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On Absolutely Simple Locally Finite Groups.

RICHARD E. PHILLIPS (*)

1. Introduction.

A well-known result of Kegel [5; pp. 172-173] (or [6; p. 115]) asserts that if G is a countably infinite locally finite simple group then

(1.1) there is an ascending chain $F_1 \subseteq \dots \subseteq F_n \subseteq \dots$ of finite subgroups of G satisfying

(a) $\bigcup F_n = G$, and

(b) for each $n > 1$ there is a maximal normal subgroup M_n of F_n such that $F_{n-1} \cap M_n = 1$.

The import of this result lies in the display of finite simple sections of unbounded orders in the finite subgroups of a countably infinite locally finite simple G . In general, the condition (1.1) does not imply simplicity [6; p. 116]. Indeed, there are countably infinite residually finite groups satisfying (1.1).

A minor adaptation of Kegel's arguments can be used to strengthen (1.1) to a condition equivalent to simplicity, and we will give such a condition in Theorems 1 and 2. In these same theorems we give a similar criteria for the absolute simplicity of G .

Recall that G is *absolutely simple* if the only composition series of G is the one consisting of 1 and G only; equivalently, G is absolutely

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simple if the only serial subgroups of G are 1 and G (see [11; I, p. 12, p. 16] or our § 2.1 for the relevant definitions). Obviously, every absolutely simple group is simple and, in general, the absolutely simple groups form a proper subclass of the class of simple groups ([1] or [11; II, 3.4]). However, it is not known whether or not every *locally finite* simple group is absolutely simple. Our Theorems 1 and 2 put the (possible) differences between these two (locally finite) classes in a «local» context.

In the sequel, we frequently encounter ascending chains $F_1 \subseteq \dots \subseteq F_n \subseteq \dots$ of finite subgroups of the countably infinite locally finite G with $\bigcup F_n = G$. Such a chain is called an *approximating sequence* of G (caution; this term is used in a different way in [6; p. 116]).

Part of Theorem 1 is stated in terms of subnormal subgroups. Recall that if $M \subseteq G$, the *standard series* of M in G is defined inductively by

$$M(0, G) = G \quad \text{and for } n \leq 1, \quad M(n, G) = M^{M(n-1), G}.$$

We also have occasion to use the subgroups

$$M(\omega, G) = \bigcap \{M(n, G) : n \geq 0\}.$$

THEOREM 1. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G .

- a) If G is simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of G satisfying
- i) $F_1 = D_{n_1}$, and for $k > 1$, $F_k = D_{n_{k-1}}(1, D_{n_k}) = D_{n_{k-1}}^{D_{n_k}}$ and
 - ii) for $k > 1$, if $V \not\subseteq F_k$ then there is an $x \in N_{F_{k-1}}(F_k)$ such that $V^x \cap F_{k-1} = 1$.
- b) If G is absolutely simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of G satisfying
- i) $F_1 = D_{n_1}$, and for $k > 1$, $F_k = D_{n_{k-1}}(\omega, D_{n_k})$, and
 - ii) for $k > 1$, if $V \not\subseteq F_k$, then $V \cap F_{k-1} = 1$.

Obviously, the condition in (ii) of part (a) implies (1.1).

A type of converse is provided in

THEOREM 2. Suppose $\{F_k\}$ is an approximating sequence of the countably infinite locally finite G .

- a) If $\{F_k\}$ satisfies the property (ii) of Theorem 1(a), then G is simple.
- b) If $\{F_k\}$ satisfies the property (ii) of Theorem 1(b), then G is absolutely simple.

An interesting interplay between Theorems 1 and 2 is that the existence of a single approximating sequence $\{F_k\}$ of G satisfying condition (ii) of Theorem 1(a) (or Theorem 1(b)) implies that a sequence with similar properties can be extracted from any approximating sequence (as in part a(i) of Theorem 1). The conditions a(ii) and b(ii) of Theorem 1 accentuate the possible differences between the countable « simple » and « absolutely simple » locally finite groups.

We also note that the condition (ii) of Theorem 1(b) is equivalent to

$$(1.2) \quad \text{for all } k \geq 1, 1 \neq x \in F_k \text{ implies } x^{F_{k+1}} = F_{k+1}.$$

While the above results are stated for countable groups, they can be extended to higher cardinalities by employing « countable » local theorems for the classes of simple and absolutely simple groups. The available theorems are recorded in

- (1.3) The infinite group G is simple (absolutely simple) if and only if G has a local system of countable simple (absolutely simple) subgroups (see [6; p. 114], [9; p. 190], [7; p. 131] for the simple case and [3; p. 529] for the absolutely simple case).

It follows immediately that Theorems 1 and 2 can be formulated in terms of the countable subgroups of the locally finite G .

Our final result gives a sufficient condition for absolute simplicity.

THEOREM 3. Let G be a countably infinite locally finite simple group and $\{D_n\}$ an approximating sequence of G . If there is a $d \geq 0$ such that every perfect subnormal subgroup of D_n has defect at most d in D_n , then G is absolutely simple.

The definition of the defect of a subnormal subgroup is given later in § 2.1; see also [11; I, p. 173].

It is not difficult to see that the perfect subnormal subgroups of a finite group L all have defect d or less if and only if the perfect subnormal subgroups of L/H have defect d or less, where H is the solvable

radical of L . Since all known countably infinite locally finite simple groups have approximating sequences $\{D_n\}$ where for $n \geq 1$, $D_n/\zeta(D_n)$ is a direct product of non-Abelian simple groups (see [10; p. 385]) we have, as a consequence of Theorem 3 and the above remarks on extensions to higher cardinals,

(1.4) all known locally finite simple groups are absolutely simple.

As a final remark, we point out, that with minor modifications, the groups $\{F_k\}$ of Theorem 1 can always be chosen to be perfect. To see this, let $\{D_n\}$ be an approximating sequence of the simple G and denote by D_n^ω the intersection of the members of the derived series of D_n . Since G cannot be locally solvable [11; p. 154] and $\bigcup \{D_n^\omega: n \geq 1\}$ is a normal subgroup of G , we must have $\bigcup \{D_n^\omega: n \geq 1\} = G$. Thus, $\{D_n^\omega\}$ is an approximating sequence of perfect subgroups of G . If the groups $\{F_k\}$ are chosen relative to $\{D_n^\omega\}$ (rather than $\{D_n\}$), the F_k 's will also be perfect.

2. Proofs.

2.1. Remarks on serial and subnormal subgroups.

The standard series of M in G has been defined in § 1. It is frequently easier to work with the commutator form

$$M(n, G) = M[G, nM]$$

where $[G, nM]$ is defined inductively by $[G, 0M] = G$ and for $n \geq 1$, $[G, nM] = [[G, (n-1)M], M]$ (c.f. [11; I, p. 173]). The subgroup M of G is subnormal in G written $M \triangleleft\triangleleft G$ if and only if $M = M(n, G)$ for some $0 \leq n < \omega$. Equivalently, $M \triangleleft\triangleleft G$ if and only if $[G, nM] \subseteq M$ for some $n \geq 0$. If $M \triangleleft\triangleleft G$, the minimal n for which $M = M(n, G)$ is called the *defect* of M in G . The symbol $M \triangleleft\triangleleft_n G$ will mean that M is subnormal in G of defect n or less.

Several useful facts are given in

- (2.1.1) (a) if $N \subseteq M \triangleleft\triangleleft_n G$, then $N(n, G) \subseteq M$.
 (b) If $N \subseteq M \triangleleft\triangleleft G$, then $N(\omega, G) \subseteq M$.
 (c) If G is finite and $M \subseteq G$, then $M^{M(\omega, G)} = M(\omega, G)$.

Our use of the term normal series coincides with the normal systems of Kurosh [8; p. 171] and is essentially equivalent to the series of Robinson [11; I, pp. 9-10]. A subgroup H of G is a serial subgroup of G (written $H \text{ ser } G$) if there is a normal series \mathcal{C} of G with $H \in \mathcal{C}$. We will need the following «local» characterization of serial subgroups [4; Theorem 2] (or in the locally finite case [2; Lemma 2]).

(2.1.2) If $H \subseteq G$, then $H \text{ ser } G$ if and only if for every finitely generated $F \subseteq G$, $F \subseteq H^F$ implies $F \subseteq H$.

An essential lemma for our arguments is

LEMMA 1. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G and suppose that for each $n \geq 1$ we have a subgroup $M_n \triangleleft\triangleleft D_n$ and that $n > m$ implies $M_m \subseteq M_n$. Then $M = \bigcup \{M_n : n \geq 1\} \text{ ser } G$. Further, if there is a $d \geq 0$ such that $M_n \triangleleft\triangleleft_d D_n$ for all $n \geq 1$ then $M \triangleleft\triangleleft_d G$.

PROOF. For the first part, we use the criterion (2.1.2). Let F be a finite subgroup of G and suppose that $F \subseteq M^F$. Then there is an n such that $F \subseteq D_n$ and $F \subseteq M_n^F$. Since $M_n \triangleleft\triangleleft \langle M_n, F \rangle \subseteq D_n$ we have $F \subseteq M_n \subseteq M$ as desired.

Suppose now that $M_n \triangleleft\triangleleft_d D_n$ for all n . Then, for $n \geq 1$, $[D_n, dM_n] \subseteq M_n$ and so

$$[G, dM] = \bigcup \{[D_n, dM_n] : n \geq 1\} \subseteq M.$$

Thus, $M \triangleleft\triangleleft_d G$ and this completes the proof.

2.2. For the proof of Theorem 1 we require the following lemma. The proof follows the lines of argument given in [6; pp.112-114].

LEMMA 2. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G and put $D = D_1$.

- a) If G is simple and $d \geq 0$ then
 - i) $\{D(d, D_n) : n \geq 1\}$ is an approximating sequence of G , and
 - ii) there is a positive integer j such that for $n \geq j$, $Y \triangleleft\triangleleft_d D_n$ implies $Y \cap D \in \{1, D\}$.
- b) If G is absolutely simple, then
 - i) $\{D(\omega, D_n) : n \geq 1\}$ is an approximating sequence of G , and

ii) there is a positive integer j such that for $n \geq j$, $Y \triangleleft \triangleleft D_n$ implies $Y \cap D \in \{1, D\}$.

PROOF. For the proof of (i) of part (a), note first that for $n \geq 1$, $D(d, D_n) \subseteq D(d, D_{n+1})$. From Lemma 1 we have $V = \bigcup \{D(d, D_n) : n \geq 1\} \triangleleft \triangleleft_a G$ and the simplicity of G forces $V = G$.

Part (i) of (b) follows similarly; in this case we have $V = \bigcup \{D(\omega, D_n) : n \geq 1\}$ ser G (by Lemma 1) and since G is absolutely simple, $V = G$.

Proceeding to part (ii) of (a), suppose that there is no j with the asserted property. There is then an approximating sequence $\{P_n\} \subseteq \{D_n\}$ and subgroups $Y_n \triangleleft \triangleleft_a P_n$ such that $Y_n \cap D \notin \{1, D\}$. Since D is finite there is a subgroup M of D with $M \notin \{1, D\}$ and an approximating sequence $\{E_n\} \subseteq \{P_n\}$ such that for $n \geq 1$ there are subgroups $X_n \triangleleft \triangleleft_a E_n$ with $X_n \cap D = M$. Now for $n \geq 1$, $M(d, E_n) \subseteq X_n$ (by (2.1.1)(a)) and so $M = D \cap M(d, E_n)$. From part (i) we have $G = \bigcup \{M(d, E_n) : n \geq 1\}$ and the contradiction $M = D \cap G$ now follows.

The proof of b(ii) is identical with that of a(ii); in the same manner we arrive at an approximating sequence $\{E_n\} \subseteq \{D_n\}$ and subgroups $X_n \triangleleft \triangleleft E_n$ with $M \notin \{1, D\}$. The fact that $\bigcup \{M(\omega, E_n) : n \geq 1\} = G$ (part (i) of (b)) together with $M = D \cap M(\omega, E_n)$ for $n \geq 1$ again yields the contradiction $D = M$.

2.3 PROOF OF THEOREM 1. Let $\{D_n\}$ be an approximating sequence of G and suppose G is simple. If F, S are finite subgroups of G there is, by Lemma 2(a) a positive integer $j = j(F, S)$ such that $\langle F, S \rangle \subseteq F(1, D_j) = \mu(F, S)$ and $Y \triangleleft \triangleleft_2 D_j$ implies $Y \cap F \in \{1, F\}$. If $V \not\triangleleft \mu(F, S)$ then $V \triangleleft \triangleleft_2 D_j$ and so $V \cap F \in \{1, F\}$. Further, if $F \subseteq V$ and $L = \text{Core}_{D_j}(V)$ then $L \cap F \in \{1, F\}$. If $F \subseteq L$, then $L = F^{D_j} = \mu(F, S)$ which contradicts the fact that $V \neq \mu(F, S)$. Thus, $L \cap F = 1$ and since for every $x \in D_j$ we have $V^x \cap F \in \{1, F\}$, there must be an $x \in D_j$ with $V^x \cap F = 1$.

Now for the construction of the desired subsequence $\{F_k\}$. Put $F_1 = D_1$, $F_2 = \mu(F_1, D_2)$ and $F_3 = \mu(F_2, D_{j_3})$ where $j_3 = \max \{3, j(F_1, D_2)\}$; for $k > 3$, let $F_k = \mu(F_{k-1}, D_{j_k})$ where $j_k = \max \{k, j(F_{k-1}, D_{j_{k-1}})\}$. One checks easily that $\{F_k\}$ has the properties listed in Theorem 1(a).

For the proof of (b), let $\{D_n\}$ be an approximating sequence of the absolutely simple G and F and S finite subgroups of G . Using Lemma 2(b) there is a positive integer $j = j(F, S)$ such that

$\langle F, S \rangle \subseteq F(\omega, D_j) = \mu(F, S)$ and $Y \triangleleft \triangleleft D_j$, implies $Y \cap F \in \{1, F\}$. If $V \not\subseteq \mu(F, S)$ and $F \subseteq V$ we have $F \subseteq V \triangleleft \triangleleft \mu(F, S)$ which forces $\mu(F, S) = V$ (by (2.1.1)(b)). From this we conclude that $F \cap V = 1$.

The sequence $\{F_k\}$ satisfying (i) and (ii) of Theorem 1(b) may now be constructed as follows:

$$F_1 = D_1, \dots, \quad F_n = \mu(F_{n-1}, D_n), \dots$$

2.4 PROOF OF THEOREM 2. Suppose G has an approximating sequence $\{F_k\}$ satisfying the property (ii) of Theorem 1(a) and let $1 \neq H \triangleleft G$. Then for some k_0 , $k \geq k_0$ implies $H \cap F_k \neq 1$. For any such k , $f_k \cap (F_{k+1} \cap H)^x = F_k \cap H = 1$ for any $x \in N(F_{k+1})$. Thus, $F_{k+1} \cap H = F_{k+1}$ and this forces $H = G$. We have shown that G is simple and this concludes the proof of part (a).

For part (b), suppose the approximating sequence $\{F_k\}$ satisfies the property (ii) of Theorem 1(b) and that $1 \neq H \text{ ser } G$. As above, there is a k_0 such that $k \geq k_0$ implies $H \cap F_k \neq 1$. Thus, if $k \geq k_0$, $1 \neq H \cap F_k \triangleleft \triangleleft F_k$ and $(H \cap F_k) \cap (H \cap F_{k+1})^{F_{k+1}} \neq 1$. This gives $(H \cap F_{k+1})^{F_{k+1}} = F_{k+1}$ and we conclude that $(H \cap F_{k+1}) = F_{k+1}$. It follows that $H = G$ and that G is absolutely simple.

2.5. Prior to our proof of Theorem 3, we need

(2.5.1) If H is a serial locally solvable subgroup of a locally finite G , then H^G is also locally solvable.

The proof of (2.5.1) is straightforward and will not be given here. For the proof of Theorem 3, let G be a countable simple locally finite group and $\{D_n\}$ an approximating sequence such that for each n , the perfect subnormal subgroups of D_n are of defect d or less. Now let $H \text{ ser } G$ with $1 \neq H$; from (2.5.1) and the fact that simple locally solvable groups are finite [11; I, p. 154], we see that H is not locally solvable. Thus, there is an n_0 such that for $n \geq n_0$, $H \cap D_n$ is not solvable. Consequently, if $n \geq n_0$, the subgroup $(H \cap D_n)^\omega$, the intersection of the terms of the derived series of $H \cap D_n$, is a non-trivial perfect subnormal subgroup of D_n . From our assumptions, we have $(H \cap D_n)^\omega \triangleleft \triangleleft_a D_n$. Lemma 1 now implies that $V = \bigcup \{(H \cap D_n)^\omega : n \geq 1\} \triangleleft \triangleleft_a G$ and the simplicity of G forces $V = G$. Since $V \subseteq H$, we have $H = G$ also, and G is absolutely simple.

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