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A Criterion for a Rational Projectively Normal Variety to be Almost-Factorial.

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SUNTO - Si dimostra che una varietà razionale proiettivamente normale $\mathcal{F} \subset \mathbf{P}^N$, $n = \dim \mathcal{F}$, è semifattoriale se e solo se ammette una parametrizzazione $\sigma_p : \mathbf{P}^n \to \mathcal{F}$ che gode della seguente proprietà: sono sottoinsieme intersezione completa di $\mathcal{F}$ tutte le componenti di $\mathcal{F} \cap S$, dove S è una ipersuperficie di $\mathbf{P}^N$ legata alla parametrizzazione σ_p. Vengono date applicazioni ed esempi.

0. Introduction.

In the forthcoming paper « Factorial singularities on rational quartic surfaces of $\mathbf{P}^3$ », written in collaboration with P. C. Craighero, the properties of such surfaces in connection with their parametric representation on a plane $\mathbf{P}^2$ have been deeply investigated. In such a research, the curves on the surfaces coming from particular points of the plane, that is the exceptional curves, play a leading role. This fact has suggested the author the investigation between the relation on almost-factoriality of rational surfaces and one of its parametric representation.

This paper presents the answer to the matter. It holds that, if $\mathcal{F}$ is a rational projectively normal variety, $\mathcal{F}$ is almost-factorial iff are set-theoretic complete intersection on $\mathcal{F}$ only a finite subset of sub-

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varieties of codimension 1 of $\mathcal{F}$ which are *referred* to the parametrization (see def. 5); however, among these there are also the exceptional subvarieties (see Prop. 1). An immediate affine version of this result is given and some applications to classes of surfaces of $\mathbb{A}^3$ or of $\mathbb{P}^3$ are enunciated.

The result is an extension of the well-known criterion of D. Gallarati for the almost-factoriality which holds for monoid hypersurfaces in $\mathbb{P}^N$. According to it, one can, for example, tackle the question of the classification of the almost-factorial rational surfaces of degree four with only double points.

On the other hand, as it is proved in [1] Prop. 2.11 p. 260, the almost-factoriality is a non local property which is unaffected by isomorphisms in the class of the projectively normal varieties. Since $\mathbb{P}^n$ is factorial (so almost-factorial), for every $n > 0$ one can find very large classes of almost-factorial varieties by means of isomorphisms: for example all the m-ple embeddings of $\mathbb{P}^n$ in $\mathbb{P}^N$, $N = \binom{n+m}{n} - 1$, $m > 1$. According to the new result, one can build more models of rational almost-factorial varieties which are not necessarily isomorphic to some $\mathbb{P}^n$ or to some monoid.

At the end of the paper a rational quartic surface with only double points which results to be 12-almostfactorial is examined as a detailed example, and some classes of rational almost-factorial surfaces of $\mathbb{A}^3$ are pointed out as well.

1. Let k be an algebraically closed field of any characteristic. $\mathbb{P}^N$ and $\mathbb{A}^N$ denote respectively the projective and the affine space of dimension N over k. Variety on $\mathbb{P}^N$ (or on $\mathbb{A}^N$), will mean always an algebraic irreducible and reduced closed subset on $\mathbb{P}^N$ (or on $\mathbb{A}^N$), $N \geqslant 3$.

A prime divisor (or more shortly) a *prime* on a variety $\mathcal{F}$, non singular in codimension 1, will be an irreducible and reduced subvariety on $\mathcal{F}$ of codimension 1. Curves surfaces, hypersurfaces on $\mathbb{P}^N$ (or on $\mathbb{A}^N$) will be varieties of dimension 1, 2, $N-1$ respectively.

Let $\mathcal{F}$ be a variety on $\mathbb{P}^N$: $J(\mathcal{F})$, $k[\mathcal{F}]$, $k(\mathcal{F})$ denote respectively the (prime and homogeneous) ideal of $\mathcal{F}$ in $k[X_0, ..., X_N]$, $(= k[X])$, the quotient ring $k[X]/J(\mathcal{F})$, the field of rational functions on $\mathcal{F}$, (= field of quotients of elements in $k[\mathcal{F}]$ of same degree).

DEFINITIONS 1. A Variety $\mathcal{F} \subset \mathbb{P}^N$ is called *projectively normal* if $k[\mathcal{F}]$ is integrally closed.

2. A prime $\mathcal{C}$ on a variety $\mathcal{F} \subset \mathbf{P}^N$, is *set-theoretic complete intersection* (S.T.C.I.) on $\mathcal{F}$ with multeplicity λ, if exists a hypersurface $\mathcal{G} \subset \mathbf{P}^N$ such that $\mathcal{F} \cdot \mathcal{G} = \lambda \mathcal{C}$, i.e. $\lambda \mathcal{C}$ is the complete intersection of $\mathcal{F}$ and $\mathcal{G}$.

3. A Variety $\mathcal{F}$ projectively normal is *almost-factorial* if every prime $\mathcal{C} \subset \mathcal{F}$ is S.T.C.I. on $\mathcal{F}$. In particular $\mathcal{F}$ is ϱ-almostfactorial if every prime $\mathcal{C}$ on $\mathcal{F}$ is S.T.C.I. on $\mathcal{F}$ with multeplicity $\lambda \leqslant \varrho$.

Let $\mathcal{F}$ be a projective variety of $\mathbf{P}^N$, with $n = \dim \mathcal{F}$. We recall some well known facts (see [4], pp. 107-124) about birational correspondences between projective spaces and in particular between $\mathbf{P}^n$ and $\mathcal{F}$.

DEFINITION 4. A *parametrization* p on $\mathbf{P}^n$ of a projective variety $\mathcal{F}$ is a set

$$H_0, \dots, H_N \in k[U_0, \dots, U_n], \qquad (= k[U])$$

of homogeneous polynomials (forms) of the same degree such that

a) substituting

$$X_i \to H_i \quad i = 0, \dots, N, \tag{1}$$

for every $P \in J(\mathcal{F})$ it follows $P(H_0, \dots, H_N) = 0$;

b) if g denotes the image of G in the canonical projection $k[X] \to k[\mathcal{F}] = k[x_0, \dots, x_N]$, for every $G \in k[X]$, there exists a set

$$F_0, \dots, F_n \in k[X]$$

of forms of the same degree such that

$$MU_j = f_j(H_0, \dots, H_N) \quad j = 0, \dots, n, \tag{2}$$

for a suitable $M \in k[U]$, $M \neq 0$.

We note that in *b*) the forms $F_0, \dots, F_n$ are chosen in $k[X]$ only $\operatorname{mod} J(\mathcal{F})$; thus the polynomial M in (2) can change with the set $F_0, \dots, F_n$; however it must be homogeneous from (2) itself. Moreover (1) and (2) together imply there exists a k-homomorphism between the fields $k(\mathcal{F})$ and $k(U)$ of rational functions on $\mathcal{F}$ and $\mathbf{P}^n$ respectively. By this, there must exist a suitable (homogeneous) polynomial $Q \in$

$\in k[X]$, $Q \notin J(\mathcal{F})$ such that

$$(3) \qquad qx_j = H_j(f_0, \ldots, f_n) \qquad j = 0, \ldots, N$$

(see also [4] p. 116). We remark also that for every point $A \in \mathbf{P}^n - \{M = 0\}$ is $M(A) \neq 0$, if exists j, $0 \leqslant j \leqslant N$, for which $U_j \neq 0$ and, by (2)

$$0 \neq M(A)\, U_j(A) = f_j\big(H_0(A), \ldots, H_n(A)\big) \Rightarrow H_i(A) \neq 0$$

for at least one i, $0 \leqslant i \leqslant n$. This proves that the parametrization p rises, by means of (1), to a map

$$\sigma_p\colon \mathbf{P}^n \to \mathcal{F} \qquad (\subset P^N)$$

which is regular in $\mathbf{P}^n - \{M = 0\}$ and σ_p is invertible by (3).

As above one sees that $\sigma_p^{-1}\colon \mathcal{F} \to \mathbf{P}^n$ is surely regular in $\mathcal{F} - \mathcal{F} \cap \cap \{Q = 0\}$, but in general nothing can be said for the points of the set $\mathcal{F} \cap \{Q = 0\}$. If we consider another parametrization q on $\mathbf{P}^n$ of $\mathcal{F}$, being $\mathcal{F}$ irreducible, we have that the map $\sigma_q\colon \mathbf{P}^n \to \mathcal{F}$ rised by q coincides with σ_p in an open suitable subset of $\mathbf{P}^n$. By this they give a birational map $\sigma\colon \mathbf{P}^n \to \mathcal{F}(\subset P^N)$ which may be biregular too. For example it is what happens in the m-ple embedding $\mathbf{P}^n \to \mathcal{F} \subset \mathbf{P}^N$ with $N = \binom{n+m}{n} - 1$.

2. In the following we suppose $\mathcal{F} \subset \mathbf{P}^N$ to be a rational variety (i.e. every variety $\mathcal{F}$ for which exists a birational map $\mathbf{P}^n \to \mathcal{F}$, $n = \dim \mathcal{F}$) and let p a parametrization of $\mathcal{F}$ on $\mathbf{P}^n$.

We are precisely concerned with the particular map $\sigma_p\colon P^n \to \mathcal{F}$ gived by p and the sets $\mathbf{P}^n - \{M = 0\}$ and $\mathcal{F} - \mathcal{F} \cap \{Q = 0\}$ which depend from p according to the previous notations, and the said situation.

DEFINITIONS 5. Given a parametrization p on $\mathbf{P}^n$ of a projective variety $\mathcal{F}$, we call *referred* to p all the subvarieties of codimension 1 in $\mathcal{F}$ which belong to $\{Q = 0\}$.

6. Rational variety of $\mathbf{A}^N$ will be a variety of $\mathbf{A}^N$ whose projective closure is a rational variety in $\mathbf{P}^N$.

The properties of the map σ in the present hypothesis are well known; we recall someone of them in the

PROPOSITION 1.*a*) *To every subvariety* $\mathcal{V} \subset \mathcal{F}$ *not referred to* p, *it corresponds a subvariety* $\sigma_p^{-1}(\mathcal{V})$ *such that* $\dim \sigma_p^{-1}(\mathcal{V}) = \dim \mathcal{V}$ *and* $\sigma_p^{-1}(V) \not\subseteq \{M = 0\}$ (*see* [4], *Satz VI*, *p.* 120)

b) *The restriction of* σ_p^{-1} *to the set of non singular points of* $\mathcal{V} - \mathcal{V} \cap \{Q = 0\}$ *is bijective, moreover to every non singular point corresponds a non singular point* (*see* [4] *Korollar*, *p.* 121).

Let $\mathcal{F}'$ and $\mathcal{F}$ be two varieties of dimension n and τ a birational map $\tau\colon \mathcal{F}' \to \mathcal{F}$. We recall that a prime $\mathcal{V} \subset \mathcal{F}$ is said *exceptional* for τ if $\dim \tau^{-1}(\mathcal{V}) < n - 1$. The rational variety $\mathcal{F}$ can have only a finite number of exceptional primes for the birational map $\mathbb{P}^n \to \mathcal{F}$: indeed they can belong among the maximal components of $\mathcal{F} \cap \{Q = 0\}$ for every parametrization p on $\mathbb{P}^n$ of $\mathcal{F}$; they are then referred to every parametrization p. On the other hand, if the birational map $\mathbb{P}^n \to \mathcal{F}$ is biregular, no prime of $\mathcal{F}$ is exceptional for it.

LEMMA 1. *Let* $\mathcal{F}$ *be a rational variety*, $\mathcal{F} \subset \mathbb{P}^N$, *and* p *be a parametrization on* $\mathbb{P}^n$ *of* $\mathcal{F}$. *For each prime* $\mathcal{V} \subset \mathcal{F}$ *not referred to* p, *it exists at least an irreducible form* $\Psi \in k[U]$ *such that*

$$\Psi(F_0, \ldots, F_n) \in J(\mathcal{V}) .$$

PROOF. Let $C_1, \ldots, C_s \in k[X]$ such that $J(\mathcal{V}) = (C_1, \ldots, C_s)$ and let be

$$D_i = C_i(H_0, \ldots, H_N) \qquad i = 1, \ldots, s .$$

Let us denote always with $\sigma_p\colon \mathbb{P}^n \to \mathcal{F}$ the rational mapping rised by p and $Q \in [X]$, $M \in k[U]$ the forms in the (3) and (2) respectively. First we have

$$\sigma_p^{-1}(\mathcal{V}) = \{D_1 = \ldots = D_s = 0\} .$$

Indeed obviously $\sigma_p^{-1}(\mathcal{V}) \subseteq \{D_1 = \ldots = D_s = 0\}$; on the other hand, by Prop. 1.*a*), $\sigma_p^{-1}(\mathcal{V})$ is irreducible and of codimension 1 in $\mathbb{P}^n$, so it must be a hypersurface of $\mathbb{P}^n$. From this $\sigma_p^{-1}(\mathcal{V}) \supseteq \{D_1 = \ldots = D_s = 0\}$ and $J(\sigma_p^{-1}(\mathcal{V}))$ will be a principal ideal generated by $\Psi = $ G.C.D. $\{D_1, \ldots, D_s\}$ and Ψ will be irreducible and Ψ does not divide M. Later, being the ring $k[U]$ U.F.D. and $\Psi = \text{G.C.D}\{D_1, \ldots, D_s\}$, there exist A_i, $B_i \in k[U]$, $i = 1, \ldots, s$, such that

$$D_i = \Psi A_i \quad \text{for } i = 1, \ldots, s\,; \qquad 1 = \sum_{1=i}^{s} A_i B_i$$

by which $\Psi = \Psi \sum_{1=i}^{s} A_i B_i = \sum_{1=i}^{s} D_i B_i = \sum_{1=i}^{s} C_i(H_0, \ldots, H_N) B_i$:

Let us consider now the form $\Psi(F_0, \ldots, F_n) \in k[X]$ obtained by substituting F_i to place of U_i, $i = 0, \ldots, n$. It results

$$(\#) \qquad \Psi(F_0, \ldots, F_n) = \\ = \sum_{1=i}^{s} C_i\big(H_0(F_0, \ldots, F_n), \ldots, H_N(F_0, \ldots, F_n)\big) B_i(F_0, \ldots, F_n)\,.$$

Now we want to calculate its image $\Psi(f_0, \ldots, f_n)$ in $k[\mathcal{F}]$. From (3) it is

$$C_i\big(H_0(f_0, \ldots, f_n), \ldots, H_N(f_0, \ldots, f_n)\big) = C_i(qx_0, \ldots, qx_N) = q^{\deg C_i} C_i(x_0, \ldots, x_N)\,,$$

by this, from (#) one gets

$$\Psi(f_0, \ldots, f_n) = \sum_{1=i}^{s} q^{\deg C_i} C_i(x_0, \ldots, x_N) B_i(f_0, \ldots, f_n)$$

which belongs to the image of $J(\mathcal{V})$ in $k[\mathcal{F}]$, whence

$$\Psi(F_0, \ldots, F_n) \in J(\mathcal{V})\,.$$

PROPOSITION 2. *Let $\mathcal{F} \subset \mathbf{P}^N$ be a rational projectively normal variety of $\dim \mathcal{F} = n$ and p a parametrization of $\mathcal{F}$ on $\mathbf{P}^n$. The following are equivalent:*

a) $\mathcal{F}$ is almost-factorial;

b) every prime on $\mathcal{F}$ referred to p is set-theoretic complete intersection on $\mathcal{F}$.

More precisely if $\mathcal{C}_1, \ldots, \mathcal{C}_t$ are the primes on $\mathcal{F}$ referred to p and $\lambda_i \mathcal{C}_i$ is the complete intersection $\mathcal{F}$ with a suitable hypersurface $\mathcal{G}_i \subset \mathbf{P}^N$, $i = 1, \ldots, t$, then for every prime $\mathcal{C} \subset \mathcal{F}$ it exists a hypersurface $\mathcal{G} \subset \mathbf{P}^N$

such that

$$\mathcal{F}\cdot\mathfrak{G}=\lambda\mathfrak{C} \quad \textit{where} \quad \lambda=\text{L.C.M.}\{\lambda_1,\ldots,\lambda_t\},$$

that is $\mathcal{F}$ is λ-almostfactorial.

PROOF. *a*) $\Rightarrow$ *b*) is obvious. So we have only to prove *b*) $\Rightarrow$ *a*) Let $\mathfrak{C}_1,\ldots,\mathfrak{C}_t$ be all the primes of $\mathcal{F}$ referred to p. By hypothesis *b*) there exist hypersurfaces $\mathfrak{L}_i=\{L_i=0\}\subset P^N$ and $\lambda_i>0$, $i=1,\ldots,t$, such that

$$\mathcal{F}\cdot\mathfrak{L}_i=\lambda_i\mathfrak{C}_i \quad i=1,\ldots,t. \tag{4}$$

Let $\lambda=\text{L.C.M.}\{\lambda_1,\ldots,\lambda_t\}$ and let $n_1,\ldots,n_t$ be positive integers such that

$$\lambda=n_i\lambda_i \quad i=1,\ldots,t. \tag{5}$$

For every prime $\mathfrak{D}\subset\mathcal{F}$ we denote with $C(\mathfrak{D})$ the affine cone of $\mathfrak{D}$ and let $C(\mathcal{F})$ be the affine cone of $\mathcal{F}$, both in $\mathbf{A}^{N+1}$. Obviously $C(\mathfrak{D})$ has codimension 1 in $C(\mathcal{F})$ for every prime $\mathfrak{D}\subset\mathcal{F}$. Moreover the ring $k[\mathcal{F}]$ can be considered as the ring of the regular functions on $C(\mathcal{F})$. Let K be the quotient field of $k[\mathcal{F}]$. For every prime $\mathfrak{C}\subset\mathcal{F}$ not referred to p it exists, by Lemma 1, an irreducible polynomial $\Psi\in k[U]$ such that

$$\Psi(F_0,\ldots,F_n)\in J(\mathfrak{V}).$$

Let be $H=\Psi(F_0,\ldots,F_n)\in k[X]$ and let h be its projection in $k[\mathcal{F}]$. We have

$$\{H=0\}\cdot\mathcal{F}=\operatorname{div}(h)=\mu\mathfrak{C}+\nu_1\mathfrak{D}_1+\ldots+\nu_r\mathfrak{D}_r, \tag{6}$$

$$\mu>0,\ \nu_i>0,\ i=1,\ldots,r,$$

where $\mu=1$ by Prop. 1.*b*) because $\mathfrak{C}$ is not referred to p, and $\mathfrak{D}_1,\ldots,\mathfrak{D}_r$ are distinct primes on $\mathcal{F}$, different from $\mathfrak{C}$, which are necessarly referred to p. Indeed, if $\mathfrak{D}_i$ is one of $\mathfrak{D}_1,\ldots,\mathfrak{D}_r$, it is or exceptional for p (and then referred to p), or, if it would not be referred to p, the ideal $J(\sigma_p^{-1}(\mathfrak{D}_i))$, by Lemma 1 is a principal ideal which contains Ψ itself. Since Ψ is irreducible, then $J(\sigma_p^{-1}(\mathfrak{D}_i))=(\Psi)=J(\sigma_p^{-1}(\mathfrak{C}))$. On the other hand $\{\Psi=0\}\not\subset\{M=0\}$ and σ_p is regular in $\mathbf{P}^n-\{M=0\}$ so we would have $\mathfrak{D}_i=\mathfrak{C}$. So $\mathfrak{D}_i$, $i=1,\ldots,r$, is in any case referred to p. Of course $r\leqslant t$, being t the number of all primes of $\mathcal{F}$ referred to p and among them there are $\mathfrak{D}_1, \dots, \mathfrak{D}_r$. We can suppose that the primes of $\mathcal{F}$ referred to p in (6) to be $\mathfrak{D}_1 = \mathfrak{C}_1, \dots, \mathfrak{D}_r = \mathfrak{C}_r$.

Let us consider now the polynomials $L_i^{n_i \nu_i} \in k[X]$, $i = 1, \dots, r$ and we denote with p_i their images in $k[\mathcal{F}]$. Since is

$$g = h^\lambda / (p_1 \dots p_r) \in K .$$

it results, by (4), (5) and (6),

$$\text{(7)} \qquad \operatorname{div}(g) = \lambda \mathfrak{C} + \lambda \nu_1 \mathfrak{C}_1 + + \dots + \lambda \nu_r \mathfrak{C}_r - [\nu_1 \lambda_1 n_1 \mathfrak{C}_1 + \dots + \nu_r \lambda_r n_r \mathfrak{C}_r] = \lambda \mathfrak{C} .$$

Note (7) means that for every valuation v_ε of the field K centered in the subvariety $\mathcal{E}$ of codimension 1 in $C(\mathcal{F})$ is

$$v_\varepsilon(g) = 0 \text{ if } \mathcal{E} \neq C(\mathfrak{C}) \quad \text{and} \quad v_\varepsilon(g) = \lambda \text{ if } \mathcal{E} = C(\mathfrak{C}) .$$

By this g is an element which belongs to the integral closure of $k[\mathcal{F}]$, by the structure theorem of noetherian integrally closed domains. On the other hand, being $k[\mathcal{F}]$ normal because $\mathcal{F}$ is projectively normal (see def. 1), $g \in k[\mathcal{F}]$. It exists then at least a homogeneous $G \in k[X]$ such that its projection in $k[\mathcal{F}]$ is g. Moreover we get

$$\text{(8)} \qquad \mathcal{F} \cdot \mathcal{G} = \lambda \mathfrak{C} .$$

We note that the integer λ in (8) does not depend on $\mathfrak{C}$ but only on all the primes referred to p. So $\mathcal{F}$ is λ-almostfactorial.

REMARK 1. Prop. 2 is an extension of a well known criterion of D. Gallarati on the monoid hypersurfaces $\mathcal{M} \subset \mathbb{P}^N$ (see [3], cap. III, 17, p. 38, and also [7], Prop. 1):

Every prime of $\mathcal{M}$ is set-theoretic complete intersection of $\mathcal{M}$ iff all the primes of the cone of the straight lines passing through the vertex of $\mathcal{M}$ are set-theoretic complete intersection.

Indeed the projection from the vertex V of $\mathcal{M}$ onto a hyperplane not passing through V, gives a parametrization of $\mathcal{M}$ on that hyperplane. The primes of $\mathcal{M}$ referred to this parametrization are just the primes of the cone of straight lines of $\mathcal{M}$ passing through V. They are all exceptional too.

REMARK 2. Prop. 2 also shows that the image $\mathcal{F}$ of a m-ple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$, $N = \binom{n+m}{n} - 1$, is m-almostfactorial (fact well known). Indeed in the usually parametrization p of $\mathcal{F}$ on $\mathbb{P}^n$ (see [4], pag. 124 for example) is referred to p a prime $\mathcal{C} \subset \mathcal{F}$ which can be supposed to be the image of a hyperplane of $\mathbb{P}^n$. It results that $m\mathcal{C}$ is just the complete intersection of $\mathcal{F}$ with a suitable hyperplane in $\mathbb{P}^N$. Since $\mathcal{F}$ is projectively normal one gets that $\mathcal{F}$ is m-almost-factorial.

We can formulate an affine version of Prop. 2 in the following

PROPOSITION 3. *Let $\mathcal{E} \subset \mathbb{A}^N$ be a rational normal variety. The following facts are equivalent*:

a) $\mathcal{E}$ is almost-factorial;

b) every prime $\mathcal{C}_a$ on $\mathcal{E}$ referred to a parametrization p of the projective closure $\bar{\mathcal{E}}$ is S.T.C.I. on $\mathcal{E}$.

PROOF. Let $\mathcal{F} = \bar{\mathcal{E}}$ be the projective closure of $\mathcal{E}$ and $\mathcal{C}_a \subset \mathcal{E}$ be a prime whose projective closure $\mathcal{C} = \bar{\mathcal{C}}_a$ is not referred to the parametrization of $\mathcal{F}$. To obtain relation (7) we argue as in Prop. 2. (7) induces on $\mathcal{E}$

$$\text{div}\,(g_a) = \lambda \mathcal{C}_a\,, \tag{7'}$$

where g_a now belongs to the quotient field of $\mathcal{E}$. Since $\mathcal{E}$ is normal, the ring $k[\mathcal{E}]$ coincides with its integral closure. The arguments as at the end of the proof of Prop. 2 prove that $g_a \in k[\mathcal{E}]$. So it exists a suitable polynomial G for which it results

$$\mathcal{E} \cdot \{G = 0\} = \lambda \mathcal{C}_a\,. \tag{8'}$$

This proves $b) \Rightarrow a)$, while $a) \Rightarrow b)$ is obvious.

3. Applications and examples.

It is well known that a hypersurface $\mathcal{F} \subset \mathbb{P}^N$ is projectively normal iff is non singular in codimension 1 (see [6] Prop. 1 p. 389 and Prop 2 p. 391) arguing, in the projective case, on the affine cone of $\mathcal{F}$. In the case of surfaces of $\mathbb{P}^3$ we can apply Prop. 2 to state the

COROLLARY 1. *A rational non singular surface $\mathcal{F} \in \mathbf{P}^3$ is almost-factorial only if it is a plane. (So it is factorial).*

PROOF. Let d be the degree of $\mathcal{F}$. Since $\mathcal{F}$ is non singular and rational, the geometric genus

$$p_g(\mathcal{F}) = p_g(\mathbf{P}^2) = (d-1)(d-2)(d-3)/6 = 0\,.$$

So, if $\mathcal{F}$ is not a plane, $\mathcal{F}$ is a non singular quadric or cubic. But these surfaces are not almost-factorial (see [3], or [7]).

COROLLARY 2. *A rational surface $\mathcal{F} \subset \mathbf{P}^3$ of degree $d > 1$ is almost-factorial if only if it has a positive (finite) number of singular points, and has a parametrization p on $\mathbf{P}^2$ such that very curve on $\mathcal{F}$ which is referred to p is S.T.C.I. of $\mathcal{F}$.*

PROOF. It follows from Prop. 2 and from what we have recalled about the condition for a hypersurface of $\mathbf{P}^N$ to be projectively normal.

EXAMPLE 1. *A quartic surface in $\mathbf{P}^3$ with only two double singular points.* Let us denote with $\{T, X, Y, Z\}$ the coordinates in $\mathbf{P}^3$. Let $\mathcal{F}$ be:

$$\mathcal{F} = \{T^2X^2 + TY^3 - Z^4 = 0\}\,.$$

The surface $\mathcal{F}$ is singular only in the double points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$.

A parametrization p of $\mathcal{F}$ is, for example, given by

$$H_0 = W^3(W^2 - U^2)^2\,, \qquad H_1 = V^6U\,, \qquad H_2 = V^4W(W^2 - U^2)\,,$$
$$H_3 = V^3W^2(W^2 - U^2) \in k[W, U, V]$$

because we have, first

$$H_0^2H_1^2 + H_0H_2^3 - H_3^4 = 0\,,$$

secondly, let us choose $F_0 = Z^2$, $F_1 = TX$, $F_2 = YZ \in k[T, X, Y, Z]$. We can consider then the rational map

$$\sigma\colon (W, U, V) \to (T = H_0,\ X = H_1,\ Y = H_2,\ Z = H_3) \subseteq \mathcal{F} \subset \mathbf{P}^3\,.$$

The restriction on $\mathcal{F}$, of the map

$$\pi\colon (T, X, Y, Z) \to (W = F_0,\ U = F_1,\ V = F_2) \subset \mathbf{P}^2$$

is just σ^{-1}. Indeed for every $F \in k[T, X, Y, Z]$ let f be the canonical projection of F in

$$k[T, X, Y, Z]/J(\mathcal{F}) = k[\mathcal{F}] = k[t, x, y, z]\,;$$

one gets the relations

$$H_0(f_0, f_1, f_2) = \ldots = (ty^6 z^6)\,t\,, \qquad H_1(f_0, f_1, f_2) = \ldots = (ty^6 z^6)\,x\,,$$
$$H_2(f_0, f_1, f_2) = \ldots = (ty^6 z^6)\,y\,, \qquad H_3(f_0, f_1, f_2) = \ldots = (ty^6 z^6)\,z$$

from which it results $Q = TY^6Z^6$. Moreover the product-map

$$(t, x, y, z) \to (W = f_0,\ U = f_1,\ V = f_2) \to$$
$$\to (t = H_0,\ x = H_1,\ y = H_2,\ z = H_3)$$

is the identity on $\mathcal{F}$, while the identities

$$F_0(H_0, \ldots, H_3) = H_3^2 = [V^6 W^3 (W^2 - U^2)^2]\,W\,,$$
$$F_1(H_0, \ldots, H_3) = H_0 H_3 = [V^6 W^3 (W^2 - U^2)^2]\,U\,,$$
$$F_2(H_0, \ldots, H_3) = H_1 H_2 = [V^6 W^3 (W^2 - U^2)^2]\,V\,,$$

show that one must assume $M = V^6 W^3 (W^2 - U^2)^2$ and they prove that the product-map

$$(W, U, V) \to (t = H_0,\ x = H_1,\ y = H_2,\ z = H_3) \to$$
$$\to (W = f_0,\ U = f_1,\ V = f_2)$$

is the identity on $\mathbf{P}^2$. The components of $\mathcal{F} \cap \{Q = 0)\}$ are

$$\mathfrak{R} = \{T = Z = 0\}\,, \qquad \mathfrak{C}_3 = \{Z = TX^2 + Y^3 = 0\}\,,$$
$$\mathfrak{C}_2 = \{Y = TX - Z^2 = 0\}\,, \qquad \mathfrak{C}_2' = \{Y = TX + Z^2 = 0\}$$

and they are the curves on $\mathcal{F}$ referred to p.

Now we have to prove that every such curves is S.T.C.I. of $\mathcal{F}$. Clearly is

$$\mathcal{F}\cdot\{T=0\}=4\mathfrak{R}\,,\qquad \mathcal{F}\cdot\{TX^2+Y^3=0\}=4\mathcal{C}_3\,,$$
$$\mathcal{F}\cdot\{TX-Z^2=0\}=3\mathcal{C}_2+2\mathfrak{R}\,,\qquad \mathcal{F}\cdot\{TX+Z^2=0\}=3\mathcal{C}_2'+2\mathfrak{R}\,.$$

Let us consider on $\mathcal{F}$ the divisor defined by the quotient

$$[(tx-z^2)^2]/t\,.$$

It is just the divisor of $2tx^2-2xz^2+y^3$ because in $k[\mathcal{F}]$ equality $z^4=t(tx^2+y^3)$ holds. From this, it follows

$$\mathcal{F}\cdot\{2TX^2-2XZ^2+Y^3=0\}=6\mathcal{C}_2\,.$$

By the same arguments one gets

$$\mathcal{F}\cdot(2TX^2+2XZ^2+Y^3=0\}=6\mathcal{C}_2'\,.$$

Since every curve of $\mathcal{F}$ referred to p is S.T.C.I. on $\mathcal{F}$ we can apply to $\mathcal{F}$ Prop. 2: $\mathcal{F}$ is then *almost-factorial*; more precisely $\mathcal{F}$ is 12-*almostfactorial.*

As example let us consider the rational curve

$$\mathcal{C}_5=\{t=(1-s)^2,\ x=4s^4(1-2s),\ y=4s^3(1-s),\ z=2s^2(1-s)\}\,.$$

$\mathcal{C}_5$ belongs to $\mathcal{F}$. The curve $\mathcal{C}_5$ belongs even to the surfaces $\{Z^3+ TXZ-TY^2=0\}$, $\{Y^2Z+TXY+2TXZ-TY^2=0\}$ etc., and even to the quadric $Q=\{TX+YZ-Z^2=0\}$. The images on $\mathbf{P}^2$, by means of σ^{-1}, of the intersections with $\mathcal{F}$ of such surfaces are curves of which a common component, not component of $\{M=0\}= \{V^6W^3(W^2-U^2)^2=0\}$, is the line $\{U+V-W=0\}$. By this we can suppose $\Psi(W,U,V)=U+V-W$, so $\Psi(F_0,F_1,F_2)=TX+ YZ-Z^2$.

On the other hand it is

$$\mathcal{F}\cdot Q=\mathcal{C}_5+\mathcal{C}_2+\mathfrak{R}\,.$$

Now $\mathcal{C}_2$ and $\mathfrak{R}$ are S.T.C.I. on $\mathcal{F}$, so it is also $\mathcal{C}_5$ with multeplicity at most 12. It is enough to consider on $\mathcal{F}$ the divisor defined by the quotient

$$[tx - z^2 + yz]^{12}/[(tx - z^2)^4]t.$$

Indeed in $k[\mathcal{F}]$ the identities

$$ty^3 = -(tx - z^2)(tx + z^2), \qquad z^4 = t(tx^2 + y^3);$$

hold, which give, first

$$[tx - z^2 + yz]^3/(tx - z^2) =$$
$$= [(tx - z^2)]^2 + 3(tx - z^2)yz + 3y^2z^2 + y^3z^3/(tx - z^2) =$$
$$= 2t^2x^2 - 2txz^2 + ty^3 + 3txyz - 3yz^3 + 3y^2z^2 - zy^3 - tx^2z - xz^3 =$$
$$= tL + zN$$

where we have assumed, for example,

$$L = 2x^2t - 2xz^2 + y^3 + 3xyz - x^2z, \qquad N = -y^3 - 3yz^2 + 3y^2z - xz^2.$$

Secondly, the divisor of the quotient

$$(tL + zN)^4/t$$

coincides with the divisor of

$$g = t^3L^4 + 4t^2zL^3N + 6tz^2L^2N^2 + 4z^3LN^3 + (tx^2 + y^3)N^4.$$

It then exists in $k[X]$ a polynomial G, homogeneous of degree 15, such that its image on $k[\mathcal{F}]$ is g; then we get $\{G = 0\}\cdot\mathcal{F} = 12C_5$.

EXAMPLE 2. *Classes of rational surfaces whith an affine part isomorphic to a plane.* It is easy to determine a class of surfaces $\mathcal{F}_n \subset \mathbf{A}^3$ of degree n, for every $n > 0$, which are *isomorphic* to a plane and having projective closure non singular in codimension 1 and n-almostfactorial.

P. C. Craighero has pointed this example to me in the case $n = 4$.

Let $a(T)$, $b(T)$, $c(T)$ be arbitrary polynomials of $k[T]$ of degree r, s, m respectively for which is

$$rs + 1 = m = n \qquad \text{or} \qquad rs = n = m + 1.$$

Let us consider the two isomorphisms of $\mathbf{A}^3$

$$\eta\colon (U, V, W) \to \big(U' = U + a(W),\ V' = V,\ W' = W\big)$$

and

$$\chi\colon (U', V', W') \to \big(U'' = U',\ V'' = V',\ W'' = W' + b(U') + c(V')\big)$$

and their product

$$\chi\circ\eta\colon (U, V, W) \to$$
$$\to \big(U'' = U + a(W),\ V'' = V,\ W'' = W + b\big(U + a(W)\big) + c(V)\big).$$

The affine surfaces of $\mathbf{A}^3$:

$$\mathcal{F}_n = \{Z + b\big(X + a(Z)\big) + c(Y) = 0\}$$

are isomorphic to a plane (and more precisely to the plane $\{W'' = 0\}$) by means of $\chi\circ\eta$ and are of degree n, and, for example, they admit the parametrization

$$X = U - a[-b(U) - c(V)]\,, \qquad Y = V\,, \qquad Z = -\,b(U) - c(V)$$

whose inverse is $\big(U = X + a(Z),\ V = Y\big)$, with $Z + b\big(X + a(Z)\big) + c(Y) = 0$. Such surfaces are then rational and factorial. Their projective closure $\overline{\mathcal{F}}_n$ is non singular in codimension 1; the section of such surfaces with the plane at the infinity is a straight line which is just the complete intersection of such two surfaces. Such a line is the only curve on the surface $\overline{\mathcal{F}}_n$ which is referred to the parametrization of $\overline{\mathcal{F}}_n$: by this $\overline{\mathcal{F}}_n$ is n-almostfactorial.

If the polynomial $b(T)$ is linear ($s = 1$) such surfaces are monoids; then one can apply Gallarati's criterion to them with the same conclusions.

EXAMPLE 3. *Classes of trinomial rational surfaces* (of the kind $\{X^a + Y^b = Z^c\}$). Let m, n be coprime positive integers, with $m < n$. Let (r_0, s_0) a integer solution of the diophantine equation

$$(\alpha) \qquad\qquad xm - yn = 1\,.$$

Every other integer solution (r, s) of such equation is

$$(\beta_1) \qquad r = r_0 + tn\,,$$

$$(\beta_2) \qquad s = s_0 + tm \quad \text{for every } t \in \mathbb{Z}\,.$$

For every integer r in (β_1), we consider the classes of affine surfaces of $\mathbb{A}^3$

$$\mathcal{E}_{m,n,r} = \{X^m + Y^n = Z^{rm-1}\}\,, \quad \mathcal{F}_{m,n,r} = \{X^m + Y^n = Z^{rm}\} \quad \text{for } r > 0\,,$$

$$\mathcal{G}_{m,n,r} = \{X^m + Y^n = Z^{-rm}\}\,, \quad \mathcal{H}_{m,n,r} = \{X^m + Y^n = Z^{1-rm}\} \quad \text{for } r < 0\,.$$

One gets a parametrization of such surfaces assuming

$$x = z^r u\,, \quad y = z^s v \quad \text{with } z = (1 - v^n)/u^m \quad \text{for the } \mathcal{E}_{m,n,r}\,,$$

$$x = z^r u\,, \quad y = z^s v \quad \text{with } z = v^n/(1 - u^m) \quad \text{for the } \mathcal{F}_{m,n,r}\,,$$

$$x = z^{-r} u\,, \quad y = z^{-s} v \quad \text{with } z = (1 - u^m)/v^n \quad \text{for the } \mathcal{G}_{m,n,r}\,,$$

$$x = z^{-r} u\,, \quad y = z^{-s} v \quad \text{with } z = u^m/(1 - v^n) \quad \text{for the } \mathcal{H}_{m,n,r}\,,$$

while for the inverse map we have to assume for the $\mathcal{E}_{m,n,r}$ and $\mathcal{F}_{m,n,r}$:

$$u = x/z^r\,, \quad v = y/z^s$$

and $u = xz^r$, $v = yz^s$ for the $\mathcal{G}_{mnr}$ and for the $\mathcal{H}_{m,n,r}$ respectively. Every surface $\mathcal{E}_{m,n,r}$ and $\mathcal{H}_{m,n,r}$ is *m-almostfactorial*, while the surfaces $\mathcal{F}_{m,n,r}$ and $\mathcal{G}_{m,n,r}$ are *n-almostfactorial*, by Prop. 2 and Prop. 3.

There are other classes of affine rational surfaces which are trinomial and almost-factorial. For example, for every pair of positive integers (m, n) the affine surfaces in $\mathbb{A}^3$:

$$\mathcal{B}_{m,n} = \{X^m + Y^n = Z^{mn+1}\} \quad \text{and} \quad \mathcal{C}_{m,n} = \{X^m + Y^n = Z^{mn-1}\}\,.$$

$\mathcal{B}_{m,n}$ admits a parametrization $x = z^n u$, $y = z^m v$, $z = u^m + v^n$ whose inverse is $u = x/z^n$, $v = y/z^m$, $\mathcal{C}_{m,n}$ admits a parametrization $x = z^n u$, $y = z^m v$, $z = 1/(u^m + v^n)$ whose inverse is $u = x/z^n$, $v = y/z^m$. The surfaces $\mathcal{B}_{m,n}$ and $\mathcal{C}_{m,n}$ are *factorial* if m, n are coprimes, $\mathcal{B}_{m,n}$ is $(mn + 1)$-*almostfactorial* and $\mathcal{C}_{m,n}$ is $(mn - 1)$-*almostfactorial* otherwise.