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Derivations and multilinear polynomials

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Derivations and Multilinear Polynomials.

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Let R be a ring and $f = f(x_1, x_2, \ldots, x_n)$ a multilinear homogeneous polynomial in n noncommuting variables.

We recall (see [7]) that R is an f-radical extension of a subring S if, for every $r_1, r_2, \ldots, r_n \in R$, there is an integer $m = m(r_1, r_2, \ldots, r_n) \geqslant 1$ such that $f(r_1, \ldots, r_n)^m \in S$.

When R is f-radical over its center $Z(R)$ we say that f is power central valued.

Rings with a power central valued polynomial have been studied in [10]. Results on f-radical extensions of rings have been obtained in [1] and [7] also.

Let now d be a nonzero derivation on R; in this paper we will study the case in which there exists a polynomial $f(x_1, \ldots, x_n)$ such that $d(f(r_1, \ldots, r_n)^m) = 0$ for all $r_i \in R$ with $m = m(r_1, \ldots, r_n) \geqslant 1$. This is equivalent to say that R is f-radical over $S = \{x \in R : d(x) = 0\}$.

Notice that when $f = x_1$ and R is a prime ring with no nonzero nil ideals then, by [6], the above condition forces R to be commutative. Moreover, if d is an inner derivation on R, a prime ring with no nonzero nil right ideals, then in [4] it was proved that f is power central valued and R satisfies the standard identity of degree $n + 2$, $S_{n+2}(x_1, \ldots, x_{n+2})$ provided an additional technical hypothesis also holds.

This is related to the following open question: « Let D be a division ring and f a polynomial power central valued in D, then is D finite dimensional over its center? » (see [10]).

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In [4] and [10] it is proved that if R is a prime ring with no nonzero nil right ideals and f is power central valued in R, then R satisfies a polynomial identity; the proof in [4] and [10] that R is P.I. holds under the assumption that f is not an identity for $p \times p$ matrices in char. $p > 0$. Hence, to apply this results in our paper we assume this extra hypothesis:

(A) *If* $\operatorname{char} R = p \neq 0$ *then* f *is not an identity for* $p \times p$ *matrices in characteristic* p.

The main result of this paper is the following.

THEOREM 1. *Let* R *be a prime ring,* $\operatorname{char} R \neq 2$, *with no nonzero nil right ideals and let* $f(x_1, \ldots, x_n)$ *be a multilinear homogeneous polynomial. Suppose that* d *is a nonzero derivation on* R *such that, for every* $r_1, \ldots, r_n \in R$, *there exists* $m \in \boldsymbol{N}$, $m = m(r_1, \ldots, r_n)$ *with*

$$d(f(r_1, \ldots, r_n)^m) = 0 .$$

If hypothesis (A) *holds, then* $f(x_1, \ldots, x_n)$ *is power central valued and* R *satisfies* $S_{n+2}(x_1, \ldots, x_{n+2})$.

Moreover if $f(x_1, \ldots, x_n)$ *is not a polynomial identity for* R *and* $d(Z(R)) \neq 0$ *then* $Z(R)$ *is infinite of characteristic* $p \neq 0$.

As a consequence we will prove the following result of independent interest on Lie ideals (see [3]).

THEOREM 2. *Let* R *be a prime ring with no nonzero nil right ideals,* $\operatorname{char} R \neq 2$, *and let* U *be a noncentral Lie ideal of* R.

Suppose that d *is a nonzero derivation on* R *such that for every* $u \in U$ *there is* $m = m(u) \geqslant 1$ *with* $d(u^m) = 0$. *Then* R *satisfies* $S_4(x_1, \ldots, x_4)$.

Throughout this paper we will use the following notation:

1) R will always be an associative algebra over C, where C is a commutative ring with 1.

2) $f(x_1, \ldots, x_n)$ will denote a multilinear homogeneous polynomial in n non commuting variables, and we will assume that

$$f(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n + \sum \alpha_\pi x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)}$$

where $\alpha_T \in C$ and $1 \neq \pi \in S_n$ the symmetric group on $\{1, \ldots, n\}$.

3) $f(x_1, \ldots, x_n)$ will often be abbreviated as f or $f(x_i)$.

4) $Z(R)$ will always denote the center of R.

5) $\mathscr{d}$ will be a nonzero derivation on R which is C-linear. (i.e. for $c \in C$, $r \in R$, $\mathscr{d}(cr) = c\mathscr{d}(r)$).

6) $S = \{x \in R : \mathscr{d}(x) = 0\}$.

Finally, in all that follows, unless stated otherwise, we will assume that *R is a prime ring,* $\operatorname{char} R \neq 2$, *and R is f-radical over S. Furthermore we will assume that hypothesis* (A) *holds.*

We now can begin a series of reductions necessary to prove our result.

LEMMA 1. *If R is a division ring then $f(x_1, \ldots, x_n)$ is power central valued.*

PROOF. Let $0 \neq x \in S = \{r \in R : \mathscr{d}(r) = 0\}$, then we have

$$0 = \mathscr{d}(1) = \mathscr{d}(xx^{-1}) = \mathscr{d}(x)x^{-1} + x\mathscr{d}(x^{-1}) = x\mathscr{d}(x^{-1})$$

which implies $\mathscr{d}(x^{-1}) = 0$, i.e. $x^{-1} \in S$, so that S is a proper subdivision ring of R. Then, by [7, Theorem 1], f is power central valued.

For the next lemma we need to recall the following:

DEFINITION 1. We say that $a \in T(R)$ if for all $r_1, \ldots, r_n$ in R there exists an integer $m = m(a, r_1, \ldots, r_n) \geqslant 1$ such that $af(r_1, \ldots, r_n)^m = f(r_1, \ldots, r_n)^m a$ (see [4]).

DEFINITION 2. Let x be a quasi regular element of R, i.e. there exists $x' \in R$ such that $x + x' + xx' = x + x' + x'x = 0$.

Notice that if R has a unit element 1 then $1 + x$ is invertible and $(1 + x)^{-1} = 1 + x'$.

Let $\varphi_x : R \to R$ be the map defined by

$$\varphi_x(r) = r + xr + rx' + xrx'.$$

φ_x is an automorphism of R, we write $\varphi_x(r) = (1 + x)r(1 + x)^{-1}$ and we say that $a = 1 + x$ is formally invertible.

We also write $r(1 + x)$ for $r + rx$ and $(1 + x)r$ for $r + xr$.

LEMMA 2. *If $a \in R$ is invertible, or formally invertible, then there exists $z \in T(R)$ depending on a such that $\mathscr{d}(a) = az$.*

PROOF. If $r_1, \ldots, r_n \in R$ let $m \geqslant 1$ be such that $f(r_i)^m$ and $f(ar_i a^{-1})^m$ are in S. Thus $d(af(r_i)^m a^{-1} a) = d(a) f(r_i)^m$ and also $d(af(r_i)^m a^{-1} a) = af(r_i)^m a^{-1} d(a)$.

Therefore $a^{-1} d(a) = z \in T(R)$ and so $d(a) = az$.

LEMMA 3. *If $T(R) = Z(R)$ and $J(R)$, the Jacobson radical of R, is non zero then R is commutative.*

PROOF. If $x \in J(R)$ then $1 + x$ is formally invertible. By Lemma 2 $d(x) = d(1 + x) = z + zx$ for some $z \in T(R) = Z(R)$, and so $d(x)$ commutes with x; that is $d(x)x = xd(x)$ for all $x \in J(R)$. Since R is prime, by [6, Lemma], R is commutative.

LEMMA 4. *Suppose that $T(R) = Z(R)$. If $t \in R$ is such that $t^2 = 0$ then $d(t) = 0$.*

PROOF. Since $1 + t$ is formally invertible, by Lemma 2, one has $d(t) = d(1 + t) = z + zt$ for some $z \in T(R) = Z(R)$.

But $0 = d(t^2) = td(t) + d(t)t = 2zt$. Since $\operatorname{char} R \neq 2$ $zt = 0$. Moreover since $z \in Z(R)$, either $z = 0$ or z is not a zero divisor in R. In any case $d(t) = 0$.

LEMMA 5. *Let R be without nonzero nil right ideals. If there exists a non trivial idempotent $e = e^2 \neq 0, 1$ in R then f is power central valued.*

PROOF. Let A be the subring of R generated by the elements of square zero. A is invariant under all automorphism of R. Since R is a prime ring with nontrivial idempotents than, by [9, Theorem], A contains a nonzero ideal I of R. On the other hand by [4, Theorem] either f is power central valued in R or $T(R) = Z(R)$. In this last case, by Lemma 4, $d(x) = 0$ for all $x \in A$ and so $d(I) = 0$.

Now, since $0 = d(I) \supseteq d(IR) = Id(R)$, by the primeness of R we obtain $d(R) = 0$ which is a contradiction.

In the next Lemma we examine the case when R is primitive.

Lemma 6. *If R is primitive then f is power central valued.*

PROOF. Let V be a faithful irreducible right R-module with endomorphism ring D, a division ring. By Lemma 1 and Lemma 5 we may assume that V is infinite dimensional over D and R does not contain a non trivial idempotent. By [8] this says that R does not have nonzero linear transformations of finite rank.

We will prove that these assumptions lead to a contradiction.

Now, (see [1, Lemma 7]), C acts on V and we may assume that both R and $S = \{x \in R\colon d(x) = 0\}$ act densely on V over D.

Let now $vr = 0$, for some $v \in V$ and $r \in R$, and suppose that $vd(r) \neq 0$.

Since r has infinite rank there exist $w_1, \ldots, w_n \in \operatorname{Im} r$ such that $vd(r), w_1, \ldots, w_n$ are linearly independent, and let $v_1, \ldots, v_n \in V$ such that $w_i = v_i r$, $1 \leqslant i \leqslant n$.

Now by the Jacobson density theorem there exist $a_1, \ldots, a_n \in R$ such that $w_i a_i = v_{i+1}$ $(i = 1, \ldots, n \bmod n)$, $w_i a_j = 0$ otherwise, and $vd(r)a_1 = v_2$, $vd(r)a_i = 0$ $(i = 2, \ldots, n)$.

Notice that for all $r_1, \ldots, r_n \in R$ we have

$$d\big(f(r_1, \ldots, r_n)^m\big) = \sum_{p+q=m-1} f(r_i)^p\, d\big(f(r_i)\big)\, f(r_i)^q$$

and also, since $f(x_1, \ldots, x_n)$ is multilinear, we have:

$$d\big(f(r_i)\big) = \sum_{t=1}^{n} f\big(r_1, \ldots, d(r_t), \ldots, r_n\big)\,.$$

Let $m \geqslant 1$ be such that $d\big(f(ra_i)^m\big) = 0$, hence one has:

$$\begin{aligned}0 = vd\big(f(ra_i)^m\big) &= \sum_{p+q=m-1} vf(ra_i)^p\, d\big(f(ra_i)\big)\, f(ra_i)^q = \\ &= vd\big(f(ra_i)\big)\, f(ra_i)^{m-1} = \sum_t vf\big(ra_1, \ldots, d(ra_t), \ldots, ra_n\big)\, f(ra_i)^{m-1} = \\ &= vf\big(d(r)a_1, ra_2, \ldots, ra_n\big)\, f(ra_i)^{m-1} = v_1 f(ra_i)^{m-1} = \ldots = v_1\,,\end{aligned}$$

a contradiction.

Hence if $vr = 0$ then $vd(r) = 0$.

Let $0 \neq v \in V$ and suppose that vr and $vd(r)$ are linearly dependent for all $r \in R$. Let $x, y \in R$ be such that vx and vy are linearly independent, then

$$vd(x) = \lambda_x vx\,, \qquad vd(y) = \lambda_y vy \qquad \text{and} \qquad vd(x+y) = \lambda_{x+y} v(x+y)\,,$$

where λ_x, λ_y and λ_{x+y} are in D.

Therefore $\lambda_{x+y} vx + \lambda_{x+y} vy = \lambda_x vx + \lambda_y vy$, thus $\lambda_x = \lambda_y$.

As a result there exists $\lambda \in D$ such that $vd(x) = \lambda vx$ for all $x \in R$

with $vx \neq 0$. On the other hand, as we said above, $vr = 0$ implies $vd(r) = 0$, hence $vd(x) = \lambda vx$ for all $x \in R$.

However since S acts densely on V there is $x \in S$ such that $vx \neq 0$ and we obtain $0 = vd(x) = \lambda vx$, hence $\lambda = 0$. By this argument, if vr and $vd(r)$ are linearly dependent for all $v \in$ and r in R then $Vd(R) = 0$ and so $d = 0$.

Therefore, we may assume that there exist $v \in V$ $r \in R$ such that vr and $vd(r)$ are linearly independent; moreover, as above r has infinite rank.

Let $w_1, \ldots, w_n \in \operatorname{Im} r$ be such that $vr, vd(r), w_1, \ldots, w_n$ are linearly independent, and let $v_1, v_2, \ldots, v_n \in V$ be such that $v_i r = w_i$ $(i = 1, \ldots, n)$.

By the density of S on V there exist $s_1, \ldots, s_n \in S$ such that $vrs_i = 0$ $(i \geqslant 1)$, $vd(r)s_1 = v_2$, $vd(r)s_i = 0$ $(i \geqslant 2)$, $w_i s_i = v_{i+1}$ $(i = 1, \ldots, n$ mod. $n)$, $w_i s_j = 0$ for $i \neq j$.

Then we have:

$$vf(rs_1, \ldots, rs_n) = 0\,, \qquad vf(d(r)s_1, rs_2, \ldots, rs_n) = v_1\,,$$

$$vf(rs_1, \ldots, d(r)s_t, \ldots, rs_n) = 0 \ \ (t \neq 1)\,, \qquad v_1 f(rs_1, \ldots, rs_n) = v_1\,.$$

Let now $m \geqslant 1$ be such that $d(f(rs_1, \ldots, rs_n)^m) = 0$; hence we have

$$\begin{aligned}0 = vd(f(rs_i)^m) &= \sum_{p+q=m-1} vf(rs_i)^p\, d(f(rs_i))\, f(rs_i)^q = \\ &= vd(f(rs_i))\, f(rs_i)^{m-1} = \sum_t vf(rs_1, \ldots, d(rs_t), \ldots, rs_n)\, f(rs_i)^{m-1} = \\ &= vf(d(r)s_1, rs_2, \ldots, rs_n)\, f(rs_i)^{m-1} = v_1 f(rs_i)^{m-1} = \ldots = v_1\,,\end{aligned}$$

a contradiction, and this proves the result.

Next we are going to examine the general case. First we will study a special kind of ideals invariant under the derivation.

Let I be any ideal of R. We define

$$I' = \{x \in I\colon d^n(x) \in I \ \ \forall n \geqslant 1\}\,.$$

Then I' is an ideal of R invariant under d; in fact I' is the largest subset of I invariant under d. We have the following:

LEMMA 7. *Let P be a primitive ideal such that* char $R/P \neq 2$. *If $f(x_1, \ldots, x_n)$ is not power central valued in R/P then*

1) char $R/P' \neq 2$;

2) $T(R/P') = Z(R/P')$;

3) *R/P' is a prime ring.*

PROOF. To prove 1), let $x \in R$ be such that $2x \in P'$; hence $d^i(2x) \in P$, $\forall i \geqslant 0$, and so $2d^i(x) \in P$, $\forall i \geqslant 0$. Since char $R/P \neq 2$ this implies that $d^i(x) \in P$ $\forall i \geqslant 0$, thus $x \in P'$.

This says that R/P' is 2-torsion free.

We now prove 2). Let

$$A = \{x \in R : x + P' \in T(R/P')\}.$$

A is a subring of R invariant under d. In fact, for $x \in A$ and $r_1, \ldots, r_n \in R$ there exists $m \geqslant 1$ such that $xf(r_i)^m - f(r_i)^m x$ is in P' and we may assume that $d(f(r_i)^m) = 0$.

Since P' is d-invariant we have:

$$P' \ni d\big(xf(r_i)^m - f(r_i)^m x\big) = d(x) f(r_i)^m - f(r_i)^m d(x)$$

and so $d(x)$ is also in A. Since f is not power central valued in R/P, then by [4, Theorem] $T(R/P) = Z(R/P)$, hence, as $P' \subseteq P$, we have $x + P \in T(R/P) = Z(R/P)$ for all $x \in A$. This says that, for $x \in A$ and $y \in R$, $[x, y] = xy - yx \in P$.

Next we claim that $[x, y] \in P'$.

In fact, for $m \geqslant 1$, we have by Leibniz's formula

$$d^m(xy - yx) = d^m(xy) - d^m(yx) = \sum_i \binom{m}{i} [d^i(x), d^{m-i}(y)].$$

Since $d^i(x) \in A$ one has, as above, that $[d^i(x), R] \subseteq P$, hence

$$d^m(xy - yx) \in P, \ \forall m \geqslant 1.$$

This says that $xy - yx \in P'$ for $x \in A$ $y \in R$ and so $T(R/P') = Z(R/P')$.

To prove 3) we first show that R/P' is a semiprime ring.

We remark that $R' = R/P'$ is a ring with induced derivation, defined by $d(x + P') = d(x) + P'$ and for all $r_1', \ldots, r_n' \in R'$ there exists $m = m(r_i') \geqslant 1$ such that $d(f(r_i')^m) = 0 \in R'$; moreover if $d = 0$ then $P = P'$ and we are done. Hence we may assume that d is nonzero *in* R$'$. Furthermore, as we said above, R' is 2-torsion free and $T(R') = Z(R')$.

If $t \in R$ and $t^2 \equiv 0 \bmod P'$ then, since $(1+t)+P'$ is formally invertible, by the argument given in Lemma 4 it follows that $d(t) \equiv z + zt$ and $0 \equiv d(t^2) \equiv 2zt$ for some $z \in R$ such that $z + P' \in Z(R')$. Therefore, since R' is 2-torsion free $zt \equiv 0$ and $d(t) \equiv z \bmod P'$.

Let $t \in R$ be such that $tRt \equiv 0 \pmod{P'}$ and $t^2 \equiv 0 \pmod{P'}$.

Then, for every $r \in R$, we have $d(t) + P' \in Z(R')$ and also $d(tr) + P' \in Z(R')$; which implies $(d(t)r + td(r)) + P' \in Z(R')$, and so

$$(d(t)^2 r + d(t)td(r)) + P' = d(t)^2 r + P' \in Z(R') .$$

Therefore, for $r, s \in R$, we have $d(t)^2(rs - sr) \equiv 0 \bmod P'$ and so $d(t)^2 R(rs - sr) \equiv 0 \bmod P'$ (recall that $d(t) + P' \in Z(R')$).

Let now

$$B = \{x \in R : xR(rs - sr) \equiv 0 \bmod P' \ \ \forall r, s \in R\} .$$

Notice that B is invariant under d; in fact

$$0 \equiv d(xR(rs - sr)) \equiv d(x)R(rs - sr) + xd(R)(rs - sr) +$$
$$+ xR(d(r)s - sd(r)) + xR(rd(s) - d(s)r) \equiv d(x)R(rs - sr) .$$

Moreover, since R/P is noncommutative, there exists r, s in R such that $rs - sr \notin P$. But, for all $x \in B$, we have $xR(rs - sr) \subseteq P' \subseteq P$; since R/P is primitive this implies $B \subseteq P$. Hence $B \subseteq P'$, the largest subset of P d-invariant.

In other words we have proved that $tRt \equiv 0$ and $t^2 \equiv 0 \bmod P'$ implies $d(t)^2 \equiv 0$ and $d(t)Rd(t) \equiv 0 \bmod P'$.

Hence, by induction, we have $d^i(t)^2 \equiv 0$ and $d^i(t)Rd^i(t) \equiv 0 \bmod P'$; since $P' \subseteq P$ and R/P is primitive this says that $d^i(t) \in P$, $\forall i \geqslant 0$, that is $t \in P'$ and $R' = R/P'$ is semiprime.

Finally, let $a, b \in R$ and suppose that $aRb \subseteq P'$.

Then, for any $x \in R$ we have $d(axb) \in P'$ and $ad(x)b \in P'$, so

$$(*) \qquad d(a)xb + axd(b) \in P' .$$

Now R/P' is a semiprime ring, hence $aRb \subseteq P'$ forces $bRa \subseteq P'$. Multiplying $(*)$ on the left by bR we obtain $bRd(a)xb \subseteq P'$ and consequently $d(a)xb$ is in P'. From (*) it follows that $axd(b)$ is also in P'.

We have proved that $d(a)Rb \subseteq P'$ and also $aRd(b) \subseteq P'$. At this stage an easy induction leads to $d^i(a)Rd^j(b) \subseteq P'$ $\forall i, j \geqslant 0$. Since $P' \subseteq P$ and R/P is primitive, we conclude as above that either $a \in P'$ or $b \in P'$. This completes the proof.

Now we are ready to prove the main result of this paper.

PROOF OF THEOREM 1. As quoted above, since R is a prime ring with no nonzero nil right ideals and hypothesis (A) holds then either f is power central valued or $T(R) = Z(R)$ (see [4]).

In the first case, by [4, Lemma 6], R satisfies S_{n+2}. In the last case, if $J(R) \neq 0$ then by Lemma 3 R is commutative.

Suppose now that R is semisimple, so that R is a subdirect product of primitive rings R_α of characteristic different from 2. Let P_α be a primitive ideal of R such that $R_\alpha \cong R/P_\alpha$; we now partition these primitive ideals into four sets:

$$\mathscr{Q}_1 = \{P\colon d(R) \subseteq P\}$$

$$\mathscr{Q}_2 = \{P\colon d(P) \subseteq P \text{ but } d(R) \nsubseteq P\}$$

$$\mathscr{Q}_3 = \{P\colon d(P) \nsubseteq P \text{ and } f \text{ is power central valued in } R/P\}$$

$$\mathscr{Q}_4 = \{P\colon d(P) \nsubseteq P \text{ and } f \text{ is not power central valued in } R/P\}$$

in addition, let $I_i = \cap P$ for $P \in \mathscr{Q}_i$ $i = 1, \dots, 4$.

Since R is semisimple $I_1 I_2 I_3 I_4 \subseteq I_1 \cap I_2 \cap I_3 \cap I_4 = 0$.

Since R is prime we must have that at least one among I_1, I_2, I_3 or I_4 is zero. However $I_1 \neq 0$, otherwise $d(R) \subseteq I_1 = 0$, a contradiction. If $I_2 = 0$ then R is a subdirect product of primitive rings on which d induces a nonzero derivation d' satisfying all the hypotheses of Lemma 6. Then f is power central valued on R/P, for each $P \in \mathscr{Q}_2$, and so by [4, Lemma 6] R/P satisfies $S_{n+2}(x_1, \dots, x_{n+2})$.

Therefore if $I_2 = 0$ then R satisfies $S_{n+2}(x_1, \dots, x_{n+2})$.

We also remark that if $P \in \mathscr{Q}_3$ then, as above, R/P satisfies S_{n+2}. Hence, if $I_3 = 0$ then R satisfies also this identity.

Finally we claim that $\mathscr{Q}_4 = \emptyset$.

Let $P \in \mathscr{Q}_4$, and let $P' = \{x \in P\colon d^i(x) \in P,\ \forall i \geqslant 1\}$. By Lemma 7 R/P' is a prime ring, char. $R/P' \neq 2$ and $T(R/P') = Z(R/P')$. Moreover d induces on $R' = R/P'$ a non zero derivation d which also satisfies $d(f(r'_1, \dots, r'_n)^m) = 0$ for all r'_i in R' for some $m = m(r'_i, \dots, r'_n) \geqslant 1$.

We remark again that $f(x_1, \dots, x_n)$ is nil valued on the nonzero ideal P/P' of $R' = R/P'$. If R' is with no nonzero nil right ideals then $f(x_1, \dots, x_n)$ is a polynomial identity for P/P' and so for R' (see [5]).

Of course, this implies that $f(x_1, \dots, x_n)$ is a polynomial identity for R/P, a contradiction since $P \in \mathcal{Q}_4$.

Therefore R' has a nonzero nil right ideal and so $J(R') \neq 0$. But, in this case, by Lemma 3 R' is commutative, and this is also a contradiction.

As a result R satisfies the standard identity S_{n+2} and $R'' = R_Z = \{rz^{-1} : r \in R,\ 0 \neq z \in Z(R)\}$ is a central simple algebra finite dimensional over F, the quotient field of $Z(R)$.

At it is well known, d extends uniquely to a derivation on R'' (which we shall also denote by d) as follows:

$$d(rz^{-1}) = d(r)z^{-1} - rd(z)z^{-2} \qquad \forall r \in R,\ 0 \neq z \in Z(R)\,.$$

If R does not satisfies f then there exist $r_1, \dots, r_n \in R$ such that $f(r_1, \dots, r_n)$ is not nilpotent [5]. If $0 \neq z \in Z(R)$ there is an $m \geqslant 1$ such that $d\big(f(zr_1, r_2, \dots, r_n)^m\big) = 0$ and $d\big(f(r_1, \dots, r_n)^m\big) = 0$. Hence, we have $0 = d\big(f(zr_1, r_2, \dots, r_n)^m\big) = d\big(z^m f(r_1, \dots, r_n)^m\big) = d(z^m) f(r_1, \dots, r_n)^m$ and so $d(z^m) = 0$.

As a result, if $s_i = r_i z_i^{-1} \in R''$ there is an $m = m(s_i) \geqslant 1$ such that

$$d\big(f(r_i)^m\big) = 0 \qquad \text{and} \qquad d(z^m) = 0\,,$$

where $z = z_1 \dots z_n$, hence $d\big(f(s_1, \dots, s_n)^m\big) = 0$.

Therefore by Lemma 6 $f(x_1, \dots, x_n)$ is power central valued in R'' and we are done. Moreover, if $d(Z(R)) \neq 0$ and f is not a polynomial identity for R we obtain, as above, $d(z^m) = 0$ for all $z \in Z(R)$. Of course this implies that $Z(R)$ is infinite of characteristic $p \neq 0$. This completes the proof.

Of some independent interest is the special case when $f(x, y) = xy - yx$. In particolar, we do not need any extra assumptions regarding the behavior of f on $p \times p$ matrices. We state this result as:

COROLLARY. *Let R be a prime ring with no nonzero nil right ideals,* char $R \neq 2$. *Let d be a nonzero derivation on R such that for every $x, y \in R$ there exists $m = m(x, y) \geqslant 1$ with $d\big((xy - yx)^m\big) = 0$. Then R satisfies $S_4(x_1, \dots, x_4)$.*

We conclude this paper with an easy application of this result to Lie ideals. This extend to arbitrary derivations a result of [3].

PROOF OF THEOREM 2. Since char $R \neq 2$ and U is a non central Lie ideal of R, by [2, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq U$.

Let $I' = \{x \in I: d^i(x) \in I, \ \forall i \geq 1\}$, I' is an ideal of R invariant under d. Moreover, by hypothesis, for every $x, y \in I$ some power of $(xy - yx)$ lies in I'. Since R has no nonzero nil right ideals and R is not commutative we must have $I' \neq 0$. Then I' is a prime ring with a nonzero derivation d satisfying all the hypothesis of the Corollary, and so I' satisfies $S_4(x_1, \dots, x_4)$. Since R is prime, R also satisfies $S_4(x_1, \dots, x_4)$.

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