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# A note on fractional integration

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# A Note on Fractional Integration.

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ABSTRACT - We give a new proof of the Hardy-Littlevood-Sobolev fractional integration theorem, based on a pointwise estimate of the rearrangement of the fractional integral.

The main purpose of this note is to give an alternative proof of the Hardy-Littlewood-Sobolev fractional integration theorem. Our proof is simple, of great generality (it applies to any locally compact unimodular group with $\sigma$-finite Haar measure) and does not depend on interpolation results or covering properties of the space. It generalizes some ideas of Hedberg (see [2]). The key step is a pointwise estimate of a variant of the nonincreasing rearrangement of the convolution $f * K_\alpha$, where $K_\alpha$ is a function in $L(n/(n-\alpha), \infty)$ (the weak-$L^{n/(n-\alpha)}$). Hedberg gives a pointwise estimate of $f * K_\alpha$ in terms of the Hardy-Littlewood maximal function of $f$. Our estimate is in terms of a sort of nonincreasing rearrangement of $f$. This is advantageous when dealing with problems connected with Lorentz spaces. The pointwise estimate of $f * K_\alpha$ we obtain allows us to generalize to kernels in $L(n/(n-\alpha), \infty)$ the other inequalities contained in Hedberg's paper.

We recall that a function $f$ is in the Lorentz space $L(n/(n-\alpha), \infty)$ if $|\{y: |f(y)| \geqslant \lambda\}| \leqslant A\lambda^{-n/(n-\alpha)}$.

Given a locally integrable function $f$, we introduce a sort of re-

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arrangement of $f$ (see [3]) in the following way

$$f^{**}(t) = \sup_{|E|\geqslant t} |E|^{-1}\int_E |f(y)|\, dy \qquad t>0\,.$$

It is well known that if $p>1$, $\left(\int_0^\infty \left((f^{**}(t))^p\right) dt\right)^{1/p}$ is comparable with the $L^p$ norm of $f$.

We refer to Hunt's paper ([3]) for more about Lorentz spaces (see also [1]).

Lemma 1. *Suppose that* $K_\alpha \in L(n/(n-\alpha), \infty)$ *and* $f \in L^p(R^n)$, $1<p<n/\alpha$. *Then, for any* $\delta>0$

$$(I_\alpha(f))^{**}(t) \leqslant A\{\delta^\alpha f^{**}(t) + \delta^{\alpha-n/p}\|f\|_p\}\,.$$

Proof. Set $F_\delta = \{y\colon K_\alpha(x-y)\geqslant \delta^{\alpha-n}\}$. Then

$$|I_\alpha(f)(x)| \leqslant \Big(\int_{F_\delta} + \int_{F_\delta^c}\Big)\, |f(y)K_\alpha(x-y)|\, dy = I_1(x) + I_2(x)\,.$$

By Holder's inequality

$$I_2(x) \leqslant \|f\|_p \Big(\int_{F_\delta^c} |K_\alpha(x-y)|^{p'} dy\Big)^{1/p'}\,.$$

Since $F_\delta^c = \bigcup_j B_{\delta,j}$, where $B_{\delta,j} = \{y\colon 2^{-j}\delta^{\alpha-n} \leqslant K_\alpha(x-y) < 2^{-j+1}\delta^{\alpha-n}\}$ and $K_\alpha \in L(n/(n-\alpha), \infty)$ we have

$$\int_{F_\delta^c} |K_\alpha(x-y)|^{p'} dy = \sum_{j=1}^{\infty} \int_{B_{\delta,j}} |K_\alpha(x-y)|\, dy \leqslant$$

$$\leqslant A \sum_{j=1}^{\infty} 2^{jn/(n-\alpha)}\delta^n (2^{-j}\delta^{\alpha-n})^{p'} \leqslant A\delta^{n+(\alpha-n)p'}$$

whence $I_2(x) \leqslant A\|f\|_p\,\delta^{\alpha-n/p}$.

Set $F_\delta \bigcup\limits_j F_{\delta,j}$, where

$$F_{\delta,j} = \{y : 2^j \delta^{\alpha-n} < K_\alpha(x-y) \leqslant 2^{j+1}\delta^{\alpha-n}\} .$$

Moreover set

$$\tilde{F}_{\delta,j} = \{y : 2^j \delta^{\alpha-n} < K_\alpha(y) \leqslant 2^{j+1}\delta^{\alpha-n}\} .$$

Then

$$|E|^{-1}\int_E I_1(x)\,dx = |E|^{-1}\int_E\int_{F_\delta} |f(y)K_\alpha(x-y)|\,dy\,dx =$$

$$= \sum_{j=1}^{\infty} 2^j\delta^{\alpha-n}\int_{\tilde{F}_{\delta,j}} |E|^{-1}\int_E |f(x-y)|\,dy\,dx = \sum_{j=1}^{\infty} 2^j\delta^{\alpha-n}\int_{\tilde{F}_{\delta,j}} \left(|E|^{-1}\int_E |f(y)|\,dy\right) dx \leqslant$$

$$\leqslant Af^{**}(t)\sum_{j=1}^{\infty} 2^j\delta^{\alpha-n}(2^j\delta^{\alpha-n})^{-n/(n-\alpha)} = A\delta^\alpha f^{**}(t) .$$

Therefore

$$|E|^{-1}\int_E I_\alpha(f)(x)\,dx \leqslant |E|^{-1}\int_E (I_1(x)+I_2(x))\,dx \leqslant A(\|f\|_p\delta^{\alpha-n/p} + \delta^\alpha f^{**}(t))$$

from which the lemma follows easily.

THEOREM 2. *Let* $0<\alpha<n$, $1<p$, $q<\infty$, $1/q = 1/p - \alpha/n$. *Then*

$$\|I_\alpha(f)\|_q \leqslant A\|f\|_p .$$

PROOF. Set $\delta = (f^{**}(t)/\|f\|_p)^{-p/n}$ and integrate both sides of the inequality in Lemma 1 to get the result.

REMARKS. (*a*) The proof of Muckenhoupt and Stein (see [4]) uses the Marcinkiewicz interpolation theorem, but gives also an endpoint result (the case $p=1$). Hedberg's proof relies on the Hardy-Littlewood maximal function theorem, which depends on a covering property of the ambient space. In general cases this property is not available or it is technically very difficult to prove.

(*b*) In the compact case slight modifications are in order; we omit the details.

(*c*) Our proof gives interesting results even when dealing with non convolution operators. Applications to such situations will be given elsewhere.

We can also prove the following generalization of Theorem 4 in [2]. We remark that this result for general kernels which belong to $L(n/(n-\alpha), \infty)$ is new.

THEOREM 3. *If* $f \geqslant 0$ *is measurable, then*

$$\|I_{\alpha\theta}(f^s)\|_r \leqslant \|f\|_p^{s-\theta} \|I_\alpha(f)\|_q^\theta$$

*whenever* $0<\alpha<n$, $0<\theta<1$, $1<p$, $q<\infty$, $\theta<s<\theta+(1-\theta)p$ *and* $1/r = (s-\theta)/p + \theta/q$.

PROOF. For simplicity we give only the proof in the case $s = 1$. For different $s$ the proof follows the lines of the proof of Theorem 4 in [2], although our calculations are more involved.

It is easy to verify that $K_{\alpha\theta} \in L(n/(n-\alpha), \infty)$ implies $K_\alpha \equiv K_{\alpha\theta}^{(n-\alpha)/(n-\alpha\theta)} \in L(n/(n-\alpha), \infty)$. Set $F_\delta$ as in the proof of Lemma 1. Then

$$\int_{F_\delta^c} |f(y) K_{\alpha\theta}(x-y)|\, dy = \int_{F_\delta^c} |f(y) K_\alpha(x-y)\, (K_{\alpha\theta}(x-y))^\beta|\, dy \leqslant \delta^{\alpha(1-\theta)} I_\alpha(f)(x)\,,$$

where $\beta = 1-(n-\alpha)/(n-\alpha\theta)$, while

$$\int_{F_\delta} |f(y) K_{\alpha\theta}(x-t)|\, dy = \int_{\{y:\, K_{\alpha\theta}(x-y) \geqslant \delta^{\alpha\theta-n}\}} |f(y) K(x-y)|\, dy\,.$$

Lemma 1 now implies:

$$[I_{\alpha\theta}(f)]^{**}(t) \leqslant \{\delta^{\alpha\theta} f^{**}(t) + \delta^{\alpha(1-\theta)} [I_\alpha(f)]^{**}(t)\}\,.$$

Choosing $\delta^\alpha = ([I_\alpha(f)]^{**}(t))/f^{**}(t)$ and using Holder's inequality we get the result.

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