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On the Convolution in the Space $\mathcal{D}'_{L^1}(M_p)$.

STEVAN PILIPOVIĆ (*)

SUMMARY. - We investigate the convolution and the Fourier transformation in the space of Beurling ultradistributions $\mathcal{D}'_{L^1}(M_p)$. We give some simple conditions on a convolutor S for the solvability and the hypoellipticity of a convolution equation $S * U = V$ in $\mathcal{D}'_{L^1}(M_p)$.

1. The space $\mathcal{D}'_{L^1}(M_p)$ introduced in [3] is a subspace of the space of Beurling ultradistributions ([2]). This is a natural generalization of the Schwartz space \mathcal{D}'_{L^1} , and we investigated them in [3] in connection with the Hilbert transformation of ultradistributions. With suitable assumptions on the sequence M_p , we determined in [4] elements of $\mathcal{D}'_{L^1}(M_p)$ as boundary values of certain holomorphic functions.

In [3] some questions on the convolution in $\mathcal{D}'_{L^1}(M_p)$ occurred. This was the motivation for our investigations of the convolution in $\mathcal{D}'_{L^1}(M_p)$. The Fourier transformation maps $\mathcal{D}'_{L^1}(M_p)$ into a subspace of L^2_{loc} . So by proving the exchange formula we obtain simple conditions for the solvability and the hypoellipticity of a convolution equation in $\mathcal{D}'_{L^1}(M_p)$.

2. Our notation is the same as in [2]. Let M_p , $p \in \mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$ be a sequence of positive numbers such that

$$(M.1) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathfrak{N};$$

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(M.2) There are constants A and H such that

$$M_p \leq AH^p \min_{0 \leq a \leq p} \{M_a M_{p-a}\}, \quad p \in \mathfrak{N}_0;$$

(M.3) There is a constant A such that

$$\sum_{a=p+1}^{\infty} M_{a-1}/M_a \leq Ap M_p/M_{p+1}, \quad p \in \mathfrak{N}.$$

We assumed in [3] instead of (M.2) and (M.3) the weaker conditions: (M.2)' and (M.3)'. (All these conditions were analysed in [2].) The reason for that is the structural theorem for $\mathcal{D}'_{L^1}^{(M_p)}$ which we need for the full characterization of the convolution in $\mathcal{D}'_{L^1}^{(M_p)}$.

The associated function M is defined by

$$M(\varrho) = \sup_{p \in \mathfrak{N}_0} \left\{ \log \frac{M_0 \varrho^p}{M_p} \right\}, \quad \varrho > 0.$$

From [2, Proposition 3.6] it follows

$$(1) \quad 2M(\varrho) \leq M(H\varrho) + \log AM_0 \quad (A \text{ and } H \text{ are from (M.2)}).$$

The space of Beurling ultradistributions is defined ([2]) as the strong dual of the space

$$\mathcal{D}^{(M_p)} = \text{injlim}_{m \in \mathfrak{N}} \text{projlim}_{n \in \mathfrak{N}} \mathcal{D}_{m,n}^{(M_p)}$$

where

$$\mathcal{D}_{m,n}^{(M_p)} = \{ \varphi \in C^\infty; \text{supp } \varphi \subset K(m), \|\varphi\|_{m,n} < \infty \},$$

$C^\infty = C^\infty(\mathfrak{R}^d)$ is the space of smooth functions on \mathfrak{R}^d , $K(m)$ is the closed ball with the center at zero and with the radius $m > 0$,

$$\|\varphi\|_{m,n} = \sup_{\substack{\alpha \in \mathfrak{N}_d^0 \\ x \in K(m)}} \left\{ \frac{n^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M^{|\alpha|}} \right\} \quad (|\alpha| = \alpha_1 + \dots + \alpha_d).$$

The space $\mathcal{D}'_{L^2}(M_p)$ is defined ([3]) as follows:

$$\mathcal{D}'_{L^2, h}(M_p) = \left\{ \varphi \in C^\infty \cap L^2; \gamma_h(\varphi) = \sum_{\alpha \in \mathfrak{N}_0^q} \frac{h^{|\alpha|} \|\varphi^{(\alpha)}\|_2}{M_{|\alpha|}} < \infty \right\}, \quad h > 0,$$

$$\mathcal{D}'_{L^2}(M_p) = \text{projlim}_{\alpha \in \mathfrak{N}} \mathcal{D}'_{L^2, \alpha}(M_p).$$

The space $\mathcal{D}'_{L^2}(M_p)$ is an *FG*-space (Gelfand space, see [1]) and

$$\mathcal{D}^{(M_p)} \hookrightarrow \mathcal{D}'_{L^2}(M_p) \hookrightarrow \mathcal{D}_{L^2} \quad ([3])$$

where « $A \hookrightarrow B$ » means that A is a dense subspace of B and the inclusion mapping is continuous. (\mathcal{D}_{L^2} is well-known Schwartz space.)

The Fourier transformation of an $f \in L^2$ is defined by

$$(\mathcal{F}_2 f)(\xi) = \text{l.i.m.}_{A \rightarrow \infty} \int_{-A}^A \dots \int_{-A}^A f(x) \exp(i \langle x, \xi \rangle) dx_1, \dots, dx_q,$$

$$\xi \in \mathfrak{R}^q (\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_q \xi_q),$$

where l.i.m. means the square-mean limit. Since we shall use in the paper the Fourier transformations of tempered distributions and of $\mathcal{D}'_{L^2}(M_p)$ -ultradistributions, we indicate these transformations by \mathcal{F}_t and \mathcal{F}_M , respectively. It is well-known that for $f \in L^2 \subset S'$, $\mathcal{F}_2 f = \mathcal{F}_t f$.

Obviously, the sequence of norms γ_k , $k \in \mathfrak{N}$, on $\mathcal{D}'_{L^2}(M_p)$ is equivalent to the following one:

$$\tilde{\gamma}_k(\varphi) = \sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{k^{|\alpha|} \|\varphi^{(\alpha)}\|_2}{M_{(\alpha)}} \right\} = (2\pi)^{-q/2} \sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{k^{|\alpha|} \|(\xi^\alpha(\mathcal{F}_2 \varphi)(\xi))\|_2}{M_{|\alpha|}} \right\}.$$

$$k \in \mathfrak{N}_0(\xi^\alpha = \xi_1^{\alpha_1}, \dots, \xi_q^{\alpha_q}).$$

This implies that $\mathcal{D}'_{L^2}(M_p)$ is isomorphic to the space $D_{L^2}(M_p) = \mathcal{F}_2(\mathcal{D}'_{L^2}(M_p))$ in which we transport the convergence structure from $\mathcal{D}'_{L^2}(M_p)$.

The inverse Fourier transformation of an $f \in L^2$ is defined by

$$(\mathcal{F}_2^{-1} f)(\xi) = (2\pi)^{-q} (\mathcal{F}_2 f)(-\xi), \quad \xi \in \mathfrak{R}^q.$$

Clearly, \mathcal{F}_2^{-1} is an isomorphism of $D_{L^2}(M_p)$ onto $\mathcal{D}'_{L^2}(M_p)$. This implies that

the adjoint mappings

$$\mathcal{F}_M: \mathcal{D}'_{L^s}{}^{(M_p)} \rightarrow \mathcal{D}'_{L^s}{}^{(M_p)}, \quad \mathcal{F}_M^{-1}: \mathcal{D}'_{L^s}{}^{(M_p)} \rightarrow \mathcal{D}'_{L^s}{}^{(M_p)}$$

are isomorphisms. One can easily prove that for $f \in L^2$

$$\mathcal{F}_M f = \mathcal{F}_2 f \quad \text{and} \quad \mathcal{F}_M^{-1} f = \mathcal{F}_2 f.$$

Similarly, we define $\mathcal{F}_M: \mathcal{D}'_{L^s}{}^{(M_p)} \rightarrow \mathcal{D}'_{L^s}{}^{(M_p)}$ and $\mathcal{F}_M^{-1}: \mathcal{D}'_{L^s}{}^{(M_p)} \rightarrow \mathcal{D}'_{L^s}{}^{(M_p)}$. Let $f \in \mathcal{D}'_{L^s}{}^{(M_p)}$ and $\varphi \in \mathcal{D}'_{L^s}{}^{(M_p)}$. We have:

$$(2) \quad \langle \mathcal{F}_M^{-1} f, \mathcal{F}_2 \varphi \rangle = \langle f, \varphi \rangle = \langle \mathcal{F}_M f, \mathcal{F}_2^{-1} \varphi \rangle.$$

Let

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \left(i \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(i \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

An operator of the form $P(D) = \sum_{\alpha \in \mathfrak{N}_0^n} a_\alpha D^\alpha$, a_α are complex numbers, is called the ultradifferential operator of class (M_p) if there are $L > 0$ and $C > 0$ such that

$$(3) \quad |a_\alpha| \leq CL^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathfrak{N}_0^n \quad ([2]).$$

It was proved in [4] that the mapping $\varphi \mapsto P(D)\varphi$ from $\mathcal{D}'_{L^s}{}^{(M_p)}$ into $\mathcal{D}'_{L^s}{}^{(M_p)}$ is continuous. We have

$$\mathcal{F}_2(P(D)\varphi)(\xi) = P(\xi) \mathcal{F}_2(\varphi)(\xi), \quad \xi \in \mathfrak{R}^n.$$

We proved in [4] the following structural theorem: $f \in \mathcal{D}'_{L^s}{}^{(M_p)}$ iff there exist an ultradifferential operator of class (M_p) , $P(D)$, and an $F \in L^2$ such that

$$(4) \quad f = P(D)F.$$

By (2) and (4) we have

$$(5) \quad (\mathcal{F}_M f)(\xi) = P(\xi) (\mathcal{F}_2 F)(\xi), \quad \xi \in \mathfrak{R}^n.$$

Thus, we see that $\mathcal{F}_M f$ is a function from L^2_{loc} :

For the analytic function $P(\zeta) = \sum_{\alpha \in \mathfrak{N}_0^q} a_\alpha \zeta^\alpha$, $\zeta \in \mathfrak{C}^q$, (a_α satisfy (3)) we have

$$(6) \quad |P(\zeta)| \leq (C/M_0) \sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{L^{|\alpha|} M_0 |\zeta|^{|\alpha|}}{M_{|\alpha|}} \right\} \leq C_1 \exp(M(L_1 |\zeta|)), \quad \zeta \in \mathfrak{C}^q,$$

where $C_1 = (C/M_0) \cdot \sum_{\alpha \in \mathfrak{N}_0^q} (L/L_1)^{|\alpha|}$ and $L_1 > L$.

3. Let $f \in \mathcal{D}'_L^{(M_p)}$ and $\varphi \in \mathcal{D}'_L^{(M_p)}$; we define

$$(f * \varphi)(x) = \langle f(t), \varphi(x-t) \rangle, \quad x \in \mathfrak{R}^q.$$

Let us put $\psi(x) = (f * \varphi)(x)$, $x \in \mathfrak{R}^q$, and assume that f is of the form (4). We have

PROPOSITION 1. *For any $\alpha \in \mathfrak{N}_0^q$, $\psi^{(\alpha)}$ is bounded and continuous. Moreover, for any $k > 0$*

$$\beta_k(\psi) = \sup_{\substack{x \in \mathfrak{R}^q \\ \alpha \in \mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|} |\psi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} < \infty.$$

PROOF. For any $x \in \mathfrak{R}^q$ we have

$$\begin{aligned} \psi^{(\alpha)}(x) &= \langle f(t), \varphi^{(\alpha)}(x-t) \rangle = \langle F(t), P(-D)\varphi^{(\alpha)}(x-t) \rangle = \\ &= \int_{\mathfrak{R}^q} F(t) P(-D)(\varphi^{(\alpha)}(x-t)) dt. \end{aligned}$$

Since $P(D)\varphi^{(\alpha)} \in L^2$, the first part of the assertion follows.

Using (6), for $x \in \mathfrak{R}^q$, $\alpha \in \mathfrak{N}_0^q$, we have

$$\begin{aligned} \frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| &\leq \|F\|_2 \frac{k^{|\alpha|}}{M_{|\alpha|}} \|(P(D)\varphi^{(\alpha)})(\xi)\|_2 \leq \\ &\leq C_1 \frac{k^{|\alpha|}}{M_{|\alpha|}} \|F\|_2 \|\exp(M(L_1 |\xi|)) \xi^\alpha (\mathcal{F}_2 \varphi)(\xi)\|_2. \end{aligned}$$

Thus by (1) we have

$$\begin{aligned} \sup_{\substack{x \in \mathfrak{R}^q \\ \alpha \in \mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| \right\} &\leq C \|\exp(M(L_1 |\xi|) + M(k|\xi|)) (\mathcal{F}_2 \varphi)(\xi)\|_2 \leq \\ &\leq C_0 \|\exp(M((L_1 + k)|\xi|)) (\mathcal{F}_2 \varphi)(\xi)\|_2 \end{aligned}$$

where C and C_0 are suitable constants. Since the sequence of norms

$$\nu_k(\varphi) = \|\exp(M(k|\xi|))(\mathcal{F}_2\varphi)(\xi)\|_2, \quad k \in \mathfrak{N},$$

is equivalent to the sequence γ_k , $k \in \mathfrak{N}$ ([4, Proposition 2.1]), the second assertion of the proposition follows.

If $\varphi, \psi \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$, then $\mathcal{F}_2(\varphi * \psi) = (\mathcal{F}_2\varphi)(\mathcal{F}_2\psi)$ ([5]). (In this case $*$ is the ordinary convolution.)

PROPOSITION 2. *Let $f \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$ and $\varphi \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$. Then*

$$\mathcal{F}_t(f * \varphi)(\xi) = (\mathcal{F}_M f)(\xi)(\mathcal{F}_2\varphi)(\xi), \quad \xi \in \mathfrak{R}^a.$$

PROOF. We assume that f is of the form (4). We have

$$\begin{aligned} (f * \varphi)(x) &= \langle P(D)F(t), \varphi(x-t) \rangle = \langle F(t), P(-D)\varphi(x-t) \rangle = \\ &= \int_{\mathfrak{R}^a} F(t)P(-D)\varphi(x-t) dt = (F * P(D)\varphi)(x). \end{aligned}$$

since $F \in L^2$ and $P(D)\varphi \in L^2$, from the remark given before Proposition 2 and (5), the assertion follows.

4. Let $g \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$ such that for every $\varphi \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$, $g * \varphi \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$. Then we call g the convolution-operator or convolutor. The space of all convolutors is denoted by $\mathcal{O}'_C(\mathcal{D}'_{L^s}(\mathbb{R}^a), \mathcal{D}'_{L^s}(\mathbb{R}^a))$ or in short, by $\mathcal{O}'_C(\mathbb{R}^a)$. (Note that we do not assume the continuity of the mapping $\varphi \rightarrow g * \varphi$, in $\mathcal{D}'_{L^s}(\mathbb{R}^a)$. This will be proved in Proposition 5.)

Proposition 2 implies.

PROPOSITION 3. *Let $g \in \mathcal{O}'_C(\mathbb{R}^a)$ and $\varphi \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$. Then*

$$\mathcal{F}(g * \varphi)(\xi) = (\mathcal{F}_M g)(\xi)(\mathcal{F}_2\varphi)(\xi), \quad \xi \in \mathfrak{R}^a.$$

PROPOSITION 4. *A $g \in \mathcal{D}'_{L^s}(\mathbb{R}^a)$ is from $\mathcal{O}'_C(\mathbb{R}^a)$ iff there exists $k > 0$ such that*

$$(7) \quad (\mathcal{F}_M g) \exp(-M(k|\cdot|)) \in L^\infty.$$

PROOF. From (7) it follows that for any $\varphi \in \mathcal{D}'_{L^2}(\mathcal{M}^p)$ and any $\alpha \in \mathfrak{N}_0^q$

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g) \mathcal{F}_2(\varphi^{(\alpha)}) \in L^2$$

and

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = \mathcal{F}_2(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g)(-i\xi)^\alpha (\mathcal{F}_2 \varphi).$$

Let $r > 0$. From (7) and (1) we obtain

$$\begin{aligned} \sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{r^{|\alpha|}}{M^{|\alpha|}} \|g * \varphi^{(\alpha)}\|_2 \right\} &\leq (2\pi)^{-q/2} \left\| \sup_{\alpha} \left\{ \frac{r^{|\alpha|}}{M^{|\alpha|}} |(\mathcal{F}_M g)(\xi)| |\xi|^\alpha |(\mathcal{F}_2 \varphi)(\xi)| \right\} \right\|_2 < \\ &\leq C \|\exp(M(r|\xi|)) \exp(M(k|\xi|))(\mathcal{F}_2 \varphi)(\xi)\|_2 \\ &\leq \bar{C} \|\exp(M((r+k)|\xi|))(\mathcal{F}_2 \varphi)(\xi)\|_2. \end{aligned}$$

Thus, we have proved

$$(8) \quad \tilde{\gamma}_r(g * \varphi) \leq \bar{C} v_{r+k}(\varphi), \quad \varphi \in \mathcal{D}'_{L^2}(\mathcal{M}^p).$$

Conversely, let $g \in \mathcal{O}'_c(\mathcal{M}^p)$ and φ be an arbitrary element from $\mathcal{D}'_{L^2}(\mathcal{M}^p)$. Since $\mathcal{F}_2(g * \varphi) = (\mathcal{F}_M g)(\mathcal{F}_2 \varphi)$ and $v_k(g * \varphi) < \infty$ for every $k > 0$, we have

$$(9) \quad \|(\mathcal{F}_M g) \exp(M(k|\cdot|))(\mathcal{F}_2 \varphi)\|_2 < \infty \quad \text{for every } k > 0.$$

It can be proved by (9) that $\mathcal{F}_M g$ is from L^∞_{loc} . Namely, for any open ball $\mathring{K}(m)$, $L^2(\mathring{K}(m))$ can be embedded into $D'_{L^2}(\mathcal{M}^p)$ in a natural way:

$$L^2(\mathring{K}(m)) \ni \psi \mapsto \tilde{\psi} = \begin{cases} \psi & \text{on } \mathring{K}(m) \\ 0 & \text{on } \mathfrak{R}^q \setminus \mathring{K}(m) \end{cases} \in D'_{L^2}(\mathcal{M}^p).$$

Since the condition

$$\|H(\xi)\psi(\xi)\|_{L^2(\mathring{K}(m))} < \infty \quad \text{for every } \psi \in L^2(\mathring{K}(m)),$$

implies $H \in L^\infty(\mathring{K}(m))$, we obtain that $\mathcal{F}_M g \in L^\infty_{\text{loc}}$.

Let us prove (7) by proving that g does not belong to $\mathcal{O}'_c(\mathcal{M}^p)$ if (7) does not hold.

If (7) does not hold there exists a sequence a_j such that $a_1 > 1$,

$a_{j+1} > a_j + 1$ and

$$|(\mathcal{F}_M g)(\xi)| \exp(-M(j|\xi|)) \geq j \quad \text{if } |\xi| \in A_j \subset (a_j, a_{j+1})$$

with $\text{mes } A_j = \varepsilon_j > 0$, $j \in \mathfrak{N}$.

Put

$$\psi_j(\xi) = \begin{cases} \varepsilon_j^{-1} \exp(-M(j|\xi|)), & |\xi| \in A_j, j \in \mathfrak{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and $\psi = \sum_{j=1}^{\infty} \psi_j$. Since $\exp(-M(|\xi|))$ decreases monotonically faster than any power of $1/|\xi|$ when, $|\xi| \rightarrow \infty$, one can prove that $\psi \in D_{L^1}^{(M_p)}$ and that $(\mathcal{F}_M g)\psi$ does not belong to $D_{L^1}^{(M_p)}$.

The proof is completed.

From (8) directly follows:

PROPOSITION 5. *If $g \in \mathcal{O}'_C^{(M_p)}$ then the mapping*

$$\varphi \mapsto g * \varphi \quad \text{from } \mathcal{D}'_{L^1}^{(M_p)} \text{ into } \mathcal{D}'_{L^1}^{(M_p)}$$

is continuous.

The last assertion enables us to define the convolution of an $f \in \mathcal{D}'_{L^1}^{(M_p)}$ and a $g \in \mathcal{O}'_C^{(M_p)}$ in a usual way:

$$\langle f \star g, \varphi \rangle = \langle f, \check{g} * \varphi \rangle \quad \text{where } \check{g}(x) = g(-x), x \in \mathfrak{R}.$$

PROPOSITION 6. *If $f \in \mathcal{D}'_{L^1}^{(M_p)}$ and $g \in \mathcal{O}'_C^{(M_p)}$ then,*

$$\mathcal{F}_M(f \star g) = (\mathcal{F}_M f)(\mathcal{F}_M g).$$

PROOF. For $\psi \in D_{L^1}^{(M_p)}$ we have

$$\begin{aligned} \langle \mathcal{F}_M(f \star g), \psi \rangle &= \langle f \star g, \mathcal{F}_2 \psi \rangle = \langle f, \check{g} * \mathcal{F}_2 \psi \rangle = \\ &= \langle \mathcal{F}_M f, \mathcal{F}_2^{-1}(\check{g} * \mathcal{F}_2 \psi) \rangle = \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-a} \mathcal{F}_2(\check{g} * \mathcal{F}_2 \psi)(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-a} (\mathcal{F}_M g)(\xi) (\mathcal{F}_2(\mathcal{F}_2 \psi))(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (\mathcal{F}_M g)(\xi) \psi(\xi) \rangle. \end{aligned}$$

This implies the assertion.

If g and f are from $\mathcal{O}'_C^{(M_p)}$ then $f \star g \in \mathcal{O}'_C^{(M_p)}$.

This follows from the definition of $\mathcal{O}'_C(\mathcal{M}_p)$ and the fact that

$$(f \star g) * \varphi = f \star (g * \varphi) = f * (g * \varphi), \quad \varphi \in \mathcal{D}'_{L^*}(\mathcal{M}_p).$$

Properties of the convolution are given in the next

PROPOSITION 7. *Let $g, h \in \mathcal{O}'_C(\mathcal{M}_p)$ and $f \in \mathcal{D}'_{L^*}(\mathcal{M}_p)$.*

- (i) $g \star h = h \star g$;
- (ii) $(f \star g) \star h = f \star (g \star h)$.

PROOF. (i) We have

$$\begin{aligned} \langle g \star h, \varphi \rangle &= \langle g, \check{h} * \varphi \rangle = (g * (\check{h} * \varphi))(0); \\ \langle h \star g, \varphi \rangle &= (h * (\check{g} * \varphi))(0). \end{aligned}$$

Proposition 3 implies that $(h * (\check{g} * \varphi))(x) = (g * (\check{h} * \varphi))(x)$, $x \in \mathfrak{R}^a$.

(ii) Since

$$\begin{aligned} \langle f \star (g \star h), \varphi \rangle &= \langle f, (g \star h) * \varphi \rangle = \langle f(x), (g \star h)(t), \varphi(x+t) \rangle = \\ &= \left\langle f(x), \langle g(t), \langle \check{h}(u), \varphi(x+t-\alpha) \rangle \rangle \right\rangle = \\ &= \langle f(x), \langle g(t), (\check{h} * \varphi)(x+t) \rangle \rangle = \langle f(x), \check{g} * (\check{h} * \varphi)(x) \rangle, \end{aligned}$$

and

$$\langle (f \star g) \star h, \varphi \rangle = \langle (f \star g), (\check{h} * \varphi) \rangle = \langle f, \check{g} * (\check{h} * \varphi) \rangle,$$

the assertion is proved.

5. Observe the convolution equation in $\mathcal{D}'_{L^*}(\mathcal{M}_p)$:

$$(10) \quad S \star U = V,$$

where $S \in \mathcal{O}'_C(\mathcal{M}_p)$ and $V \in \mathcal{D}'_{L^*}(\mathcal{M}_p)$ are known ultradistributions and $U \in \mathcal{D}'_{L^*}(\mathcal{M}_p)$ is the unknown one.

Denote by $\mathcal{O}'_{C, \mathfrak{A}}(\mathcal{M}_p)$ the space of all convolutors $S \in \mathcal{O}'_C(\mathcal{M}_p)$ for which $\mathcal{F}_M S$ is a smooth function which has the analytic continuation onto the whole \mathbb{C}^a .

PROPOSITION 8. *Let $S \in \mathcal{O}'_C(\mathcal{A}^p)$. The sufficient condition that the equation (10) is solvable in $\mathcal{D}'_{L^2}(\mathcal{M}_p)$ for any $V \in \mathcal{D}'_{L^2}(\mathcal{M}_p)$ is the following one: There exist $C > 0$ and $k > 0$ such that*

$$(11) \quad |(\mathcal{F}_M S)(\zeta)| \geq \frac{C}{\exp(M(k|\zeta|))}, \quad \zeta \in \mathbb{C}^q.$$

PROOF. Assume that (11) holds. Put

$$u(\xi) = \frac{(\mathcal{F}_M V)(\xi)}{(\mathcal{F}_M S)(\xi)}, \quad \xi \in \mathfrak{R}^q.$$

From (5) it follows that for some ultradifferential operator P of class (M_p) and some $v \in L^2$

$$u(\xi) = \frac{P(\xi)}{(\mathcal{F}_M S)(\xi)} v(\xi), \quad \xi \in \mathfrak{R}^q.$$

Let $P_1(\xi) = P(\xi)/(\mathcal{F}_M S)(\xi)$, $\xi \in \mathfrak{R}^q$. From [2, Proposition 4.5] it follows that $P_1(D)$ is an ultradifferential operator of class (M_p) . Thus, by (5) again, we obtain that the solution of (10) is

$$U = P_1(D) \mathcal{F}_2^{-1} v.$$

PROPOSITION 9. *Let $S \in \mathcal{O}'_C(\mathcal{M}_p)$. Then the necessary condition for the solvability of (10) for any $V \in \mathcal{D}'_{L^2}(\mathcal{M}_p)$ is the following one: There exist $C > 0$, $D > 0$ and $k > 0$ such that*

$$(12) \quad |(\mathcal{F}_M S)(\xi)| \geq C \exp(-M(k|\xi|)), \quad |\xi| \geq D.$$

PROOF. Assume that (10) is solvable for any $V \in \mathcal{D}'_{L^2}(\mathcal{M}_p)$ but (12) does not hold. This implies that there exists a sequence of sets A_j , $j \in \mathfrak{N}$, such that $\text{mes } A_j = \varepsilon_j > 0$, $A_j \subset (a_j, a_{j+1})$, $a_1 > 1$, $a_{j+1} > a_j + 1$, $j \in \mathfrak{N}$ and

$$|(\mathcal{F}_M S)(\xi)| \leq j^{-1} \exp(-M(j^2|\xi|)), \quad |\xi| \in A_j, \quad j \in \mathfrak{N}.$$

As in the proof of Proposition 4, put

$$v_j = \begin{cases} \varepsilon_j^{-1} \exp(-M(j|\xi|)), & |\xi| \in A_j, \quad j \in \mathfrak{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$v = \sum_{j=1}^{\infty} v_j.$$

Clearly $v \in \mathcal{D}'_{L^s}(\mathbb{R}^n)$ but

$$v/(\mathcal{F}_M S) \notin \mathcal{D}'_{L^s}(\mathbb{R}^n).$$

Thus, for such $V = \mathcal{F}_M^{-1}(v)$ and S the solution of (10) does not exist in $\mathcal{D}'_{L^s}(\mathbb{R}^n)$.

6. We say that equation (10) satisfies the hypoellipticity condition if the existence of the solution $U \in \mathcal{D}'_{L^s}(\mathbb{R}^n)$ of (10) and $V \in \mathcal{D}'_{L^s}(\mathbb{R}^n)$ imply that $U \in \mathcal{D}'_{L^s}(\mathbb{R}^n)$. In this case we say that U is the hypoelliptic solution of (10).

PROPOSITION 10. *Let $S \in \mathcal{O}'_C(\mathbb{R}^n)$. Equation (10) is hypoelliptic iff (12) holds.*

PROOF. Let $\tilde{\psi}(\xi) = 1$ for $|\xi| \leq D + 1$ and $\tilde{\psi}(\xi) = 0$ for $|\xi| > D + 1$. This is an element of $\mathcal{D}'_{L^s}(\mathbb{R}^n)$.

Put

$$\tilde{P}(\xi) = \frac{1 - \tilde{\psi}(\xi)}{(\mathcal{F}_M S)(\xi)}.$$

We have

$$|\tilde{P}(\xi)| \leq C \exp M(k|\xi|), \quad \xi \in \mathbb{R}^n.$$

Obviously, $P = \mathcal{F}_M^{-1} \tilde{P} \in \mathcal{O}'_C(\mathbb{R}^n)$ and

$$S \star P = \delta - \psi, \quad \text{where } \psi = \mathcal{F}^{-1}(\tilde{\psi}).$$

Since

$$U = U * \delta = V \star P + U \star \psi$$

one can easily prove that $U \in \mathcal{D}'_{L^s}(\mathbb{R}^n)$.

By the same arguments as in the second part of the proof of Proposition 9 one can prove that if (10) is hypoelliptic then (12) holds for S .

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Added in proof. Proposition 4 implies that $(\mathcal{F}_M S)(\xi) = s(\xi) \exp [M(k|\xi|)]$, $\xi \in \mathfrak{R}^n$, for some $k > 0$ and some $s \in L^\infty$, and from (4) we have $(\mathcal{F}_M V)(\xi) = = P(\xi)v(\xi)$, $\xi \in \mathfrak{R}^n$, where P is an ultradifferential operator of class (M_p) and $v \in L^2$. So, by using the Fourier transformation we have:

« (10) is solvable in $\mathcal{D}_L^{(M_p)}$ if the equation $su = v$ has a solution $u \in L^2$ ».

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