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Carter subgroups and injectors in a class of locally finite groups

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Carter Subgroups and Injectors in a Class of Locally Finite Groups.

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1. Introduction.

The class of $\mathcal{U}$ of locally finite groups was introduced in [4], where a theory of saturated formations was developed in an arbitrary subclass of $\mathcal{U}$, closed under subgroups and homomorphic images. Many other results from the theory of finite soluble groups have since been extended to $\mathcal{U}$, and our main aim here is to develop the basic theory of Fitting classes and their associated injectors.

The class $\mathcal{U}$ was originally defined as the largest subgroup closed class of locally finite groups satisfying the conditions:

(U1) If $G \in \mathcal{U}$, then $G$ has a series $1 = G_0 < G_1 < \ldots < G_n = G$ with locally nilpotent factors,

(U2) If $G \in \mathcal{U}$ and $\pi$ is any set of primes, then the Sylow (that is maximal) $\pi$-subgroups of $G$ are conjugate in $G$.

It was shown in [7] that the first condition is redundant, as it is implied by the second. In fact a much stronger result was obtained Lemma 4.2. of [7] shows that

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LEMMA 1.1. If $G \in \mathcal{U}$, then $G$ has a series of normal subgroups

$$1 < N < A < B < G$$

with $N$ locally nilpotent, $A/N$ abelian of finite rank, $B/A$ abelian with finite primary components, and $G/B$ finite.

In particular, $G/N$ is hyperfinite, in the sense that it has an ascending series of normal subgroups with finite factors.

We shall need to consider more general series. Let $\Omega$ be a totally ordered set. A series of type $\Omega$ of a group $G$ is a set $(U_\sigma, V_\sigma : \sigma \in \Omega)$ of pairs of subgroups of $G$, indexed by $\Omega$, such that

1. $V_\sigma \triangleleft U_\sigma$ for all $\sigma \in \Omega$,
2. $U_\tau \triangleleft V_\sigma$ if $\tau < \sigma$,
3. $G - 1 = \bigcup_{\sigma \in \Omega} (U_\sigma - V_\sigma)$.

Such a series is called a normal series of $G$ if the subgroups $U_\sigma$, $V_\sigma$ are all normal in $G$. If the index set $\Omega$ is well ordered, then $\Omega$ can be taken to be a set of ordinals and the above reduces to the usual concept of an ascending series, with $U_\sigma = V_{\sigma+1}$ and, for limit ordinals $\sigma$,

$$\bigcup_{\tau < \sigma} V_\tau = \bigcup_{\tau < \sigma} U_\tau = V_\sigma.$$

A subgroup $H$ of $G$ is said to be serial in $G$ (written $H \text{ ser } G$) if $H$ is a member of some series of $G$, and ascendant in $G$ (written $H \text{ asc } G$) if $H$ is a member of some ascending series of $G$. If $H$ is serial in a locally finite group $G$ and $N \triangleleft G$, then $HN \text{ ser } G$ (see [6], and also [5, Corollary E1]). This makes it easy to see that a serial subgroup of a hyperfinite locally finite group is ascendant, and hence that if $G$ is locally finite, $N \triangleleft G$, $G/N$ is hyperfinite and $H \text{ ser } G$, then $HN \text{ asc } G$. This remark will be crucial in the proof of our main result on injectors.

For our results on injectors we work within an arbitrary but fixed subclass $\mathcal{K}$ of $\mathcal{U}$ satisfying

(K1) $\mathcal{K}$ is closed under taking subgroups.

(K2) If $G \in \mathcal{K}$ and $C_p$ is a cyclic group of prime order $p$, then $G \times C_p \in \mathcal{K}$.

(Classes of groups, as usual, are taken to be closed under isomorphisms and to contain the trivial groups.) It follows that $\mathcal{K}$ contains all cyclic groups of prime order. From now on, $\mathcal{K}$ denotes a class satisfying
these conditions. Among the many possibilities for $\mathcal{K}$ are the class of all finite soluble groups, the class of all periodic locally soluble groups having a locally nilpotent subgroup of finite index, the class of periodic soluble linear groups, the class of soluble Černikov groups, and $\mathcal{K}$ itself. A Fitting class of $\mathcal{K}$-groups (or a $\mathcal{K}$-Fitting class) is a subclass $\mathcal{X}$ of $\mathcal{K}$ such that

\[(F1)\] Every serial subgroup of an $\mathcal{X}$-group belongs to $\mathcal{X}$.

\[(F2)\] Every $\mathcal{K}$-group, generated by serial $\mathcal{X}$-subgroups, belongs to $\mathcal{X}$.

When $\mathcal{K}$ is the class of all finite soluble groups this coincides with the usual definition as given in [3], and for the class of soluble Černikov groups it agrees with [1].

Examples of subgroup-closed Fitting classes are easily obtained from Fitting classes of finite soluble groups.

**Lemma 1.2.** Let $\mathcal{X}$ be a subgroup-closed Fitting class of finite soluble groups. Then the class $L\mathcal{X} \cap \mathcal{K}$ of all locally-$\mathcal{X}$ groups in $\mathcal{K}$ is a Fitting class of $\mathcal{K}$-groups.

**Proof.** The class $L\mathcal{X} \cap \mathcal{K}$ is clearly closed under taking serial subgroups. Let $G$ be a $\mathcal{K}$-group generated by serial $L\mathcal{X} \cap \mathcal{K}$-subgroups $H_i$ $(i \in I)$. If $F$ is a finite subgroup of $G$, then $F \leq \langle F_1, \ldots, F_n \rangle = L$, where $F_r$ is a finite subgroup of $H_i$ $(1 \leq r \leq n)$. Thus $F \leq L = \langle L \cap H_i, \ldots, L \cap H_n \rangle$. Each $L \cap H_i$ is a subnormal $\mathcal{X}$-subgroup of $L$, whence $L \in \mathcal{X}$, and $F \in \mathcal{X}$.

In particular, we have the Fitting class $(LN)^k \cap \mathcal{K}$ of $\mathcal{K}$-groups of locally nilpotent length at most $k$. For examples of Fitting classes that are not subgroup-closed (working relative to the class of soluble Černikov groups) see [1].

If $\mathcal{X}$ is any class of groups, and $G \in \mathcal{K}$, the join $G_\mathcal{X}$ of all serial $\mathcal{X}$-subgroups of $G$ is a characteristic $\mathcal{X}$-subgroup $G_\mathcal{X}$, the $\mathcal{X}$-radical of $G$. A routine argument gives

**Lemma 1.3.** If $\mathcal{X}$ is a $\mathcal{K}$-Fitting class, $G \in \mathcal{K}$ and $H \ser G$, then $H_\mathcal{X} = H \cap G_\mathcal{X}$.

If $\mathcal{X}$ is any class of groups, then an $\mathcal{X}$-injector of the group $G$ is an $\mathcal{X}$-subgroup $V$ of $G$ such that $V \cap H$ is a maximal $\mathcal{X}$-subgroup of $H$ whenever $H \ser G$. This agrees with the definition used for soluble Černikov groups in [1], and in particular is consistent with the finite case. Our main result is
Theorem. Let $\mathcal{X}$ be a Fitting class of $\mathcal{K}$-groups. Then every $\mathcal{K}$-group $G$ has $\mathcal{X}$-injectors, and any two $\mathcal{X}$-injectors of $G$ are conjugate.

The proof is roughly similar to the finite case [3], but several technical difficulties have to be overcome. The proof in the finite case depends on the conjugacy of the self-normalizing nilpotent (or Carter) subgroups of a finite soluble group, and this result, a version of which is known for $\mathcal{U}$-groups, must first be recast into the appropriate form.

2. Carter subgroups.

If $\mathcal{X}$ is any class of groups, then an $\mathcal{X}$-projector of a group $G$ is an $\mathcal{X}$-subgroup $X$ of $G$ such that $XK = H$ whenever $XH \triangleleft G$, $K \triangleleft H$ and $H/K \in \mathcal{X}$. Though the Carter subgroups of a finite soluble group were originally defined as the self-normalizing nilpotent subgroups [2], they are of course now known to be the nilpotent projectors. In [4], the Carter subgroups of a $\mathcal{U}$-group $G$ were defined as its locally nilpotent projectors. They were shown to exist and form a conjugacy class, and it was shown [4, Lemma 5.8] that they are the self-normalizing locally nilpotent subgroups of $G$, provided that the locally nilpotent subgroups of $G$ are all hypercentral. However they do not have this description in general, since a locally nilpotent group may possess proper self-normalizing subgroups.

To remedy this, let us say that a subgroup $H$ of $G$ is self-serializing in $G$, if $H$ is the only subgroup of $G$ containing $H$ as a serial subgroup. Then the Carter subgroups of a $\mathcal{U}$-group $G$ have the following characterization, which is important for us.

Theorem 2.1. The Carter subgroups of a $\mathcal{U}$-group $G$ are precisely its self-serializing locally nilpotent subgroups.

Proof. Let $C$ be a Carter subgroup of $G$ and suppose that $C \ser K \triangleleft G$. If $C < K$, then we have subgroups $C < V < U < K$ with $V \triangleleft U$. By (U1), we can choose $W$ with $V < W < U$ and $W/V$ locally nilpotent, contradicting the fact that $C$ is a locally nilpotent projector of $G$. Thus $C = K$ and $C$ is self-serializing.

Conversely, let $C$ be a self-serializing locally nilpotent subgroup of $G$. We prove that $C$ is a Carter subgroup of $G$ by induction on the locally nilpotent length of $G$. If $G$ is locally nilpotent, the result is
clear, since every subgroup of a locally nilpotent group is serial. We
need to consider separately the case \( G \in (LN)^2 \). In this case, let \( R \)
be the locally nilpotent radical of \( G \), so that \( G/R \) is locally nilpotent. 
If \( R = \{ R_p \} \) and \( C = \{ C_p \} \) are the unique Sylow bases of \( R \) and \( C \) respectively, then \( C \) is contained in the basis normalizer \( D \) of the Sylow basis \( \{ R_p C_p \} \) of \( RC \). However, \( C \) is then serial in \( D \), and so \( C = D \). By Lemma 1.1, \( G/R \) is hyperfinite and so hypercentral, and so if \( CR < G \), we have \( CR < N = N_0(CR) \). Using the conjugacy of the basis normalizers of \( CR \) and the Frattini argument, we have \( N = CRN_N(C) \). But \( C \) is certainly self-normalizing, whence \( N = CR \). 
This contradiction shows that \( CR = G \), and hence \( C \) is a Carter subgroup of \( G \) [4, Theorem 5.1].

Now let \( G \in (LN)^k \), where \( k > 3 \), and again let \( R \) be the \( LN \)-radical of \( G \). Suppose that \( CR/R \) is an \( H/R \)-simple group. Then \( CR/R \) lies in the locally nilpotent radical \( K/R \) of \( H/R \). Since \( C \) is a self-serializing locally nilpotent subgroup of the \( (LN)^2 \)-group \( K \), \( C \) is a Carter subgroup of \( K \) as we have seen. By the Frattini argument, \( H = K N_H(C) = K \). Thus \( C \) is a Carter subgroup of \( H \) and, as \( H/R \) is locally nilpotent, \( H = CR \). This shows that \( CR/R \) is a self-serializing locally nilpotent subgroup of \( G/R \), and a Carter subgroup of \( G/R \) by induction. As \( C \) is also a Carter subgroup of \( CR \), the « Gaschütz Lemma » [4, Lemma 5.3] shows that \( C \) is a Carter subgroup of \( G \).

This characterization of Carter subgroups enables us to prove 
an appropriate form of the main lemma of [3].

**LEMMA 2.2.** Let \( \mathcal{X} \) be a Fitting class of \( \mathcal{K} \)-groups. Let \( G \in \mathcal{K} \) and \( N \) be a normal subgroup of \( G \) such that \( G/N \) is locally nilpotent. If \( U, V \) are maximal \( \mathcal{X} \)-subgroups of \( G \) such that \( U \cap N = V \cap N \), then \( U \) and \( V \) are conjugate in \( G \).

**PROOF.** We may clearly assume that \( G = \langle U, V \rangle \), so that \( U \cap N = V \cap N \trianglelefteq G \). Let bars denote homomorphic images in \( G/U \cap N \), let \( \bar{M} = N_a(\bar{U}) \), and let \( S = \{ S_p \} \) be a Sylow basis of \( \bar{M} \). Then \( \{ U \cap S_p \} \) is a Sylow basis of \( \bar{U} \), and, for \( q \neq p \), \( \bar{U} \cap S_p, S_q < \bar{U} \cap N = 1 \). Hence \( \bar{U} \) normalizes \( S \) and so \( \bar{U} \) is contained in a Carter subgroup \( \bar{C} = C/(U \cap N) \) of \( \bar{M} \) [4, Theorem 5.9]. If \( C \) is serial in some subgroup \( H \) of \( G \), then \( U \) is serial in \( H \) and so \( U < H_\mathcal{X} \). By the maximality of \( U \), we have \( U = H_\mathcal{X} \trianglelefteq H \), and so \( H < \bar{M} \), and since \( \bar{C} \) is self-serializing in \( \bar{M} \), we have \( \bar{C} = \bar{H} \). Thus \( \bar{C} \) is a self-serializing locally nilpotent subgroup of \( \bar{C} \), that is, by Theorem 2.1, a Carter subgroup of \( \bar{G} \).

Similarly, we have a Carter subgroup \( \bar{D} = D/U \cap N \) of \( \bar{C} \) with
The subgroups $C$ and $D$ are conjugate, and so $U$, for some $x \in G$. But $U$ and $V^z$ are serial $\mathcal{X}$-subgroups of $C$, which belongs to $\mathcal{K}$, and so $\langle U, V^z \rangle \in \mathcal{X}$. Finally, the maximality of $U$ and $V^z$ gives $U = \langle U, V^z \rangle = V^z$.

**Corollary 2.3.** Let $\mathcal{X}$ be a Fitting class of $\mathcal{K}$-groups and $G \in \mathcal{K}$. Let $N, M$ be normal subgroups of $G$ such that $N < M$ and $G/N$ is locally nilpotent, and assume that each of $M$ and $N$ has a unique conjugacy class of $\mathcal{X}$-injectors. Let $U$ be an $\mathcal{X}$-injector of $N$ and let $V$ be any maximal $\mathcal{X}$-subgroup of $G$ containing $U$. Then $V \cap M$ is an $\mathcal{X}$-injector of $M$.

**Proof.** The hypotheses imply easily that $U$ is contained in an $\mathcal{X}$-injector $W_0$ of $M$. Now if we form any tower of $\mathcal{X}$-subgroups of $G$ containing $W_0$, its union is generated by serial $\mathcal{X}$-subgroups and so belongs to $\mathcal{X}$. Hence, by Zorn's Lemma, $W_0$ is contained in a maximal $\mathcal{X}$-subgroup $W$ of $G$. Now $V \cap N = U = W \cap N$ and so by Lemma 2.2, $V = W^z$ for some $x \in G$. Therefore $V \cap M = W^z \cap M = W_0^z$, as required.

**Corollary 2.4.** If $\mathcal{X}$ is a Fitting class of $\mathcal{K}$-groups and $G \in \mathcal{K}$, then any two $\mathcal{X}$-injectors of $G$ are conjugate in $G$.

**Proof.** This follows by using Lemma 2.2 and induction on the $\mathcal{L}_\mathcal{N}$-length, exactly as in the finite case [3].

### 3. Injectors.

Let $\mathcal{X}$ be a Fitting class of $\mathcal{K}$-groups. Then $O(\mathcal{X})$, the characteristic of $\mathcal{X}$, is defined to be the set of primes $p$ such that $\mathcal{X}$ contains a cyclic group of order $p$. Standard arguments show that if $O(\mathcal{X}) = \pi$, then every $\mathcal{X}$-group is a $\pi$-group, and $\mathcal{X}$ contains every locally nilpotent $\pi$-group in $\mathcal{K}$. Details of these arguments can be found in [1]. They are similar to the finite case, and it is for them that (K2) is needed.

In the rest of the paper, $\mathcal{X}$ denotes a Fitting class of $\mathcal{K}$-groups, $\pi = C(\mathcal{X})$ and $R = G_{\mathcal{L}_\mathcal{N}}$. By the above remarks, $O_\pi(R) < G_\mathcal{X}$, so $RG_\mathcal{X}/G_\mathcal{X}$ is a $\pi$' group, and $G/RG_\mathcal{X}$ is hyperfinite, by Lemma 1.1.

By the remarks in the introduction, if $H$ ser $G$, then $HRG_\mathcal{X}$ asc $G$, and much of our proof of the main theorem will consist of an induction argument on an ascending series from $HRG_\mathcal{X}$ to $G$. Limit ordinals are dealt with by the following, in which $\mathcal{U}$-group properties are not involved.
LEMMA 3.1. Let $G$ be the union $G = \bigcup_{\lambda \in \Lambda} G_{\lambda}$ of a set of serial subgroups $G_{\lambda}$ ($\lambda \in \Lambda$). Then $V$ is an $\mathcal{X}$-injector of $G$ if and only if $V \cap G_{\lambda}$ is an $\mathcal{X}$-injector of $G_{\lambda}$, for each $\lambda \in \Lambda$.

PROOF. If $V$ is an $\mathcal{X}$-injector of $G$, the definition shows that $V \cap G_{\lambda}$ is an $\mathcal{X}$-injector of $G_{\lambda}$, for each $\lambda \in \Lambda$.

Conversely, suppose $V \cap G_{\lambda}$ is an $\mathcal{X}$-injector of $G_{\lambda}$, for each $\lambda \in \Lambda$, and let $H \subseteq G$. Then $H \cap G_{\lambda}$ is an $\mathcal{X}$-injector of $G_{\lambda}$, and so $V \cap H \cap G_{\lambda}$ is maximal among the $\mathcal{X}$-subgroups of $G_{\lambda}$. If $V \cap H \subseteq W \subseteq H$ and $W \subseteq G_{\lambda}$, then $W \cap G_{\lambda} \subseteq H$, whence we find $V \cap H \cap G_{\lambda} = W \cap G_{\lambda}$, and $W = \bigcup_{\lambda \in \Lambda} (V \cap H \cap G_{\lambda}) = V \cap H$. Hence $V \cap H$ is a maximal $\mathcal{X}$-subgroup of $H$.

The following is useful in dealing with serial subgroups not containing $G_{\mathcal{X}}$.

LEMMA 3.2. If $W$ is an $\mathcal{X}$-subgroup of the serial subgroup $H$ of $G$ and $W \supset H_{\mathcal{X}}$, then $WG_{\mathcal{X}} \subseteq G_{\mathcal{X}}$.

PROOF. Let $(U_{\sigma}, V_{\sigma} : \sigma \in \Omega)$ be a series from $H$ to $G$. Since $G_{\mathcal{X}} \cap H = H_{\mathcal{X}} \subseteq W \subseteq H$, we have $[W, G_{\mathcal{X}} \cap U_{\sigma}] \subseteq G_{\mathcal{X}} \cap V_{\sigma}$, and so

$$WG_{\mathcal{X}} \cap V_{\sigma} = W(G_{\mathcal{X}} \cap V_{\sigma}) \circ W(G_{\mathcal{X}} \cap U_{\sigma}) = WG_{\mathcal{X}} \cap U_{\sigma}.$$  

Thus, intersecting with $WG_{\mathcal{X}}$ gives a series from $H \supset WG_{\mathcal{X}} \subseteq H_{\mathcal{X}} \subseteq W \subseteq H$, therefore $W$ is a serial $\mathcal{X}$-subgroup of $WG_{\mathcal{X}}$ and hence $WG_{\mathcal{X}} \subseteq G_{\mathcal{X}}$.

The main lemma is as follows.

LEMMA 3.3. Let $M$ be a normal subgroup of finite index of $G$ containing $G_{\mathcal{X}}$. If $M$ has an $\mathcal{X}$-injector, then $G$ has an $\mathcal{X}$-injector.

PROOF. By induction on $|G/M|$, we may assume that $M$ has prime index $p$. Taking account of Corollary 2.4, our hypothesis implies that every serial subgroup of $M$ has a unique conjugacy class of $\mathcal{X}$-injectors.

Let $U$ be an $\mathcal{X}$-injector of $M$ and $V$ be maximal among the $\mathcal{X}$-subgroups of $G$ containing $U$. Since $U$ has index at most $p$ in any $\mathcal{X}$-subgroup of $G$ containing it, the existence of $V$ is clear. We shall show that $V$ is an $\mathcal{X}$-injector of $G$. If $H \subseteq G$, then certainly $V \cap H \subseteq G$; the problem is to show that $V \cap H$ is a maximal $\mathcal{X}$-subgroup of $H$. Since $V \cap M = U$, an $\mathcal{X}$-injector of $M$, we have that $V \cap H \cap M = U \cap H$ is an $\mathcal{X}$-injector of $H \cap M$. 

CASE (i) $HG_X = G$. Let $W$ be an $X$-subgroup of $H$ containing $V \cap H$. Now since $M > G_X$, we have $U > G_X$, and so $V \cap H > U \cap H > G_X \cap H = H_X$, by Lemma 1.3. By Lemma 3.2, $WG_X \in X$. But $WG_X > (V \cap H)G_X = V$ and so, by the maximality of $V$, we deduce that $WG_X = V$. Hence $W < V$, as required.

CASE (ii) $HG_XR = G$. Recall that $\pi = C(X)$ and $G_XR/G_X$ is a $\pi'$-group. Let $S$ be a Sylow $\pi$-subgroup of $G$ containing $V$. Then $S \cap HG_X$ is a Sylow $\pi$-subgroup of $HG_X$, since this subgroup is serial [5, Theorem E], and since $G/G_X$ is the product of $HG_X/G_X$ and the normal $\pi'$-subgroup $G_XR/G_X$, it follows that $S \cap HG_X/G_X$ is also a Sylow $\pi$-subgroup of $G/G_X$. This gives $V \leq S < HG_X$. Now we need only apply Case (i) to $HG_X$.

CASE (iii) $HG_XR < G$. By the remarks in the introduction, there is an ascending series

$$HG_XR = H_0 < \cdots < H_\lambda < \cdots < H_\alpha = G$$

and after refining if necessary, we may assume that each factor is finite abelian. Let $\lambda$ be minimal such that $V \cap H_\lambda$ is a maximal $X$-subgroup of $H_\lambda$. If $\lambda = 0$, then $V \cap HG_XR$ is a maximal $X$-subgroup of $HG_XR$ containing the $X$-injector $V \cap M \cap HG_XR$ of $M \cap HG_XR$. By Case (ii) applied to $HG_XR$, we obtain that $V \cap H$ is a maximal $X$-subgroup of $H$.

Thus we may assume that $\lambda > 0$, so that $V \cap H_\lambda$ is a maximal $X$-subgroup of $H$ while $V \cap H_\alpha$ is not a maximal $X$-subgroup of $H_\alpha$ if $\alpha < \lambda$.

CASE (iii) $\lambda - 1$ exist. Put $L = H_{\lambda - 1} \cap M$, and note that $H_\lambda/L$ is finite abelian. We have $V \cap L = U \cap H_{\lambda - 1}$, which is a maximal $X$-subgroup of $L$. Let $W_0$ be a maximal $X$-subgroup of $H_{\lambda - 1}$ containing $U \cap H_{\lambda - 1}$, and $W$ be a maximal $X$-subgroup of $H_\lambda$ containing $W_0$. Then $W \cap L = U \cap H_{\lambda - 1} = V \cap L$, and so by Lemma 2.2, $W$ is conjugate to $V \cap H_{\lambda - 1}$ in $H_{\lambda}$: Therefore $V \cap H_{\lambda - 1} = W^x \cap H_{\lambda - 1}$ for some $x \in H_\lambda$, contrary to the fact that $V \cap H_{\lambda - 1}$ is not a maximal $X$-subgroup of $H_{\lambda - 1}$.

CASE (iii) $\lambda$ is a limit ordinal. Now $U \cap H_0$ is certainly not a maximal $X$-subgroup of $H_0$, but it is a maximal $X$-subgroup of $H_0 \cap M$, so we have $U \cap H_0 < W_0$ for some maximal $X$-subgroup $W_0$ of $H_0$. 


We now construct subgroups $W_\alpha (\alpha < \lambda)$ such that $W_\alpha < W_\beta$ for $\alpha < \beta$ and $W_\alpha$ is a maximal $\mathcal{X}$-subgroup of $H_\alpha$. Having obtained $W_\alpha$, we can obtain $W_{\alpha+1}$ since $H_{\alpha+1}/H_\alpha$ is finite, or as in Corollary 2.3. If $\beta$ is a limit ordinal and the previous $W_\alpha$ have been obtained, we put $W_\beta = \bigcup_{\alpha < \beta} W_\alpha$, which is the join of ascendant $\mathcal{X}$-subgroups and so belongs to $\mathcal{X}$, and is clearly a maximal $\mathcal{X}$-subgroup of $H_\beta$. Now we show by induction that $W_\alpha \cap M$ is an $\mathcal{X}$-injector of $H_\alpha \cap M$, for each $\alpha < \lambda$. Since $W_0 \cap M = U \cap H_0 \cap M$, the case $\alpha = 0$ is clear. Lemma 3.1. deals with the passage to limit ordinals. If $W_\alpha \cap M$ is known to be an $\mathcal{X}$-injector of $H_\alpha \cap M$, then Corollary 2.3 shows that $W_{\alpha+1} \cap M$ is an $\mathcal{X}$-injector of $H_{\alpha+1} \cap M$. Finally, we find that $W_\lambda \cap M$ is an $\mathcal{X}$-injector of $H_\lambda \cap M$. Therefore $W_\lambda \cap M = (U \cap H_\lambda)^x = (V \cap H_\lambda)^x \cap M$ for some $x \in H_\lambda \cap M$, since this group has conjugate $\mathcal{X}$-injectors. By Lemma 2.2, $W_\lambda$ and $(V \cap H_\lambda)^x$ are conjugate in $H_\lambda$. But $W_\lambda$ contains $W_0$, so $W_\lambda \not< M$, while $V \cap H_\lambda = \bigcup_{\alpha < \lambda} (V \cap H_\alpha)$.

Since $V \cap H_\alpha$ is not a maximal $\mathcal{X}$-subgroup of $H_\alpha$ if $\alpha < \lambda$, and $|V \cap H_\alpha: U \cap H_\alpha|$ is either 1 or $p$, we have $V \cap H_\alpha = U \cap H_\alpha \not< M$. Therefore $V \cap H_\lambda \not< M$ and $(V \cap H_\lambda)^x \not< M$, a contradiction.

This completes the proof that $V$ is an $\mathcal{X}$-injector of $G$.

**Proof of Main Theorem.** The conjugacy of $\mathcal{X}$-injectors is given in Corollary 2.4. For the existence, we first note that $G_{\mathcal{X}}$ is the unique $\mathcal{X}$-injector of $G_{\mathcal{X}} \cdot R$. For Lemma 3.2 shows that if $H$ is a $\mathcal{X}$-injector of $G$, then $H \cap G_{\mathcal{X}}$ is a maximal $\mathcal{X}$-subgroup of $H$. Now by the remarks in the introduction, we have an ascending series $(G_\alpha: \alpha < \beta)$ of $G$ with finite abelian factors and such that $G_\alpha = G_{\mathcal{X}} \cdot R$. We show by induction on $\alpha$ that $G_\alpha$ (and hence all its serial subgroups) has an $\mathcal{X}$-injector. For $\alpha = 0$ this has been remarked. The step from $\alpha$ to $\alpha + 1$ follows from Lemma 3.3. The limit ordinal step is made by forming a tower of injectors and using Lemma 3.1.

The following results can be deduced exactly as in the finite case [3].

**Theorem 3.4** (i). Let $1 = G_0 \not< G_1 \not< \ldots \not< G_n = G$ be a series of $G$ with locally nilpotent factors and $V \not< G$. Then $V$ is an $\mathcal{X}$-injector of $G$ if and only if $V \cap G_i$ is a maximal $\mathcal{X}$-subgroup of $G_i$ for $i = 0, 1, \ldots, n$.

(ii) If $V$ is an $\mathcal{X}$-injector of $G$ and $V \not< L \not< G$, then $V$ is an $\mathcal{X}$-injector of $L$.

(iii) The $\mathcal{X}$-injectors of $G$ are pronormal in $G$. 
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