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Proper Holomorphic Mappings between Reinhardt Domains and Pseudoellipsoids.

GILBERTO DINI - ANGELA SELVAGGI PRIMICERIO (*)

In recent years several results have been obtained on the conditions for the existence of proper holomorphic mappings between two domains D_1 and D_2 in \mathbb{C}^n and particularly for mappings of polynomial type. It is a conjecture, due to Bell [2], that if R_1 and R_2 are Reinhardt domains related by a proper holomorphic mapping then there is such a map which is polynomial. We recall that a Reinhardt domain (respect to O) in \mathbb{C}^n is an open connected set R such that if $z \in R$ for any $\theta \in \mathbb{R}^n$

$$T_\theta(z) = (\exp [i\theta_1] z_1, \dots, \exp [i\theta_n] z_n) \in R.$$

If such a condition holds only when $\theta_1 = \theta_2 = \dots = \theta_n$ then R is said to be a circular domain.

A Reinhardt domain R is complete if for any $z^0 = (z_1^0, \dots, z_n^0) \in R$ the closed polydisc $A_{z^0} = \{z \in \mathbb{C}^n : |z_i| \leq |z_i^0| \ i = 1, \dots, n\}$ is contained in R .

For any $\alpha \in \mathbb{N}^n$ the pseudoellipsoid

$$\Sigma_n(\alpha) = \left\{ z \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^{2\alpha_i} < 1 \right\}$$

is a complete bounded Reinhardt domain.

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If $a \in (\mathbf{C}^*)^n$, $T_a: \mathbf{C}^n \rightarrow \mathbf{C}^n$, defined as $T_a(z_1, \dots, z_n) = (a_1 z_1, \dots, a_n z_n)$ is a linear automorphism of \mathbf{C}^n such that for any Reinhardt domain R , $T_a(R)$ is still a Reinhardt domain. So, we will say that two Reinhardt domains R_1 and R_2 in \mathbf{C}^n are T_a -equivalent, $R_1 \cong R_2$, if there exists $a \in (\mathbf{C}^*)^n$ such that $R_1 = T_a(R_2)$. It is clear that any T_a does not affect the polynomial feature of a map $f: R_1 \rightarrow R_2$.

In this note we prove that Bell conjecture holds when $R_2 = \Sigma_n(\alpha)$. More exactly

THEOREM 1. Let R_1 be a Reinhardt domain in \mathbf{C}^n with $0 \in R_1$. If there exists a proper holomorphic mapping $T: R_1 \rightarrow R_2 \cong \Sigma_n(\alpha)$ then there exists a proper polynomial holomorphic one.

In [4] the authors proved

THEOREM 2. Let R_1 be a Reinhardt domain in \mathbf{C}^n and

$$f: R_1 \rightarrow R_2 \cong \Sigma_n(\alpha),$$

a proper polynomial holomorphic mapping then $R_1 \cong \Sigma_n(\beta)$ where $\beta_i/\alpha_i \in \mathbb{N}$ for $i = 1, \dots, n$.

The previous theorems allow to characterize in the following corollary Reinhardt domains properly related to pseudoellipsoids.

COROLLARY 3. Let R_1 be a Reinhardt domain in \mathbf{C}^n with $0 \in R_1$, $R_1 \cong \Sigma_n(\beta)$ if and only if there exists a proper holomorphic mapping $F: R_1 \rightarrow \Sigma_n(\alpha)$ on a pseudoellipsoid $\Sigma_n(\alpha)$.

PROOF OF THEOREM 1. First consider $R_2 = \mathbb{B}_n(0, 1)$, the unit ball in \mathbf{C}^n . We require two key facts.

FACT 1 (Alexander [1]). Let N be a neighborhood of $p \in b\mathbb{B}_n$ and F a non-constant mapping holomorphic in $N \cap \mathbb{B}_n$ and C^∞ in $N \cap \overline{\mathbb{B}_n}$. If $F(N \cap b\mathbb{B}_n) \subseteq b\mathbb{B}_n$ then F extends holomorphically to an automorphism of \mathbb{B}_n .

FACT 2 (Bell [2]). A proper holomorphic mapping F between bounded complete Reinhardt domains extends holomorphically past the boundary and if $F^{-1}(0) = \{0\}$ then F is a polynomial mapping.

To apply Bell's results let us see that R_1 is complete and bounded. R_1 is complete: in fact if $z^0 \in R_1$ T will extend to a holomorphic map $\hat{T}: \overline{\Delta}_{z^0} \rightarrow \mathbf{C}^n$ (see for example [5] theorem 2.4.6). The existence of $z \in \overline{\Delta}_{z^0} \cap (\mathbf{C}^n - R_1)$ would contradict the maximum principle for the function $\sum_{i=1}^n |\hat{T}_i(z)|^2$, where \hat{T}_i are the components of \hat{T} .

R_1 is bounded otherwise by Liouville theorem T would not be proper.

For any given proper mapping $T: R_1 \rightarrow \mathbf{B}_n$ and for any $g \in \text{Aut}(R_1)$ we claim that there exists $\Phi_g \in \text{Aut}(\mathbf{B}_n)$ such that $T \circ g = \Phi_g \circ T$ on R .

As T and g extend holomorphically past the boundary, one can find a point $P \in bR_1$ and a neighborhood U of P in \mathbf{C}^n such that

- i) $J_T(z) \neq 0 \quad z \in U$,
- ii) g is a biholomorphism on U ,
- iii) $J_T(\xi) \neq 0, \xi \in g(U)$, where

$$J_T(z) = \det(\partial T_i(z)/\partial z_j) \quad j, i = 1, \dots, n.$$

(By the way, one could show that J_T can vanish only on coordinate hyperplanes.)

Furthermore, for any $z \in U \cap bR$, $g(z) \in bR$ and $T(z) \in b\mathbf{B}_n$ and if $\zeta \in g(U) \cap bR$, $T(g(\zeta)) \in b\mathbf{B}_n$.

Hence one can define a biholomorphism $\varphi = T \circ g \circ T^{-1}: T(U) \rightarrow T(g(U))$ such that $\varphi(T(U) \cap b\mathbf{B}_n) \subseteq T(g(U)) \cap b\mathbf{B}_n$.

By fact 1 such a map extends to $\Phi_g \in \text{Aut}(\mathbf{B}_n)$ and $\Phi_g \circ T$ and $T \circ g$ agree on U , hence on R_1 .

As $\text{Aut}(\mathbf{B}_n)$ acts transitively one can find $\psi \in \text{Aut}(\mathbf{B}_n)$ such that $\psi \circ T \equiv F: R_1 \rightarrow \mathbf{B}_n$ is a proper map and $F(0) = 0$.

For any $\theta \in \mathbf{R}^n$ let Φ_θ be the automorphism of \mathbf{B}_n such that $\Phi_\theta \circ F = F \circ T_\theta$.

$$\Phi_\theta(0) = \Phi_\theta(F(0)) = F(T_\theta(0)) = F(0) = 0.$$

This implies $F^{-1}(0) = \{0\}$ and by fact 2 F is polynomial. In fact if there exists $0 \neq a \in F^{-1}(0)$, for any $\theta \in \mathbf{R}^n$, $F(T_\theta(a)) = \Phi_\theta(F(a)) = 0$ hence F would not be proper.

In the general case $R_2 \cong \Sigma_n(\alpha) \neq \mathbf{B}_n(0, 1)$ consider

$$H_\alpha: \Sigma_n(\alpha) \rightarrow \mathbf{B}_n(0, 1),$$

defined as $H_\alpha(w_1, \dots, w_n) = (w_1^{\alpha_1}, \dots, w_n^{\alpha_n})$.

$H_\alpha \circ T: R_1 \rightarrow \mathbf{B}_n$ is a proper holomorphic map hence R_1 can be properly mapped on \mathbf{B}_n by a polynomial map and $R_1 \cong \Sigma_n(\beta)$ for suitable β by theorem 2.

By results of Landucci [6] $\gamma_i = \beta_i/\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$ and

$$(z_1, \dots, z_n) \rightarrow (z_1^{\gamma_1}, \dots, z_n^{\gamma_n})$$

s the required map from R_1 on R_2 .

REMARK. One can obtain the same conclusion of theorem 1 for circular domains D under suitable conditions (see [3]) that imply the extendibility of $T: D \rightarrow R_2 \cong \Sigma_n(\alpha)$ applying results analogous to fact 2, for circular domains, due to Bell [3].

The following example shows anyhow that there are circular domains D in \mathbb{C}^n such that there exists proper polynomial holomorphic mapping $P: D \rightarrow \mathbb{B}_n$ but which are not T_α -equivalent to pseudoellipsoids.

$$\begin{aligned} (z_1 + z_2, 2z_1 - 2z_2, z_3^2): \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ maps } D = \\ = \{z \in \mathbb{C}^n: 5|z_1|^2 + 5|z_2|^2 - 6 \operatorname{Re} z_1 z_2 + |z_3|^4 < 1\} \end{aligned}$$

on the ball, but D is not a Reinhardt domain.

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