma_2$ .

in  $\mathbb{R}^M$  in such a way that the body intersects itself along  $\Gamma_1$ ; more precisely we consider a map  $z: \Omega \rightarrow \mathbb{R}^M$  such that

$$(1.3) \quad \begin{cases} \forall x \in \Gamma_1, & z(x) = z(\alpha(x)); \\ \forall x, y \in \Omega \setminus (\Gamma_1 \cup \alpha(\Gamma_1)), & z(x) = z(y) \quad \text{only if } z = y; \end{cases}$$

$z$  can be regarded as a discontinuous parametrization of the set  $z(\Omega) \subset \mathbb{R}^M$ . For instance  $M = 2, N = 1, \Omega = ]0, 1[, \Gamma_1 = \{\frac{1}{3}\}, \alpha(\frac{1}{3}) = \frac{2}{3}$ , and  $z(\Omega)$  is as in fig. 2a.

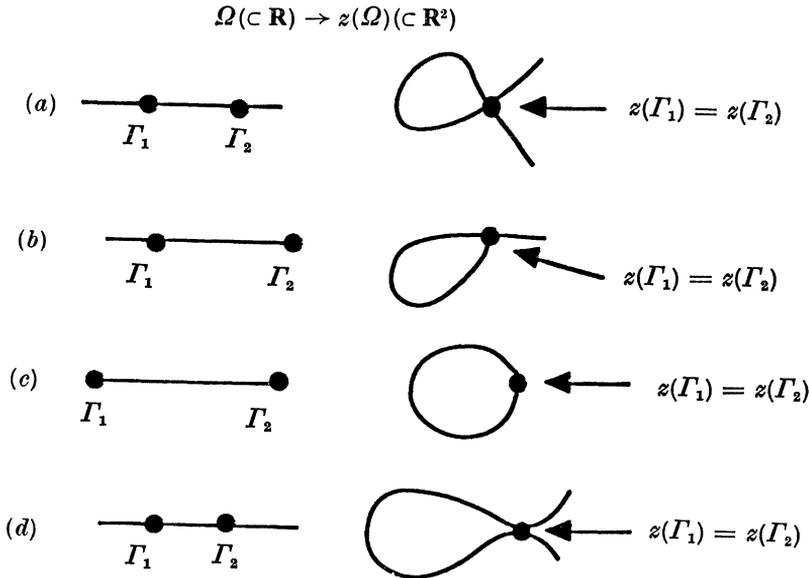


Fig. 2. The left-hand side of this figure shows examples of 0-dimensional manifolds ( $\Gamma_1$  and  $\Gamma_2$ ), namely points, included in 1-dimensional sets ( $\Omega$ ), namely segments. In the right-hand side the corresponding identifications of  $\Gamma_1$  and  $\Gamma_2$  are represented by means of deformations  $z$  in the ambient space  $\mathbb{R}^2$ ; cf. (1.3) and (1.6). In (a) and (b) the identifications are of 0-order; in (c) and (d) they are of 1st-order. In (a) and (d)  $\Gamma_1, \Gamma_2 \subset \Omega$ ; in (b)  $\Gamma_1 \subset \Omega, \Gamma_2 \subset \partial\Omega$ ; in (c) (periodicity conditions)  $\Gamma_1, \Gamma_2 \subset \partial\Omega$ .

Under suitable coerciveness assumptions,  $E$  has a minimum under the constraint (1.2). In section 2 we study the corresponding Euler-Lagrange equations; in particular a discontinuity condition holds on  $\Gamma := \Gamma_1 \cup \alpha(\Gamma_1)$ .

Let us now consider a functional of the form

$$(1.4) \quad F(v) := \int_{\Omega} [\mathcal{P}_2(v, \nabla v, \Delta v) - fv] dx, \quad \forall v \in H^2(\Omega),$$

still with  $\Psi_2$  convex function and  $f \in L^2(\Omega)$  given. Here we can choose between two constraints: either the 0-order identification (1.2), or the 1st-order one

$$(1.5) \quad v(\sigma) = v(\alpha(\sigma)), \quad \nabla v(\sigma) = \nabla v(\alpha(\sigma)) \quad \text{on } \Gamma_1.$$

The latter corresponds, for instance, to a deformation  $z \in W^{1,1}(\Omega)$  which fulfills (1.3) and such that

$$(1.6) \quad \nabla z(\sigma) = \nabla z(\alpha(\sigma)) \quad \text{on } \Gamma_1.$$

Under each of the constraints (1.2) and (1.5),  $F$  has a minimum. In section 3 we study the corresponding Euler-Lagrange conditions; in particular in each case two discontinuity conditions arise on  $\Gamma$ .

Of course several generalizations could be taken into account. For instance one could consider non-differentiable convex functions  $\Psi_i$ 's; this would lead to variational inequalities. One could replace  $E$  with

$$(1.7) \quad \tilde{E}(v) := E(v) - \int_{\Gamma_1} gv \, dx \quad \forall v \in H^1(\Omega)$$

with  $g \in L^2(\Gamma_1)$ , and similarly for  $F$ ; this would yield different jump conditions on  $\Gamma_1$ . Also the evolution case  $\partial u / \partial t + \partial \mathcal{E}(u) = 0$  (with  $\mathcal{E} = E$  or  $F$ ) could be easily treated. And so on.

The developments of the present paper are just simple extensions of some classical results of [2], and do not convey any essentially new mathematical idea. However they extend the range of applications of boundary value problems to cases which seem to be of practical interest, such as the heat (or electric) diffusion in lattices, or the similar.

## 2. Second order equations.

Let  $\Omega$  be a domain of  $\mathbb{R}^N$  ( $N \geq 1$ ), and  $\Gamma_1 \subset \Omega$  be a Lipschitz,  $(N-1)$ -dimensional manifold, possibly with boundary; we endow  $\Gamma_1$  with the  $(N-1)$ -dimensional Hausdorff measure. At almost every  $\sigma \in \Gamma_1$  one can fix a unit normal vector  $\nu_{r_1}(\sigma)$ ; we assume that locally all the  $\nu_{r_1}$  point toward the same side of  $\Gamma_1$ .

Let  $\alpha: \Gamma_1 \rightarrow \Omega$  be an injective, Lipschitz-continuous map; then also  $\Gamma_2 := \alpha(\Gamma_1)$  is a Lipschitz,  $(N-1)$ -dimensional manifold, which we endow with the  $(N-1)$ -dimensional Hausdorff measure. Also here we can introduce a field  $\nu_{\Gamma_i}$  of unit normal vectors, with the same restriction as above; however we do not need any relationship between the orientation of  $\nu_{\Gamma_1}(\sigma)$  and that of  $\nu_{\Gamma_2}(\alpha(\sigma))$ . We assume that  $\Gamma_1 \cap \Gamma_2 = \emptyset$ , just for convenience of presentation. We set  $\Gamma := \Gamma_1 \cup \Gamma_2$ .

We introduce

$$(2.1) \quad V_\alpha := \{v \in H_0^1(\Omega) : v(\sigma) = v(\alpha(\sigma)) \text{ a.e. on } \Gamma_1\};$$

this is a nonlocal Hilbert subspace of  $H^1(\Omega)$ ; more precisely  $V_\alpha$  is the closure of

$$(2.2) \quad \mathcal{D}_\alpha^0 := \{v \in \mathcal{D}(\Omega) : v(\sigma) = v(\alpha(\sigma)) \text{ on } \Gamma_1\}$$

with respect to the topology of  $H^1(\Omega)$ .

Let  $\Phi_0: \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\Phi_1: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper, lower semicontinuous, convex functions such that

$$(2.3) \quad \Phi_0(\xi) \geq C_1|\xi|^2 - C_2 \quad \forall \xi \in \mathbb{R}$$

$$(2.4) \quad \Phi_1(\xi) \geq C_3|\xi|^2 - C_4 \quad \forall \xi \in \mathbb{R}^N$$

( $C_1, \dots, C_4$ : constants  $> 0$ ). We assume that  $\Phi_0$  and  $\Phi_1$  are Gâteaux-differentiable, and denote their derivatives by  $\varphi_0$  and  $\varphi_1$ . We fix an  $f \in L^2(\Omega)$  and set

$$(2.5) \quad E(v) := \int_{\Omega} [\Phi_0(v) + \Phi_1(\nabla v) - fv] dx \quad \forall v \in H_0^1(\Omega).$$

Under the previous assumptions, there exists at least one  $u \in V_\alpha$  such that

$$(2.6) \quad E(u) \leq E(v) \quad \forall v \in V_\alpha,$$

and this condition is equivalent to the variational equation

$$(2.7) \quad \int_{\Omega} [\varphi_0(u)v + \varphi_1(\nabla u) \cdot \nabla v - fv] dx = 0 \quad \forall v \in V_\alpha;$$

in turn this yields

$$(2.8) \quad \varphi_0(u) - \nabla \cdot \varphi_1(\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega \setminus \Gamma).$$

Now we want to deduce the conditions fulfilled by  $u$  on  $\Gamma$ .

We denote by  $\pi_\Gamma: \Omega \rightarrow \Gamma$  the (possibly multivalued) projection onto  $\Gamma$ . For any « smooth » function  $w: \Omega \rightarrow \mathbf{R}$  and almost any  $\sigma \in \Gamma$  we set

$$\begin{aligned} (v_\Gamma^\pm w)(\sigma) &:= \lim_{\substack{\Omega \ni x \rightarrow \sigma \\ \pm(x-\sigma) \cdot \nu_\Gamma(\pi_\Gamma x) > 0}} \nu_\Gamma(\pi_\Gamma x) w(x), \\ \llbracket v_\Gamma w \rrbracket(\sigma) &:= (v_\Gamma^+ w)(\sigma) - (v_\Gamma^- w)(\sigma), \end{aligned}$$

when both limits exist. Notice that the signs of these quantities are independent of the orientation of  $\nu_\Gamma$ .

Let us fix any open subset  $B_1$  of  $\Gamma_1$  such that its closure is disjoint from the boundary of  $\Gamma_1$ , if existing, and let us set  $B := B_1 \cup \cup \alpha(B_1)$ ; then let us take any  $v \in \mathcal{D}(\Omega)$  such that  $v = 0$  on  $\Gamma \setminus B$ . For a moment let us assume that  $\varphi_1(\nabla u)$  is so smooth that the Green formula can be applied on both sides of  $\Gamma$ ; this restriction will be removed later on. Thus we have

$$(2.9) \quad \int_{\Omega} \varphi_1(\nabla u) \cdot \nabla v \, dx = \int_{\Omega \setminus \Gamma} \varphi_1(\nabla u) \cdot \nabla v \, dx = \\ = - \int_{\Omega \setminus \Gamma} \nabla \cdot \varphi_1(\nabla u) v \, dx - \int_B \llbracket v_\Gamma \cdot \varphi_1(\nabla u) \rrbracket(\sigma) v(\sigma) \, d\sigma.$$

By (2.8), ..., (2.9) we get

$$(2.10) \quad \int_B \llbracket v_\Gamma \cdot \varphi_1(\nabla u) \rrbracket(\sigma) v(\sigma) \, d\sigma = 0.$$

As  $\alpha$  was assumed to be Lipschitz-continuous, its Jacobian determinant  $|\nabla \alpha|$  is in  $L^\infty(\Gamma_1)$ ; hence

$$(2.11) \quad \int_{\alpha(B_1)} \llbracket v_\Gamma \cdot \varphi_1(\nabla u) \rrbracket(\sigma) v(\sigma) \, d\sigma = \int_{B_1} \llbracket v_\Gamma \cdot \varphi_1(\nabla u) \rrbracket(\alpha(\sigma)) v(\alpha(\sigma)) |\nabla \alpha(\sigma)| \, d\sigma.$$

Now let assume that  $v \in \mathcal{D}_\alpha^0(\Omega)$ , namely  $v(\sigma) = v(\alpha(\sigma))$  on  $\Gamma_1$ ; then (2.10)

becomes,

$$(2.12) \quad \int_{B_1} \{ [\nu_r \cdot \varphi_1(\nabla u)](\sigma) + [\nu_r \cdot \varphi_1(\nabla u)](\alpha(\sigma)) |\nabla \alpha(\sigma)| \} v(\sigma) d\sigma = 0,$$

whence, as  $B_1$  and  $v$  are generic, we finally get

$$(2.13) \quad [\nu_r \cdot \varphi_1(\nabla u)](\sigma) + [\nu_r \cdot \varphi_1(\nabla u)](\alpha(\sigma)) |\nabla \alpha(\sigma)| = 0, \quad \text{on } \Gamma_1.$$

So far we assumed the solution  $u$  to be regular. However it is possible to deduce a more rigorous formulation of the discontinuity condition (2.13), under further assumptions on the data. Let us require that

$$(2.14) \quad |\varphi_0(\xi)| \leq C_5 |\xi| + C_6 \quad \forall \xi \in \mathbf{R}$$

$$(2.15) \quad |\varphi_1(\xi)| \leq C_7 |\xi| + C_8 \quad \forall \xi \in \mathbf{R}^N$$

( $C_5, \dots, C_8$ : constants  $> 0$ ). Then by comparison in (2.8) we get

$$(2.16) \quad \varphi_1(\nabla u) \in L^2(\Omega)^N, \quad \nabla \cdot \varphi_1(\nabla u) \in L^2(\Omega);$$

hence, cf. [2, chapter 2], [1; Appendix 4],

$$(2.17) \quad \nu_r^\pm \cdot \varphi_1(\nabla u) \in H^{-\frac{1}{2}}(\Gamma_1) \quad (= H^{\frac{1}{2}}(\Gamma_1)').$$

Now let us also assume that

$$(2.18) \quad |\nabla \alpha| \in W^{1,\infty}(\Gamma_1);$$

then

$$(2.19) \quad \nu_r^\pm \cdot \varphi_1(\nabla u) |\nabla \alpha| \in H^{-\frac{1}{2}}(\Gamma_1),$$

and the previous Green formulae hold, if the integrals are replaced by the proper duality pairings. Thus we can conclude that (2.13) effectively holds, in the sense of  $H^{-\frac{1}{2}}(\Gamma_1)$ .

The previous results are summarized in the following statement:

**PROPOSITION 1.** The functional  $E$  has at least one minimum in  $V_\alpha$ , and the minimum condition is equivalent to the variational equation (2.7). This also corresponds to the equation (2.8) and to a weak

form of the discontinuity condition (2.13). Under the further assumptions (2.14), (2.15), (2.18), the condition (2.13) holds in the sense of  $H^{-1}(\Gamma_1)$ .

REMARK. Set

$$(2.20) \quad J(v, \mu) := E(v) + {}_{H^{-1}(\Gamma_1)}\langle \mu(\sigma), v(\sigma) - v(\alpha(\sigma)) \rangle_{H^1(\Gamma_1)} \\ \forall v \in H_0^1(\Omega), \forall \mu \in H^{-1}(\Gamma_1).$$

Let  $u \in V_\alpha$  be a minimum of the functional  $E$  in  $V_\alpha$ , and set, cf. (2.13),

$$\lambda(\sigma) := \llbracket \nu \cdot \varphi_1(\nabla u) \rrbracket(\sigma) = - \llbracket \nu \cdot \varphi_1(\nabla u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)|$$

in the sense of  $H^{-1}(\Gamma_1)$  (hence not a.e. on  $\Gamma_1!$ ). Then, as it is easy to check by applying the Green formulae as above,  $(u, \lambda) \in H_0^1(\Omega) \times H^{-1}(\Gamma_1)$  is a stationary point of  $J$ . That is  $\lambda$  is the *Lagrange multiplier* for the functional  $E$  corresponding to the constraint (1.2).

### 3. Fourth order equations.

Let  $\Omega$ ,  $\alpha$ ,  $\Gamma$  be as in section 2. We set

$$(3.1) \quad W_\alpha^0 := \{v \in H_0^2(\Omega) : v(\sigma) = v(\alpha(\sigma)) \text{ a.e. on } \Gamma_1\},$$

$$(3.2) \quad W_\alpha^1 := \{v \in H_0^2(\Omega) : v(\sigma) = v(\alpha(\sigma)), \nabla v(\sigma) = \nabla v(\alpha(\sigma)) \\ \text{a.e. on } \Gamma_1\};$$

both are nonlocal Hilbert subspaces of  $H^2(\Omega)$ ; more precisely, recalling (2.2) and setting

$$(3.3) \quad \mathcal{D}_\alpha^i(\Omega) := \{v \in \mathcal{D}(\Omega) : v(\sigma) = v(\alpha(\sigma)), \nabla v(\sigma) = \nabla v(\alpha(\sigma)) \text{ on } \Gamma_1\},$$

$W_\alpha^i$  is the closure of  $\mathcal{D}_\alpha^i(\Omega)$  with respect to the topology of  $H^2(\Omega)$ , for  $i = 0, 1$ . The exponents 0 and 1 indicate the order of the derivatives which are « identified » by  $\alpha$ .

Besides  $\Phi_0$  and  $\Phi_1$  (introduced in section 1), let also  $\Phi_2: \mathbf{R} \rightarrow \mathbf{R} \cup \{+\infty\}$  be a proper, lower semi-continuous convex function such that

$$(3.4) \quad \Phi_2(\xi) \geq C_9 |\xi|^2 - C_{10} \quad \forall \xi \in \mathbf{R}$$

( $C_9, C_{10}$ : constants  $> 0$ ). We assume that also  $\Phi_2$  is Gâteaux-differentiable and denote its derivative by  $\varphi_2$ . We set

$$(3.5) \quad F(v) := \int_{\Omega} [\Phi_0(v) + \Phi_1(\nabla v) + \Phi_2(\Delta v) - fv] dx \quad \forall v \in H_0^2(\Omega).$$

Obviously, for  $i = 0, 1$ , there exists at least one  $u_i \in W_{\alpha}^i$  such that

$$(3.6) \quad F(u_i) \leq F(v) \quad \forall v \in W_{\alpha}^i;$$

this condition is equivalent to the variational equation

$$(3.7) \quad \int_{\Omega} [\varphi_0(u_i)v + \varphi_1(\nabla u_i) \cdot \nabla v + \varphi_2(\Delta u_i)\Delta v - fu_i] dx = 0 \quad \forall v \in W_{\alpha}^i,$$

which yields

$$(3.8) \quad \varphi_0(u_i) - \nabla \cdot \varphi_1(\nabla u_i) + \Delta \varphi_2(\Delta u_i) = f \quad \text{in } \mathcal{D}'(\Omega \setminus \Gamma).$$

Also here we want to deduce the conditions fulfilled by  $u_i$  on  $\Gamma$ . To this aim, let us fix any open subset  $B_1$  of  $\Gamma_1$  such that its closure does not intersect the boundary of  $\Gamma_1$ , if existing, and let us set  $B := B_1 \cup \cup \alpha(B_1)$ ; then let us take any  $v \in \mathcal{D}(\Omega)$  such that  $v = 0$  on  $\Gamma \setminus B$ . For a moment we also assume that  $\varphi_1(\nabla u)$  and  $\varphi_2(\Delta u)$  are so smooth that the Green formulae can be applied on both sides of  $\Gamma$ ; thus we have

$$(3.9) \quad \begin{aligned} \int_{\Omega} [\varphi_1(\nabla u) \cdot \nabla v + \varphi_2(\Delta u)\Delta v] dx &= \int_{\Omega \setminus \Gamma} [\varphi_1(\nabla u) \cdot \nabla v + \varphi_2(\Delta u)\Delta v] dx = \\ &= - \int_{\Omega \setminus \Gamma} \{ [\nabla \cdot \varphi_1(\nabla u)]v + [\nabla \varphi_2(\Delta u)] \cdot \nabla v \} dx - \\ &\quad - \int_B [[\nu_{\Gamma} \cdot \varphi_1(\nabla u)]](\sigma)v(\sigma) d\sigma - \int_B [[\nu_{\Gamma} \cdot \varphi_2(\Delta u)]](\sigma) \cdot \nabla v(\sigma) d\sigma, \end{aligned}$$

$$(3.10) \quad - \int_{\Omega \setminus \Gamma} [\nabla \varphi_2(\Delta u)] \cdot \nabla v dx = \int_{\Omega \setminus \Gamma} \Delta \varphi_2(\Delta u)v d\sigma + \int_B [[\nu_{\Gamma} \cdot \nabla \varphi_2(\Delta u)]](\sigma)v(\sigma) d\sigma;$$

we also notice that a.e. on  $\Gamma$   $\nabla u$  cannot be discontinuous across  $\Gamma$ ,

for  $\Delta u \in L^2(\Omega)$ ; hence

$$(3.11) \quad \llbracket \nu_r \cdot \varphi_1(\nabla u) \rrbracket(\sigma) = 0 \quad \text{a.e. on } \Gamma.$$

By assembling (3.9), ..., (3.11) we get

$$(3.12) \quad \int_{\Omega} [\varphi_1(\nabla u) \cdot \nabla v + \varphi_2(\Delta u) \Delta v] dx = \int_{\Omega \setminus \Gamma} [-\nabla \cdot \varphi_1(\nabla u) + \Delta \varphi_2(\Delta u)] v dx + \\ + \int_B \{ \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\sigma) v(\sigma) - \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\sigma) \cdot \nabla v(\sigma) \} d\sigma.$$

By (3.8) and (3.12), the integral over  $B$  vanishes; hence, using a transformation formula similar to (2.11), we get

$$(3.13) \quad \int_{B_1} \{ \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\sigma) v(\sigma) + \\ + \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) v(\alpha(\sigma)) |\nabla \alpha(\sigma)| \} d\sigma + \\ + \int_{B_1} \{ \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\sigma) \cdot \nabla v(\sigma) + \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) \cdot \nabla v(\alpha(\sigma)) |\nabla \alpha(\sigma)| \} d\sigma = 0.$$

Now let us distinguish between  $u_0$  and  $u_1$ . For  $u_0$  we assume that  $v \in \mathcal{D}_x^0(\Omega)$ , namely  $v(\sigma) = v(\alpha(\sigma))$  on  $\Gamma_1$ ; instead  $\nabla v(\sigma)$  and  $\nabla v(\alpha(\sigma))$  are independent on  $\Gamma_1$ . Thus, as  $B_1$  and  $v$  are arbitrary, we get

$$(3.14) \quad \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\sigma) + \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)| = 0 \quad \text{on } \Gamma_1,$$

$$(3.15) \quad \llbracket \nu_r \varphi_2(\nabla u) \rrbracket(\sigma) = \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)| = 0 \quad \text{on } \Gamma_1.$$

For  $u_1$  we assume that  $v \in \mathcal{D}_x^1(\Omega)$ , namely  $v(\sigma) = v(\alpha(\sigma))$ ,  $\nabla v(\sigma) = \nabla v(\alpha(\sigma))$  on  $\Gamma_1$ . Thus we get (3.14) and

$$(3.16) \quad \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\sigma) + \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)| = 0 \quad \text{on } \Gamma_1.$$

Also here it is possible to precise the discontinuity conditions, under further assumptions on the data. Let us assume that (2.14) holds and that

$$(3.17) \quad |\varphi_1(\xi_1) - \varphi_1(\xi_2)| \leq C_{11} |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N,$$

$$(3.18) \quad |\varphi_2(\xi)| \leq C_{12} |\xi| + C_{13} \quad \forall \xi \in \mathbb{R}$$

( $C_{11}, \dots, C_{13}$ : constants  $> 0$ ). Then by comparison in (3.8) we get

$$(3.19) \quad \varphi_2(\Delta u), \quad \Delta \varphi_2(\Delta u) \in L^2(\Omega);$$

hence (3.14) holds in  $H^1(\Gamma_1)$ , (3.15) and (3.16) in  $H^{\frac{3}{2}}(\Gamma_1)$ ; this justifies the previous Green formulae.

Thus we have the following result:

**PROPOSITION 2.** For  $i = 0, 1$  the functional  $F$  has a minimum in  $W_\alpha^i$ , and the minimum condition is equivalent to the variational equation (3.7). This also corresponds to the equation (3.8) and to the discontinuity conditions (3.14) and (3.15) for  $i = 0$ , to (3.14) and (3.16) for  $i = 1$ . Under the further assumptions (2.14), (3.17) and (3.18), the conditions (3.14) holds in the sense of  $H^1(\Gamma_1)$ , and the conditions (3.15), (3.16) hold in the sense of  $H^{\frac{3}{2}}(\Gamma_1)$ .

**REMARKS.** (i) Set

$$(3.20) \quad I^0(v, \mu_1) := F(v) + \int_{\Gamma_1} \mu_1(\sigma) [v(\sigma) - v(\alpha(\sigma))] d\sigma$$

$$\forall u \in H^2(\Omega), \quad \forall \mu_1 \in L^2(\Gamma_1),$$

$$(3.21) \quad I^1(v, \mu_1, \mu_2) := I^0(v, \mu_1) + \int_{\Gamma_1} \mu_2(\sigma) \cdot [\nabla v(\sigma) - \nabla v(\alpha(\sigma))] d\sigma$$

$$\forall u \in H^2(\Omega), \quad \forall \mu_1 \in L^2(\Gamma_1), \quad \forall \mu_2 \in L^2(\Gamma_1)^3.$$

For  $i = 0, 1$ , let  $u_i$  be a minimum of the functional  $F$  in  $W_\alpha^i$ , and set, cf. (3.14) and (3.16),

$$(3.22) \quad \lambda_1(\sigma) := \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\sigma) = - \llbracket \nu_r \cdot \nabla \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)|$$

a.e. on  $\Gamma$ ,

$$(3.23) \quad \lambda_2(\sigma) := \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\sigma) = - \llbracket \nu_r \varphi_2(\Delta u) \rrbracket(\alpha(\sigma)) |\nabla \alpha(\sigma)|$$

a.e. on  $\Gamma_1$ .

Then, as can be checked by applying the Green formulae as above,

$$(u_0, \lambda_1) \in H^2(\Omega) \times H^1(\Gamma_1) \quad \text{and} \quad (u_1, \lambda_1, \lambda_2) \in H^2(\Omega) \times H^1(\Gamma_1) \times H^{\frac{3}{2}}(\Gamma_1)$$

are stationary points of  $I^0$  and  $I^1$ , respectively. That is  $\lambda_1$  and  $(\lambda_1, \lambda_2)$  are the *Lagrange multipliers* for the functional  $F$  corresponding to the constraint (1.2) and (1.2), (1.5), respectively.

(ii) So far we assumed  $\Gamma_1, \Gamma_2 \subset \Omega$ ; however one can also deal with  $\Gamma_1, \Gamma_2 \subset \bar{\Omega}$  (closure of  $\Omega$ ). For instance in fig. 2b  $\Gamma_1 \subset \Omega, \Gamma_2 \subset \partial\Omega$ ; in fig. 2c (periodicity conditions)  $\Gamma_1, \Gamma_2 \subset \partial\Omega$ . The developments of sections 2 and 3 hold also here; for any function  $\xi$ , it is sufficient to replace  $[[\nu_r \xi]]$  with  $\nu_{\bar{r}}$  on  $\Gamma \subset \partial\Omega$ , if  $\nu_r$  is oriented outward  $\Omega$ .

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