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SAMUEL ZAIDMAN

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**Existence of Almost-Periodic Ultra-Weak Solutions
to the Equation $w'(t) = a(t)Au(t) + f(t)$
in Hilbert Spaces.**

SAMUEL ZAIDMAN (*)

ABSTRACT - In this work we consider the non-homogeneous first order differential equation: $du/dt - a(t)Au(t) = f(t)$ in a separable Hilbert space H , under a few assumptions about the complex-valued almost-periodic function $a(t)$ and the linear operator A in H . We establish a sufficient condition, ensuring, for almost-periodic $f(t)$, $\mathbf{R} \rightarrow H$, the existence (and uniqueness) of an almost-periodic ultra-weak solution of the above equation.

Introduction.

This paper is intended as a continuation to [3], [4]. As in [3], let H be a Hilbert space over \mathbf{C} and then A be a linear hermitian operator with dense domain, $D(A) \subset H \rightarrow H$. We assume also the existence of a complete sequence $(e_k)_1^\infty$ in H , the e_k 's being orthonormal eigen-vectors of A corresponding to (real) nonzero eigen-values λ_k .

Next, let $a(t)$ be a complex-valued continuous almost-periodic

(*) Indirizzo dell'A.: Dept. de Mathématiques et de Statistique, Université de Montréal, C.P. 6128, Succ. A, Montréal, QUE H3C 3J7, Canada.

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(Bohr) function such that

$$(0.1) \quad a^* = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T \operatorname{Re} a(\sigma) d\sigma \right)$$

belongs to $\mathbf{R}/\{0\}$.

Finally, let $f(t)$ be a H -valued (Bochner) almost-periodic function. We shall indicate a sufficient condition involving the above considered entities in order that an almost-periodic solution of the equation in the title exists, at least in the ultra-weak sense. (The unicity would then follow from [3]).

The « simplest » case when $H = \mathbf{C}$ and A is the identity operator, that is the scalar ordinary differential equation $u'(t) = a(t)u(t) + f(t)$ is explained as Theorem 6.6 in [1]. We had to extend the arguments used there so as to apply them in the general case of operator differential equations.

1. Let us define the complex-valued almost-periodic functions $f_j(t)$ given by the scalar product $(f(t), e_j)_H$, $\forall j = 1, 2, \dots$.

Then let us consider (as in [1]) the following functions:

$$(1.1) \quad u_j(t) = \int_{-\infty}^t \exp \left(\int_s^t \lambda_j a(u) du \right) f_j(s) ds, \quad \text{for } \lambda_j a^* < 0,$$

$$(1.2) \quad u_j(t) = - \int_t^{\infty} \exp \left(\int_s^t \lambda_j a(u) du \right) f_j(s) ds, \quad \text{for } \lambda_j a^* > 0.$$

These functions are solutions on the real line of the ordinary differential equations

$$(1.3) \quad u_j'(t) = \lambda_j a(t) u_j(t) + f_j(t)$$

and they are almost-periodic (this follows from Favard's theorem, see [1]). For our purposes some very careful estimates of (1.1) and (1.2) are quite essential; we shall give them in the following lines.

i) Assume $\lambda_j > 0$ and $a^* < 0$; then

$$(1.4) \quad \left| \exp \left(\int_s^t \lambda_j a(u) du \right) \right| = \exp \left(\int_s^t \lambda_j (\operatorname{Re} a)(u) du \right) = \exp \left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) du \right)$$

where we denote: $\sigma = t - s$.

On the other hand we know that:

$$\lim_{\sigma \rightarrow \infty} \frac{1}{\sigma} \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du = a^*,$$

uniformly with respect to $t \in \mathbf{R}$. Hence, given $\varepsilon = \frac{1}{2}|a^*|$, there is a positive number T_0 such that

$$(1.5) \quad \frac{1}{\sigma} \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du < \frac{1}{2} a^*, \quad \text{for } \sigma > T_0 \text{ and } \forall t \in \mathbf{R}.$$

(Note that T_0 is independent of $j \in \mathbf{N}$).

Consequently, we see that

$$(1.6) \quad \exp\left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du\right) < \exp\left(\frac{1}{2} \lambda_j a^* \sigma\right), \quad \sigma > T_0, t \in \mathbf{R}$$

Consider now the integral:

$$\int_0^\infty \exp\left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du\right) f_j(t-\sigma) \, d\sigma = I_j,$$

and write: $I_j = I_{j,1} + I_{j,2}$ where

$$(1.7) \quad I_{j,1} = \int_0^{T_0} \exp\left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du\right) f_j(t-\sigma) \, d\sigma,$$

$$(1.8) \quad I_{j,2} = \int_{T_0}^\infty \exp\left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du\right) f_j(t-\sigma) \, d\sigma.$$

We see the estimate (for $\lambda_j > 0$, $\sigma > 0$, $0 < \sigma < T_0$, $t \in \mathbf{R}$).

$$(1.9) \quad \lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) \, du \leq \lambda_j \|\operatorname{Re} a\|_\infty \cdot \sigma \leq \lambda_j \|a\|_\infty T_0$$

hence

$$(1.10) \quad I_{j,1} \leq T_0 \exp(\lambda_j \|a\|_\infty T_0) \|f_j\|_\infty$$

where $\|\varphi\|_\infty = \sup_{\xi \in \mathbf{R}} |\varphi(\xi)|$ in (1.9) and (1.10) above.

Next, we have, using (1.6)

$$(1.11) \quad I_{j,2} \leq \left(\int_{T_0}^{\infty} \exp\left(\frac{1}{2} \lambda_j a^* \sigma\right) d\sigma \right) \|f_j\|_\infty = \frac{2}{|a^* \lambda_j|} \exp\left(\frac{1}{2} \lambda_j a^* T_0\right) \|f_j\|_\infty.$$

Thus, altogether, we obtain

$$(1.12) \quad \left| \int_0^{\infty} \exp\left(\lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) du\right) f_j(t-\sigma) d\sigma \right| = |I_{j,1} + I_{j,2}| \leq \\ \leq \|f_j\|_\infty \left(T_0 \exp(\lambda_j \|a\|_\infty T_0) + \frac{2}{|a^* \lambda_j|} \exp\left(\frac{1}{2} \lambda_j a^* T_0\right) \right)$$

which is the desired estimate for $u_j(t)$ in (1.1) in the case $\lambda_j > 0$, $a^* < 0$.

ii) Consider now the case where $a^* < 0$ and $\lambda_j < 0$, and estimate the expression (1.2) which becomes (with the substitution $s = \sigma + t$)

$$(1.13) \quad - \int_0^{\infty} \exp\left(- \int_t^{t+\sigma} \lambda_j a(u) du\right) f_j(\sigma + t) d\sigma.$$

Again we see that $\left| \exp\left(- \int_t^{t+\sigma} \lambda_j a(u) du\right) \right| = \exp\left(|\lambda_j| \int_t^{t+\sigma} (\operatorname{Re} a)(u) du\right)$ and

$$(1.14) \quad \int_t^{t+\sigma} (\operatorname{Re} a)(u) du < \frac{1}{2} a^* \sigma, \quad \text{for } \sigma \geq T_0, t \in \mathbf{R}.$$

It follows that the function $u_j(t)$ given by (1.2) is estimated by

$$(1.15) \quad \|f_j\|_\infty \left(T_0 \exp(|\lambda_j| \|a\|_\infty T_0) + \frac{2}{|a^* \lambda_j|} \exp\left(\frac{1}{2} |\lambda_j| a^* T_0\right) \right).$$

iii) Take now $\alpha^* > 0$ and $\lambda_j > 0$; again we estimate the integral (1.13). We see that

$$(1.16) \quad \frac{\alpha^*}{2} \leq \frac{1}{\sigma} \int_t^{t+\sigma} (\operatorname{Re} a)(u) du \quad \text{and} \quad -\lambda_j \int_t^{t+\sigma} (\operatorname{Re} a)(u) du \leq -\frac{1}{2} \sigma \alpha^* \lambda_j,$$

for $\sigma \geq T_0$, $t \in \mathbf{R}$.

Decomposing (1.13) as in i) above, we get the integrals from 0 to T_0 and from T_0 to $+\infty$. The first is estimated by

$$\|f_j\|_\infty \cdot T_0 \exp(\lambda_j \|a\|_\infty T_0)$$

while the second by

$$\|f_j\|_\infty \frac{2}{\alpha^* \lambda_j} \exp\left(-\frac{1}{2} \alpha^* \lambda_j T_0\right),$$

so that we get an estimate by

$$(1.17) \quad \|f_j\|_\infty \left(T_0 \exp(\lambda_j \|a\|_\infty T_0) + \frac{2}{\alpha^* \lambda_j} \exp\left(-\frac{1}{2} \alpha^* \lambda_j T_0\right) \right).$$

iv) In the last case where $\alpha^* > 0$ and $\lambda_j < 0$ we must estimate the integral (1.1) which is also

$$\int_0^\infty \exp\left(\int_{t-\sigma}^t \lambda_j a(u) du\right) f_j(t-\sigma) d\sigma.$$

As above we see that $\frac{1}{2} \sigma \alpha^* \lambda_j \geq \lambda_j \int_{t-\sigma}^t (\operatorname{Re} a)(u) du$ for $\sigma \geq T_0$, $t \in \mathbf{R}$ and we get for (1.1) an upper bound equal to

$$(1.18) \quad \|f_j\|_\infty \left(T_0 \exp(\lambda_j \|a\|_\infty T_0) + \frac{2}{\alpha^* |\lambda_j|} \exp\left(\frac{1}{2} \alpha^* \lambda_j T_0\right) \right).$$

Let us define, for any $j = 1, 2, 3, \dots$, the numbers

$$(1.19) \quad \omega_j = \left(T_0 \exp(|\lambda_j| \|a\|_\infty T_0) + \frac{2}{|\alpha^* \lambda_j|} \exp\left(-\frac{1}{2} T_0 |\alpha^* \lambda_j|\right) \right).$$

We can see then that

$$(1.20) \quad |u_j(t)| < \omega_j \|f_j\|_\infty, \quad \forall t \in \mathbf{R}, j = 1, 2, \dots$$

2. Let us make now the

MAIN HYPOTHESIS. *The numerical series $\sum_{j=1}^{\infty} \omega_j^2 \|f_j\|_\infty^2$ is convergent.*
We state the following:

THEOREM. *Assuming the M-H to be true, the vector-series in H : $\sum_{j=1}^{\infty} u_j(t) e_j$ is uniformly convergent on the real line and is the unique almost-periodic ultra-weak solution of the abstract differential equation*

$$(2.1) \quad u'(t) = a(t) Au(t) + f(t)$$

over the whole real line.

Note that « ultra-weak » solution of the above equation means here, as in the paper [3], that the integral identity

$$(2.2) \quad \int_{\mathbf{R}} (u(t), \varphi'(t) + \bar{a}(t) A^* \varphi(t))_H dt = 0 \quad \forall \varphi \in K_{A^*}(\mathbf{R})$$

holds. Here

$$K_{A^*}(\mathbf{R}) = \{\varphi(t), \mathbf{R} \rightarrow D(A^*), \varphi \in C_0^1(\mathbf{R}; H), A^* \varphi \in C(\mathbf{R}; H)\}$$

where $A^* \supset A$ is the adjoint operator to A , and $\bar{a}(t)$ is the complex conjugate of $a(t)$, $\forall t \in \mathbf{R}$.

The uniqueness of all almost-periodic solutions follows from [3]. Hence, it remains to prove existence, which is done essentially as in [4].

The uniform (over \mathbf{R}) convergence of the series $\sum_1^{\infty} u_j(t) e_j$ in H -norm follows from the relation

$$(2.3) \quad \left\| \sum_N^{N+p} u_j(t) e_j \right\|_H^2 = \sum_N^{N+p} |u_j(t)|^2$$

from the Main Hypothesis and the estimates in section 1. Hence its sum $u(t)$ is H -almost-periodic.

Now, let $v_j(t) = u_j(t) e_j$; we have

$$v'_j(t) = u'_j(t) e_j = \lambda_j a(t) u_j(t) e_j + f_j(t) e_j = \lambda_j a(t) v_j(t) + f_j(t) e_j,$$

hence

$$(2.4) \quad v'_j(t) = a(t) A v_j(t) + f_j(t) e_j$$

holds, $\forall j = 1, 2, \dots$, in H -sense. (Due to $A v_j(t) = A(u_j(t) e_j) = u_j(t) A e_j = u_j(t) \lambda_j e_j = \lambda_j v_j(t)$).

Next, put $w_N = v_1(t) + \dots + v_N(t)$. We see that the equality

$$(2.5) \quad w'_N(t) = a(t) A w_N(t) + g_N(t)$$

in H -sense, over \mathbb{R} holds, where

$$g_N(t) = \sum_1^N f_j(t) e_j.$$

Taking any $\varphi \in K_{A^*}(\mathbb{R})$ we obtain

$$(2.6) \quad \int_{\mathbb{R}} (w'_N(t) - a(t) A w_N(t), \varphi(t))_H dt = \int_{\mathbb{R}} (g_N(t), \varphi(t))_H dt.$$

By partial integration we derive

$$(2.7) \quad \int_{\mathbb{R}} (w'_N(t), \varphi(t)) dt = - \int_{\mathbb{R}} (w_N(t), \varphi'(t)) dt.$$

Also

$$(2.8) \quad \int_{\mathbb{R}} (a(t) A w_N(t), \varphi(t))_H dt = \int_{\mathbb{R}} (A w_N(t), \bar{a}(t) \varphi(t)) dt = \int_{\mathbb{R}} (w_N(t) \bar{a} A^* \varphi)(t) dt$$

Accordingly we see that for any natural N the identity

$$(2.9) \quad \int_{\mathbb{R}} (w_N(t), \varphi'(t) + \bar{a}(t) A^* \varphi(t))_H dt = - \int_{\mathbb{R}} (g_N(t), \varphi(t))_H dt$$

holds $\forall \varphi \in K_{A^*}(\mathbb{R})$.

Now, let us remember that $g_N(t) \rightarrow f(t) (N \rightarrow \infty)$, *uniformly* on \mathbf{R} , due to the relative compactness of the trajectory of $f(t)$. Also, as previously seen,

$$w_N(t) \rightarrow u(t) = \sum_1^{\infty} u_j(t) e_j,$$

uniformly on \mathbf{R} .

It follows that u is ultra-weak almost-periodic solution of $u'(t) = a(t)Au(t) + f(t)$. This proves the Theorem.

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