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Nonexistence of Solutions for Differential Inclusions with Upper Semicontinuous Nonconvex Right-Hand Side.

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This note is concerned with the «feedback» differential inclusion

$$\dot{x}(t) = f\big(x(t), u(t)\big), \qquad x(0) = 0 \in \mathbb{R}^n, \tag{1}$$

$$u(t) \in R\big(x(t)\big), \tag{2}$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and $R: \mathbb{R}^n \to \mathbb{R}^m$ is an upper semicontinuous multifunction with compact, convex values. Under various additional conditions on f and R, the existence of a Caratheodory solution for the Cauchy problem (1) with the constraints (2) was proved in [2]. The two counterexamples given in this note show that, in the general case, the problem (1), (2) need not have solutions. In the first example, f is a quadratic function of u, independent of x. In the second example, the upper semicontinuous map R is «best possible» and f is C^∞. Still, the viability conditions used in [2] fail, and no solution is found.

EXAMPLE 1. Consider the function

$$\varphi(x) = x \cdot \cos\frac{1}{x} \quad \text{if } x \neq 0, \qquad \varphi(0) = 0. \tag{3}$$

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Define the map $f(u) = (1 - u^2, u)$ and the upper semicontinuous multifunction

$$(4) \qquad R(x_1, x_2) = \begin{cases} \{-1\} & \text{if } x_2 > \varphi(x_1)\,, \\ [-1, 1] & \text{if } x_2 = \varphi(x_1)\,, \\ \{1\} & \text{if } x_2 < \varphi(x_1)\,. \end{cases}$$

Then the problem (1), (2) has no solutions. Indeed, any trajectory $x(\cdot)$ satisfying (1), (2) must be a nonconstant, Lipschitz continuous function of t, taking values inside the graph of φ. However, this is impossible because any continuous arc on the graph of φ, connecting the origin to any other point $(x_1, \varphi(x_1))$, has infinite length.

EXAMPLE 2. Let φ be as in (3) and let $\psi\colon \mathbf{R}^2 \to \mathbf{R}$ be a C^∞ function such that

$$\begin{cases} \psi(x_1, x_2) > 0 & \text{if } x_2 > \varphi(x_1)\,, \\ \psi(x_1, x_2) = 0 & \text{if } x_2 \leqslant \varphi(x_1)\,. \end{cases}$$

Let $\gamma\colon [-2, 2] \to \mathbf{R}$ be a C^∞ function such that

$$\begin{cases} \gamma(u) < 0 & \text{if } u \in [-2, -1)\,, \\ \gamma(u) = 0 & \text{if } u \in [-1, 1]\,, \\ \gamma(u) > 0 & \text{if } u \in (1, 2]\,. \end{cases}$$

Consider the C^∞ function $f\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3$,

$$f(x, u) = (f_1, f_2, f_3) =$$
$$= \big(1 - u^2, u, \gamma(u) + (u + 1)\psi(x_1, x_2) + (u - 1)\psi(-x_1, -x_2)\big)\,,$$

and define the upper semicontinuous map $R\colon \mathbf{R}^3 \to \mathbf{R}$,

$$R(x_1, x_2, x_3) = \begin{cases} \{-2\} & \text{if } x_3 > 0\,, \\ [-2, 2] & \text{if } x_3 = 0\,, \\ \{2\} & \text{if } x_3 < 0\,. \end{cases}$$

For these maps f and R, the Cauchy problem (1), (2) on $\mathbf{R}^3$ has no solution. Indeed, if $x_3 > 0$,

$$f(x, u) = f(x, -2) = \big(-3, -2, \gamma(-2) - \Psi(x_1, x_2) - 3\psi(-x_1, -x_2)\big),$$

hence $f_3 < 0$. If $x_3 < 0$,

$$f(x, u) = f(x, 2) = \big(-3, 2, \gamma(2) + 3\psi(x_1, x_2) + \psi(-x_1, -x_2)\big),$$

hence $f_3 > 0$. No solution can thus leave the plane $x_3 = 0$.

Any solution of (1), (2) therefore has the form $x(t) = \big(x_1(t), x_2(t), 0\big)$. This yields the two-dimensional problem

$$(\dot{x}_1, \dot{x}_2) = (1 - u^2, u), \tag{5}$$

$$u \in [-2, 2] \cap \{u; f_3(x, u) = 0\}. \tag{6}$$

The choice of the functions ψ and γ, used in the definition of f, implies that the problem (5), (6) is precisely the one considered in Example 1, hence it has no solutions.

REFERENCES

[1] J. P. Aubin - A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.
[2] J. P. Aubin - H. Frankowska, *Trajectories lourdes de systèmes contrôlés*, C. R. Acad. Paris Sér. I Math., **298** (1984), pp. 521-524.

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