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## ***C*-Uniform Distribution of Entire Functions.**

MICHAEL DRMOTA - ROBERT F. TICHY (\*)

**SUMMARY** - It is proved that under certain conditions on the growth of the entire function  $f(z)$  the curve  $f(t)$  (for real  $t$ ) is uniformly distributed modulo 1 in the complex plane. The same is valid for two-dimensional flows  $f(s + it)$ . Furthermore two uniformly distributed functions, the exponential function  $f(t) = \exp[at]$  and the Weierstraß  $\sigma$ -function  $f(t) = \sigma(t)$  (not satisfying the growth condition) are investigated.

### **1. Introduction.**

A continuous function  $f: [0, \infty) \rightarrow \mathbf{R}^d$  is said to be uniformly distributed modulo 1 (for short: u.d.) if

$$(1.1) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi_I(\{f(t)\}) dt = \lambda(I)$$

holds for all boxes  $I = [a_1, b_1] \times \dots \times [a_d, b_d] \subseteq [0, 1]^d$ ;  $\chi_I$  is the characteristic function of  $I$  and  $\lambda(I)$  its Lebesgue measure,  $\{f(t)\} = f(t) - [f(t)]$  denotes the componentwise fractional part of  $f(t)$ . If  $\{f(t)\}$  is interpreted as a particle's motion on the  $d$ -dimensional torus  $\mathbf{R}^d/\mathbf{Z}^d$ , definition (1.1) means that the ratio of the particle's stay in any box to the whole time converges to the volume of the box. A quantitative

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measure for the convergence in (1.1) is the discrepancy

$$(1.2) \quad D_T(f) = \sup_I \left| \frac{1}{T} \int_0^T \chi_I(\{f(t)\}) dt - \lambda(I) \right|.$$

It is well known that  $f(t)$  is u.d. if and only if  $D_T(f)$  tends to 0 (for  $T \rightarrow \infty$ ); cf. [4]. By a famous criterion due to H. Weyl [10] (1.1) is equivalent to

$$(1.3) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \exp[2\pi i \langle h, f(t) \rangle] dt = 0$$

for all integral lattice points  $h \neq 0$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $\mathbb{R}^d$ .

As general references for the theory of uniformly distributed functions we propose the monographs by E. Hlawka [5] and by L. Kuipers and H. Niederreiter [6].

In this article we study the distribution behaviour of an (in general complex valued) entire function  $f(t)$  considered as a mapping  $[0, \infty) \rightarrow \mathbb{R}^2$ . If all coefficients  $f_n$  of the Taylor expansion of  $f(z)$  are real we consider  $f$  as a function  $[0, \infty) \rightarrow \mathbb{R}$ .

In the special case  $f_n \geq 0$  (for  $n \geq n_0$ ) Satz 8 of E. Hlawka [4] immediately yields

$$(1.4) \quad D_T(f) = \mathcal{O}\left(\frac{1}{T}\right),$$

since  $f(t)$  is in this case an increasing and convex function (for  $t \geq t_0$ ). In section 2 we are interested in entire functions  $f$  of very small growth; more precisely we assume

$$(1.5) \quad \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log \log r} < \frac{3}{2},$$

where  $M(r) = \max_{|z| \leq r} |f(z)|$ . We will prove: If  $f$  is of type (1.5) and has real Taylor coefficients, then  $f(t)$  is u.d. in  $\mathbb{R}$ . An analogon for complex  $f_n$  is an immediate consequence of this result. We remark that similar theorems (with  $4/3$  instead of  $3/2$ ) for the uniform distribution of sequences are due to G. Rauzy [7], G. Rhin [8] and

R. C. Baker [1], [2]. The examples of G. Rauzy [7] and R. C. Baker [1] show also in the case of uniformly distributed functions that the constant  $3/2$  cannot be replaced by a constant  $c > 2$ .

In section 3 we discuss the exponential function  $\exp[at]$  ( $a \in \mathbb{C}$ ) and the Weierstrass  $\sigma$ -function and give estimates for the discrepancy. In the final section 4 we extend the previous results to the case of two-dimensional flows  $f(s + it)$ .

## 2. Entire functions of very small growth.

**THEOREM 1.** *Let  $f(z)$  be a (non constant) entire function satisfying (1.5) such that all Taylor coefficients of  $f$  are real. Then  $f(t)$  (considered as a function  $[0, \infty) \rightarrow \mathbb{R}$ ) is u.d.*

**COROLLARY 1.** *Let  $f(z)$  be an entire function satisfying (1.5) such that either the quotient  $\operatorname{Re}(f_n)/\operatorname{Im}(f_n)$  is irrational for some Taylor coefficient  $f_n$  with  $n \geq 1$  or there are two Taylor coefficients  $f_n, f_m$  ( $1 \leq m < n$ ) with  $\operatorname{Re}(f_m)/\operatorname{Im}(f_m) \neq \operatorname{Re}(f_n)/\operatorname{Im}(f_n)$ . Then  $f(t)$  (considered as a function  $[0, \infty) \rightarrow \mathbb{R}^2$ ) is u.d.*

**PROOF OF COROLLARY 1.** We apply Theorem 1 to the function

$$g(t) = \sum_{n=0}^{\infty} (h_1 \operatorname{Re}(f_n) + h_2 \operatorname{Im}(f_n)) \cdot t^n, \quad h = (h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

Then Weyl's criterion (1.3) immediately yields the result of the corollary.

**REMARK 1.** In the case of uniformly distributed sequences R. C. Baker [2] has shown that, in general, there is no estimate for the discrepancy of entire functions satisfying (1.5). Since we use similar techniques for the proof of Theorem 1 as have been used for sequences it is not possible to obtain a general estimate for the discrepancy by this method. Nevertheless estimates can be proved for special functions, compare (1.4).

**PROOF OF THEOREM 1.** We will apply Weyl's criterion and set

$$S(T) = \int_0^T \exp[2\pi i h f(t)] dt, \quad h \in \mathbb{Z} \setminus \{0\}.$$

As in [7] we make use of increasing sequences  $(n_k)$ ,  $(P_k)$ ,  $(Q_k)$  tending to infinity and satisfying  $P_{k+1} < Q_k$  for sufficiently large  $k$  (for details see Lemma 1). In Lemma 3 we will prove that

$$(2.1) \quad |S(T + P_k) - S(T)| < \varepsilon P_k$$

holds for every  $\varepsilon > 0$ , for all sufficiently large  $k \geq k_0(\varepsilon)$  and all  $T < \leq Q_k - P_k$ . Choosing  $\varepsilon$ ,  $k$  such that (2.1) holds and  $C = C(\varepsilon) = Q_{k_0(\varepsilon)}$  we obtain by induction

$$(2.2) \quad |S(T)| < \varepsilon T + C \quad \text{for all } T \in [0, Q_{k-1}].$$

Trivially (2.2) is valid for  $k = k_0$ . Assume that (2.2) holds for some  $k \geq k_0$ . If  $T \in [0, Q_k]$  then  $T = [T/P_k]P_k + R$  with  $R < P_k < Q_{k-1}$  and (2.1) combined with the assumption yields

$$|S(T)| \leq |S(T) - S(R)| + |S(R)| < \varepsilon \left[ \frac{T}{P_k} \right] P_k + \varepsilon R + C = \varepsilon T + C.$$

Thus (2.2) is proved for all  $\varepsilon > 0$  and all  $k \geq k_0(\varepsilon)$ .

Since  $\lim_{k \rightarrow \infty} Q_k = \infty$ , we derive from (2.2) for every  $\varepsilon > 0$

$$\limsup_{T \rightarrow \infty} \left| \frac{S(T)}{T} \right| < \varepsilon;$$

hence  $f(t)$  is u.d.

In the following we give a detailed definition of the above sequence  $(n_k)$ ,  $(P_k)$ ,  $(Q_k)$  and establish some essential properties. From condition (1.5) we obtain by Cauchy's inequality

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{\log(1/|f_n|)}{n^c} = +\infty$$

for some  $c > 3$  ( $f_n$  denote the Taylor coefficients of  $f$ ). If  $f$  is a non constant polynomial estimate (1.4) can be applied; hence  $f$  is u.d. in this case. In the following we assume that  $f_n \neq 0$  for infinitely many  $n$ . Set  $n_0 = \min \{n: f_n \neq 0\}$ . If  $n_k$  is defined set  $l_k = \log(1/|f_{n_k}|)$  and

$$m_k = \sup \{m: \log(1/|f_n|) \geq l_k + m(n - n_k) \text{ for all } n \geq n_k\}$$

and define

$$(2.4) \quad n_{k+1} = \max \{n > n_k : \log (1/|f_n|) = l_k + m_k(n - n_k)\}.$$

(By (2.3) the sequences  $(m_k)$  and  $(n_k)$  are well-defined.) Furthermore we set

$$(2.5) \quad \begin{cases} p_k = \frac{n_k l_k}{n_k^2 - 1}, & P_k = \exp [p_k], \\ M_k = \exp [m_k], & Q_k = \frac{4^{n_k+1}}{M_k}. \end{cases}$$

LEMMA 1.

- (i)  $\log (1/|f_n|) \geq l_k + m_k(n - n_k)$  for all  $n \geq n_k$
- (ii)  $l_{k+1} = l_k + m_k(n_{k+1} - n_k)$
- (iii)  $n_k < n_{k+1}, \quad m_k < m_{k+1}$
- (iv)  $\lim_{k \rightarrow \infty} \frac{m_k}{n_k^{c-1}} = \infty, \quad \lim_{k \rightarrow \infty} \frac{l_k - p_k}{n_k^c} = \infty$
- (v)  $\lim_{k \rightarrow \infty} \frac{m_k - p_{k+1}}{n_{k+1}^{c-2}} = \infty$
- (vi)  $\lim_{k \rightarrow \infty} \frac{m_k + l_k - (n_k + 1)p_k}{n_k^{c-2}} = \infty$
- (vii)  $\lim_{k \rightarrow \infty} \frac{p_k - (l_k/n_k)}{n_k^{c-3}} = \infty$
- (viii)  $P_{k+1} < Q_k$  for every  $k \geq k_0$
- (ix)  $\lim_{k \rightarrow \infty} 2^{n_k+2}(|f_{n_k}|P_k^{n_k+1})/M_k = 0$
- (x)  $\lim_{k \rightarrow \infty} (|f_{n_k}|/2)^{-1/n_k} P_k^{-1} = 0.$

The proof of (i) to (vii) can be given by verbally the same arguments as in [7]. Properties (viii), (ix), (x) are immediate consequences of the former ones.

In order to show (2.1) we make use of the following lemma.

LEMMA 2. Let  $p(t) = at^N + a_1t^{N-1} + \dots + a_{N-1}t + a_N$  be a polynomial of degree  $N$  with real coefficients. Then for all  $A < B$

$$\left| \int_A^B \exp [2\pi i p(t)] dt \right| \leq \frac{26}{|a|^{1/N}}.$$

PROOF. Using the substitution  $u = t(N|a|)^{1/N}$  for  $N \geq 2$  we have to prove

$$(2.6) \quad \left| \int_{\alpha}^{\beta} \exp [2\pi i q(u)] du \right| \leq 26$$

for any  $\alpha, \beta$  and a polynomial  $q(u) = (1/N)u^N + b_1u^{N-1} + \dots + b_N$ . Applying Theorem 3.4.1 of R. P. Boas [3] we have  $|q'(u)| \leq K$  for every  $K > 0$  and all  $u \in S$ , where the set  $S$  is the union of at most  $(N-1)$  intervals and its measure is  $\leq 12K^{1/(N-1)}$ .

Therefore there are at most  $N$  intervals (contained in  $[\alpha, \beta]$ ) where  $q$  is monotone and  $|q'| > K$ . Hence the second mean value theorem yields on such an interval  $I$

$$\left| \int_I \exp [2\pi i q(u)] du \right| \leq \frac{2}{K}.$$

Combining this with the trivial bound

$$\left| \int_S \exp [2\pi i q(u)] du \right| \leq 12K^{1/(N-1)}$$

and choosing  $K = N$  yields (2.6). Thus the proof of Lemma 2 is finished, since the case  $N = 1$  is trivial.

To complete the proof of Theorem 1 it remains to show estimate (2.1). This is worked out in the following Lemma.

LEMMA 3. For every  $\varepsilon > 0$  and sufficiently large  $k \geq k_0(\varepsilon)$  we have

$$|S(T + P_k) - S(T)| < \varepsilon P_k \quad \text{for } T \leq Q_k - P_k.$$

PROOF. Set

$$g(t) = \sum_{j=0}^{n_k} \frac{(t-T)^j}{j!} f^{(j)}(T),$$

$$\alpha_k = \frac{f^{(n_k)}(T)}{n_k!}, \quad S' = \int_T^{T+P_k} \exp [2\pi i h g(t)] dt \quad (h \in \mathbb{Z} \setminus \{0\}).$$

By Lemma 1 (i) we obtain for  $n \geq n_k$

$$|f_n| \leq |f_{n_k}| M_k^{n_k-n};$$

therefore

$$(2.7) \quad \left| \frac{f^{(n_k+1)}(t)}{(n_k+1)!} \right| \leq \left| \sum_{n=n_k+1}^n \binom{n}{n_k+1} f_n t^{n-(n_k+1)} \right| \leq \frac{|f_{n_k}|}{M_k} \left( \frac{1}{1-Q_k/M_k} \right)^{n_k+2}$$

for  $t \in [0, Q_k]$ . Hence by Taylor's formula and  $Q_k \leq M_k/2$

$$(2.8) \quad |f(t) - g(t)| \leq 2^{n_k+2} \frac{f_{n_k}}{M_k} P_k^{n_k+1}$$

for  $t \in [T, T + P_k]$ . We have

$$|\alpha_k - f_{n_k}| = \frac{1}{n_k!} |f^{(n_k)}(T) - f^{(n_k)}(0)| \leq (n_k+1) 2^{n_k+1} \frac{|f_{n_k}| Q_k}{M_k} \leq \frac{|f_{n_k}|}{2}.$$

Therefore we get by Lemma 1 and (2.8)

$$|S(T + P_k) - S(T)| \leq |S(T + P_k) - S(T) - S'| + |S'| \leq$$

$$\leq \left( 2^{n_k+2} \frac{|f_{n_k}|}{M_k} P_k^{n_k+1} + 26 \left( \frac{|f_{n_k}|}{2} \right)^{-1/n_k} P_k^{-1} \right) P_k.$$

Thus, applying Lemma 1 (ix) and (x), the proof of Lemma 3 is complete.

### 3. Some special entire functions.

As a first example (not satisfying (1.5)) we want to consider the entire function

$$(3.1) \quad f(t) = \exp [at] \quad (a \in \mathbb{C}, \operatorname{Re}(a) > 0).$$

If  $\text{Im}(a) = 0$  we can apply Satz 8 of [4] and obtain estimate (1.4) for the discrepancy  $D_T(f)$ . In the case  $\text{Im}(a) \neq 0$  we will apply the inequality of Erdős-Turan (for complex functions)  $f(t) = f_1(t) + if_2(t)$

$$(3.2) \quad D_T(f) \leq c \left( \frac{1}{H} + \sum_{0 < \|h\| \leq H} \left( \max(|h_1|, 1) \max(|h_2|, 1) \right)^{-1} \cdot \left| \frac{1}{T} \int_0^T \exp[2\pi i(h_1 f_1(t) + h_2 f_2(t))] dt \right| \right) \quad (H \text{ an arbitrary positive integer})$$

with an absolute constant  $c > 0$ ; note that  $\|h\| = \max(|h_1|, |h_2|)$ . We set  $a = \alpha + i\beta$  and obtain for some  $\gamma$

$$(3.3) \quad h_1 \text{Re}(\exp[at]) + h_2 \text{Im}(\exp[at]) = \sqrt{h_1^2 + h_2^2} \exp[\alpha t] \sin(\beta t + \gamma) =: g(t).$$

In order to estimate the integrals in (3.2) we apply the second mean value theorem on at most  $|\beta T|/(2\pi) + 2$  intervals  $I$ , where  $g(t)$  is strictly monotone and

$$|g'(t)| \geq \varepsilon \exp[j2\pi\alpha/|\beta|] \sqrt{h_1^2 + h_2^2}$$

(for an  $\varepsilon > 0$  which is chosen later). Observe that the length of the remaining intervals in  $[0, T]$  is  $\mathcal{O}(\varepsilon T)$ . Hence we have

$$\int_0^T \exp[2\pi i(h_1 f_1(t) + h_2 f_2(t))] dt = \mathcal{O}\left(\varepsilon T + \frac{1}{\varepsilon \sqrt{h_1^2 + h_2^2}}\right).$$

Applying (3.2) and choosing  $\varepsilon = T^{-\frac{1}{2}}(h_1^2 + h_2^2)^{-\frac{1}{2}}$  and  $H = [T^{\frac{1}{2}}]$  yields

$$(3.4) \quad D_T(f) = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right).$$

In the following we consider the Weierstrass  $\sigma$ -function. Let  $\Omega[\omega_1, i\omega_2]$  be a lattice in the complex plane generated by two positive real numbers  $\omega_1, \omega_2$ . Then  $\sigma(z)$  is an entire function and can be defined by

$$(3.5) \quad \sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left[\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right],$$

which is real valued for real  $z$ . We also assume that the real constant

$$(3.6) \quad \delta_1 = \frac{\sigma'(z + \omega_1)}{\sigma(z + \omega_1)} - \frac{\sigma'(z)}{\sigma(z)}$$

(cf. [9]) is positive. In order to establish an estimate for the discrepancy of  $\sigma(t)$ ,  $t \in [0, \infty)$  we use the functional equation

$$(3.7) \quad \sigma(z + \omega_1) = -\sigma(z) \exp \left[ \delta_1 \left( z + \frac{\omega_1}{2} \right) \right].$$

We consider intervals  $J_k = [k\omega_1, (k + 1)\omega_1]$ ,  $k = 0, 1, 2, \dots$ . Since the  $\wp$ -function has at most two zeroes in  $J_k$ , the Weierstrass  $\zeta$ -function (with  $\zeta' = -\wp$ ) has at most 3 zeroes in  $J_k$  (note that all involved functions are real under the above assumptions). Since  $\zeta(t) = \sigma'(t)/\sigma(t)$ ,  $\sigma(t)$  consists of at most 4 strictly monotone pieces on  $J_k$ . Applying (3.7), the inequality of Erdős-Turan and the second mean value theorem, we obtain as in the previous example

$$(3.8) \quad D_T(\sigma) = \mathcal{O} \left( \frac{1}{\sqrt{T}} \right).$$

#### 4. Two dimensional flows.

It is also of some interest to consider the distribution behaviour of two dimensional flows  $f(z)$ ,  $z = s + it$ . Generalizing definition (1.1) we call such a (complex valued) flow u.d. (mod 1) if

$$(4.1) \quad \lim_{s, T \rightarrow \infty} \frac{1}{ST} \int_0^s \int_0^T \chi_I(\{f(s + it)\}) dt ds = \lambda(I)$$

holds for all two dimensional intervals  $I \subseteq [0, 1]^2$ ; note that  $S, T$  tend independently to infinity. By similar arguments as in section 2 the following result can be established.

**THEOREM 2.** *Let  $f(z)$  be an non constant entire function satisfying (1.5) such that  $f_n$  is real or the quotient  $\text{Re}(f_n)/\text{Im}(f_n)$  is irrational for almost all  $n \geq 1$ . Then the flow  $f(s + it)$  is u.d. mod 1.*

REMARK 2. For some applications it might be useful to consider a two dimensional flow as u.d. mod 1 if

$$(4.2) \quad \lim_{s, T \rightarrow \infty} \frac{1}{4ST} \int_{-s}^s \int_{-T}^T \chi_I(\{f(s + it)\}) dt ds = \lambda(I)$$

holds for all  $I \subseteq [0, 1]^2$ . Obviously, Theorem 2 is true for this notion of uniform distribution, too.

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