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## Constant Mean Curvature Surfaces in 4-Space Forms.

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### 1. Introduction: constant mean curvature in 3-space.

Consider the following classical theorems about constant mean curvature surfaces in euclidean 3-space:

**THEOREM (a)** (Ricci) [2, 12]. Let  $(M^2, ds^2)$  be a simply connected surface with Gaussian curvature  $K$ . Then there exists an isometric immersion  $f: M \rightarrow \mathbb{R}^3$  with constant mean curvature  $H$  without umbilic points if and only if  $H^2 > K$  and the metric  $d\tilde{s}^2 := (H^2 - K)^{\frac{1}{2}} ds^2$  is flat.

**THEOREM (b)** (H. Hopf) [9]. Let  $M^2$  be a closed surface with Euler characteristic  $\chi$  and  $f: M \rightarrow \mathbb{R}^3$  an immersion with constant mean curvature. Then either  $f(M)$  is a round sphere or  $2\chi = -N$  where  $N$  is the number of umbilic points, counted with multiplicities.

**REMARK.** Hopf does not state explicitly this theorem but only its corollary: A topological sphere of constant mean curvature is round ([9], Ch. VI, 2.1). However, the more general statement can be read off from VI, 2.3., p. 139: In the non-umbilic case the singularities of the principal line fields, i.e. the umbilic points are isolated and of index  $j = -n/2$  where  $n$  is the order of the umbilic point (see below).

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This implies Theorem (b) by the Poincaré-Hopf index theorem which was discussed before ([9], Ch. III, 2.2).

At first sight, the two theorems seem to be unrelated since in (a) the umbilics are excluded, in (b) they are counted. But it turns out that a sharpened version of (a) in fact implies (b). To see this, we must extend the local characterization of minimal surfaces given in (a) to the umbilic points. First observe that the flatness of a metric  $d\tilde{s}^2 = a ds^2$  is expressed by the differential equation

$$\Delta \log a = 2K$$

where  $\Delta$  is the Laplacian with respect to the metric  $ds^2$ . Unfortunately, in general the left hand side is undefined at the zeros of the function  $a$ , and the zeros of  $a = (H^2 - K)^\dagger$  are precisely the umbilic points. However, for a certain type of zeros the expression  $\Delta \log a$  makes still sense:

**DEFINITION.** A function  $a: M \rightarrow [0, \infty)$  on a surface  $(M, ds^2)$  is called of *absolute value type* (*AVT*) if every point in  $M$  has a neighborhood  $U$  such that  $a = a_0|h|$  on  $U$  where  $a_0$  is smooth and positive and  $h \circ z^{-1}$  is holomorphic for some conformal coordinate  $z: U \rightarrow \mathbf{C}$ .

Here, *conformal* means that  $ds^2 = \lambda^2 dz d\bar{z}$  for some positive function  $\lambda$ , called *conformal factor*. Note that then  $\Delta = 4\lambda^{-2} \partial_z \partial_{\bar{z}}$ .

If an absolute value type function  $a$  does not vanish entirely its zeros are isolated, and around each zero we have  $\Delta \log a = \Delta \log a_0$  which is still defined and smooth at the zeros. The *order*  $\text{ord}_p(a)$  of  $a$  at a zero  $p \in M$  is, by definition, the order of its holomorphic part  $h$  at  $p$ . If  $M$  is a compact surface, we define the number of zeros of  $a$  (with multiplicities) to be

$$N(a) := \sum_{a(x)=0} \text{ord}_p(a).$$

Then we get as an easy consequence of the divergence theorem (see [7], 4.1)

**LEMMA 1.** Let  $(M, ds^2)$  be a compact surface and  $a \neq 0$  an *AVT* function on  $M$ . Then

$$\int_M \Delta \log a = -2\pi N(a).$$

REMARK. One may generalize the notion of  $AVT$  functions by allowing the holomorphic part  $h$  to have poles. Then the lemma is still correct with  $N(a) :=$  number of zeros – number of poles.

Now we can give the sharpened version of Ricci's theorem (a) for any 3-space form of constant sectional curvature  $c$ ,  $Q_c^3$ :

THEOREM 0. Let  $(M, ds^2)$  be a simply connected surface with Gaussian curvature  $K$ . Then there exists an isometric immersion  $f: M \rightarrow Q_c^3$  with constant mean curvature  $H$  if and only if  $K \leq c + H^2$  and  $a := (c + H^2 - K)^{\frac{1}{2}}$  is  $AVT$  satisfying

$$(1) \quad \Delta \log a = 2K$$

unless  $a \equiv 0$  and  $f$  is totally umbilic. In fact, there is exactly a one-parameter family of such immersions, up to local congruence. Of course, for the «only if» part we do not have to assume simple connectivity. By local congruence we mean that the lifts to the universal cover  $\tilde{Q}_c^3$  are congruent.

This theorem has been stated by Lawson [12], but avoiding umbilic points. However, for global applications the umbilic points are essential as we saw. In fact, from Theorem 0 and Lemma 1 we get immediately Theorem (b), using the Gauss-Bonnet theorem:

COROLLARY. Let  $M$  be a compact surface and  $f: M \rightarrow Q_c^3$  an immersion with constant mean curvature  $H$ . Then either  $f$  is totally umbilic or

$$(2) \quad 2\chi = -N(a)$$

where  $\chi$  is the Euler number of  $M$  and  $a = (c + H^2 - K)^{\frac{1}{2}}$ . Note that the zeros of  $a$  are precisely the umbilic points, by Gauss equations.

As a consequence, a topological sphere of constant mean curvature in  $Q_c^3$  is totally umbilic. This is well known: See Almgren [1] and Calabi [4] for  $Q_c^3 = S^3$  and Chern [6] for general space forms.

Theorem 0 turns out to be a special case of a theorem in dimension 4 which we will discuss in the next two sections. But the theorems (a) and (b) stated at the beginning may serve as a model case as well: Local characterization in terms of curvatures leads to global results. In the present paper we want to carry out this program for minimal surfaces in 4-space of constant sectional curvature (see also [7])

for the case of constant holomorphic curvature). We wish to mention that everything can be done for branched immersions as well which we will discuss in a subsequent paper (\*).

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## 2. Constant mean curvature in 4-space.

In the following sections, let  $Q_c^4$  be any 4-dimensional space of constant sectional curvature  $c$  and  $(M, ds^2)$  a surface with Gaussian curvature  $K$  which is always assumed to be simply connected if we claim existence of an immersion. One way of carrying over the notion of constant mean curvature surfaces to 4-space is the assumption of parallel mean curvature vector: If  $f: M \rightarrow Q_c^4$  is an immersion with 2nd fundamental form  $\alpha(x, y) = (D_x y)^\perp$ , then  $\eta := \frac{1}{2} \text{trace } \alpha$  is called the mean curvature vector field, and this is called parallel if it is a parallel section of the normal bundle, i.e.  $(D\eta)^\perp = 0$ .

**THEOREM 1.** There exists an isometric immersion  $f: M \rightarrow Q_c^4$  with parallel mean curvature vector of length  $H > 0$  if and only if  $K \leq c + H^2$  and  $a := (c + H^2 - K)^\dagger$  is  $AVT$  satisfying

$$(1) \quad \Delta \log a = 2K.$$

In fact, up to local congruence, there is exactly a two-parameter family  $f_{\sigma\tau}$ ,  $-\pi \leq \sigma, \tau \leq \pi$  of such immersions.

**REMARK.** It is known by Yau [17] that a surface of nonzero parallel mean curvature vector always lies in a totally umbilic hypersurface  $Q_{c'}^3 \subset Q_c^4$  with constant mean curvature  $H' = (H^2 - (c' - c))^\dagger$ , for some  $c' \geq c$ . Therefore, Theorem 0 and Theorem 1 are essentially the same. The two parameters  $\sigma$  and  $\tau$  are easy to explain geometrically: One parameter  $\tau$  is obtained by rotating the second fundamental form in the tangent plane, i.e. replacing  $\alpha$  with  $R_\tau^* \alpha$  where  $R_\tau$  is the rotation of angle  $\tau$ . The other parameter  $\sigma$  arises as the angle between the

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normal vector of  $Q_c^3$ , in  $Q_c^4$  and the mean curvature vector  $\eta$  in the normal plane of  $M$  in  $Q_c^4$ . In other words, we have

$$H' = H \sin \sigma, \quad (c' - c)^{\frac{1}{2}} = H \cos \sigma.$$

(Observe that  $(c' - c)^{\frac{1}{2}}$  is the mean curvature of  $Q_{c'}^3$  in  $Q_c^4$ ). In fact, we see that for fixed  $|\sigma|$ , the immersions  $f_{\sigma\tau}$  all belong to the same totally umbilic hypersurface  $Q_{c'}^3$ , up to local congruence, and that  $f_{\sigma\tau}$  is congruent to  $f_{-\sigma,\tau}$  under a reflection in  $Q_c^3$ .

There is yet another way of generalizing the notion of constant mean curvature to higher codimension, namely by only assuming that  $\|\eta\| = H = \text{const}$ . S. T. Yau has investigated this situation [17]. In particular he claims: If  $M$  is diffeomorphic to a sphere, then any immersion  $f: M \rightarrow Q_c^4$  with  $\|\eta\| = \text{const} > 0$  is totally umbilic. However, his proof relies on the claim that for any surface  $M \subset Q_c^4$  with  $\|\eta\| = \text{const}$ , like in the minimal case  $\|\eta\| = 0$ , the quartic form  $\Phi(x) = \|\alpha'(x, x)\|^2 - \|\alpha'(x, Jx)\|^2$  where  $J$  denotes the  $90^\circ$ -rotation on  $M$  is the real part of a holomorphic quartic form; here we let  $\alpha'(x, y) := \alpha(x, y) - \langle x, y \rangle \eta$ , the trace free part of the 2nd fundamental form  $\alpha$ . We will show by a counterexample that in general this is wrong (see Appendix). So the question remains open whether there exists a non-trivial immersion of  $S^2$  into  $Q_c^4$  with  $\|\eta\| = \text{const} \neq 0$ .

### 3. Minimal surfaces in 4-space.

In this case, as a new ingredience we have the normal curvature  $K_N$ , which is defined as follows. Let  $f: M \rightarrow Q_c^4$  be any isometric immersion. Let  $D^N$  be the induced connection on the normal bundle  $NM$  and  $R^N$  its curvature tensor. Then  $R^N$  is determined by a single function, called *normal curvature*,

$$K_N(p) = \langle R^N(e_1, e_2)e_4, e_3 \rangle$$

where  $p \in M$  and  $(e_1, e_2, e_3, e_4)$  is an oriented orthonormal basis of  $T_{f(p)}Q_c^4$  with  $e_1, e_2$  tangent and  $e_3, e_4$  normal. ( $TM$  and  $NM$  are always considered as subbundles of  $f^*TQ_c^4$ .) Thus  $K_N$  can be defined only if the bundle  $f^*TQ_c^4$  over  $M$  is orientable. If the mean curvature vector  $\eta$  is parallel and non-zero, the plane bundle  $NM$  has a parallel section and hence  $K_N = 0$ . But now we consider minimal surfaces, i.e. immersions with  $\eta = 0$ . Then it follows from Gauss and Ricci

equation that

$$(3) \quad |K_N| \leq c - K$$

with equality exactly at those points  $p \in M$  where the so called *ellipse of curvature*

$$E_p := \{\alpha(x, x); x \in T_p M, |x| = 1\}$$

in  $N_p M$  is a circle (see ch. 4); these points are called *circular* (or *semi-umbilic*). A minimal immersion is called *superminimal* if every point is circular.

**THEOREM 2.** Let  $K_N: M \rightarrow \mathbf{R}$  be a smooth function satisfying (3). Then there exists an isometric minimal but not superminimal immersion  $f: M \rightarrow Q_c^4$  with normal curvature  $K_N$  if and only if the functions  $a_{\pm} := (c - K \pm K_N)^{\frac{1}{2}}$  are *AVT* with

$$(4) \quad \Delta \log a_{\pm} = 2K \mp K_N.$$

In fact, up to local congruence, there is exactly a one-parameter family  $f_{\tau}$ ,  $-\pi \leq \tau \leq \pi$ , of such immersions.

This theorem was obtained in [16] under the assumption  $|K_N| < c - K$  and no circular points.

If  $M$  is closed and oriented with oriented normal bundle, the Gauss-Bonnet-Chern-Weil theorem gives the Euler number  $\chi_N$  of the normal bundle:

$$\int_M K_N dv = 2\pi \chi_N.$$

Hence integrating (2) and using Lemma 1, we get immediately:

**COROLLARY.** Let  $M$  be a closed oriented surface and  $f: M \rightarrow Q_c^4$  a minimal but not superminimal immersion with oriented normal bundle. Then we get

$$(5) \quad 2\chi \mp \chi_N = -N(a_{\pm})$$

and in particular

$$(6) \quad 4\chi = -N(a_+ a_-)$$

where  $a_{\pm} = (c - K \pm K_N)^{\frac{1}{2}}$ .

Recall that  $N(a_+ a_-)$  is the number of circular points (counted with multiplicities). One may interpret (6) as follows: At any  $p \in M$  consider the longest principal axis of the ellipse of curvature  $E_p$ . Since the mapping  $x \rightarrow \alpha(x, x): T_p^1 M \rightarrow E_p$  has degree two, there are exactly four vectors  $\pm x_1 \perp \pm x_2$  with  $\|\alpha(x_i, x_i)\|$  maximal. This defines a field of two perpendicular lines outside the circular points, and in the circular points the index of this two-line field is  $-\text{ord}_p(a_+ a_-)$  (see [8]). Thus we may get (6) from the Poincaré-Hopf index theorem. This argument is very similar to Hopf's proof of Theorem (b) (see ch. 1).

In particular it follows from (6) that a minimal sphere in  $Q_c^4$  is always superminimal which was proved first by Calabi [4] and that a minimal torus is either superminimal or has no circular points at all.

Now consider superminimal immersions in  $Q_c^4$ . For these we have

$$(3') \quad |K_N| = c - K$$

and the following intrinsic characterization:

**THEOREM 3.** There exists an isometric superminimal immersion  $f: M \rightarrow Q_c^4$  which is not totally geodesic if and only if  $K \leq c$  and the function  $a := (c - K)^{\frac{1}{2}}$  is *AVT* with

$$(7) \quad \Delta \log a = 3K - c = 2K - |K_N|.$$

Equation (7) was known to Calabi [4]. The theorem was obtained by Tribuzy and Guadalupe [16] in the case  $K < c$  (i.e. no points for which  $\alpha$  vanishes). For  $c = 0$ ,  $Q_c^4 = \mathbb{R}^4$ , the theorem shows that the superminimal immersions are exactly those which are holomorphic with respect to a suitable isometry between  $\mathbb{R}^4$  and  $\mathbb{C}^2$ , and they are also totally real with respect to another such isometry (see 3.6, 3.8 and the concluding note in [7]). For  $c = 1$ ,  $Q_c^4 = S^4$ , we see that there is only one superminimal immersion of constant curvature which is not totally geodesic, namely the Veronese embedding of  $\mathbb{R}P^2$  with  $K = \frac{1}{3}$ . The superminimal surfaces in  $S^4$  have been investigated by R. Bryant ([3], see also [10]).

As a global consequence of (3), (3') and Theorem 3 we get immediately a theorem of Rodriguez and Guadalupe [13]:

**COROLLARY** (see [13]). Let  $M$  be a closed oriented surface and  $f: M \rightarrow Q_c^4$  a minimal immersion with orientable normal bundle. Then

$$c \cdot \text{area}(M) \geq 2\pi(\chi + |\chi_N|)$$

with equality if and only if  $f$  is superminimal. In this case we have

$$(8) \quad |\chi_N| = 2\chi + N(a)$$

where  $a = (c - K)^\dagger$ , unless  $f$  is totally geodesic.

Recall that  $N(a)$  counts the number of totally geodesic points ( $\alpha = 0$ ) with multiplicities.

One consequence of this corollary is a theorem of E. Ruh [14]: A minimal 2-sphere (which has to be superminimal as we saw) with trivial normal bundle ( $\chi_N = 0$ ) is totally geodesic. E.g. it is well known that  $\chi_N = 0$  for an embedded oriented closed surface in  $S^4$  ([11], Cor. 3.2); hence an embedded minimal sphere in  $S^4$  is a great sphere.

#### 4. Structure equations of surfaces in 4-space forms.

Let  $M$  be a surface and  $f: M \rightarrow Q_c^4$  an immersion. Let  $(e_1, e_2, e_3, e_4)$  be a Darboux frame along  $f$ , i.e. an orthonormal frame with  $e_1, e_2$  tangent and  $e_3, e_4$  normal, and which is understood to be oriented if orientations of  $TM$  and  $NM$  are given. Then the 1-forms on  $M$

$$\omega_a := \langle df, e_a \rangle, \quad \omega_{ab} := \langle De_a, e_b \rangle,$$

$1 \leq a, b \leq 4$ , satisfy  $\omega_3 = \omega_4 = 0$ ,  $\omega_{ab} = -\omega_{ba}$  and the Cartan structure equations

$$(C) \quad \begin{aligned} d\omega_a &= \sum_b \omega_{ab} \wedge \omega_b, \\ d\omega_{ab} &= \sum_d \omega_{ad} \wedge \omega_{db} - c\omega_a \wedge \omega_b, \end{aligned}$$

and the induced metric is  $ds^2 = \omega_1^2 + \omega_2^2$ . Vice versa, if 1-forms  $\omega_a, \omega_{ab}$  with these properties are given on a simply connected surface  $M$  and if the quadratic form  $ds^2 = \omega_1^2 + \omega_2^2$  is nondegenerated, there exists an immersion  $f: M \rightarrow Q_c^4$  with Darboux frame  $e_1, \dots, e_4$  such that  $\omega_a, \omega_{ab}$  are the corresponding 1-forms, and this immersion is unique up to local congruence (e.g. see Spivak [15]). We will call two immersions  $f, \tilde{f}: M \rightarrow Q_c^4$  *locally congruent* if their lifts to the universal cover  $\tilde{Q}_c^4$  are congruent.

Let  $h_{ij}^\alpha = \langle D_{e_i} e_j, e_\alpha \rangle = \langle \alpha(e_i, e_j), e_\alpha \rangle$  where we assume always  $i, j \in \{1, 2\}$ ,  $\alpha, \beta \in \{3, 4\}$ .

Then we have

$$\omega_{i\alpha} = \sum h_{ij}^\alpha \omega_j.$$

Now we put

$$\begin{aligned} h_\alpha &= \frac{1}{2}(h_{11}^\alpha + h_{22}^\alpha), & k_\alpha &= \frac{1}{2}(h_{11}^\alpha - h_{22}^\alpha) - ih_{12}^\alpha, \\ h_\pm &= h_3 \pm ih_4, & k_\pm &= k_3 \pm ik_4. \end{aligned}$$

Note that the mean curvature vector is  $\eta = \sum h_\alpha e_\alpha$ , so  $\|\eta\| = h_\pm$ . The functions  $k_\pm$  are closely related to the ellipse of curvature which consists of the normal vectors

$$\alpha(ce_1 + se_2, ce_1 + se_2) = (c^2 - s^2)\alpha'(e_1, e_1) + 2cs\alpha(e_1, e_2) + \eta$$

where  $c = \cos \tau$ ,  $s = \sin \tau$  for arbitrary angles  $\tau$ . This ellipse is a circle if and only if  $\alpha'(e_1, e_1)$  and  $\alpha(e_1, e_2)$  are perpendicular and of the same length, i.e. iff  $k_+ = 0$  or  $k_- = 0$  at the particular point.

Now we introduce the complex valued 1-forms

$$\varphi = \omega_1 + i\omega_2,$$

$$\psi_\alpha = \omega_{1\alpha} - i\omega_{2\alpha} = k_\alpha \varphi + \bar{h}_\alpha \bar{\varphi}, \quad \psi_\pm = \psi_3 \pm i\psi_4 = k_\pm \varphi + h_\pm \bar{\varphi}.$$

Then one easily checks that the Cartan equations (C) are equivalent to the following set of equations:

$$(9) \quad d\varphi = -i\omega_{12} \wedge \varphi,$$

$$(10) \quad 0 = \psi_+ \wedge \varphi + \overline{\psi_- \wedge \bar{\varphi}},$$

$$(11) \quad 2id\omega_\mp = \bar{\psi}_\pm \wedge \psi_\pm - c\bar{\varphi} \wedge \varphi,$$

$$(12) \quad d\psi_\pm = i\omega_\mp \wedge \psi_\pm$$

with  $\omega_\pm := \omega_{12} \pm \omega_{34}$ .

REMARK. Under the usual isomorphism of the Lie algebra  $SO(4)$  onto the Lie algebra  $Sp(1) \times Sp(1)$ , the connection form  $\omega = (\omega_{ab})$  is mapped onto the pair

$$\left(-\frac{1}{2}(\omega_+ i + \bar{\psi}_- j); \frac{1}{2}(\omega_- i + \psi_+ j)\right)$$

where  $(i, j, ij)$  is the usual basis of  $Sp(1)$ , viewed as the space of imaginary quaternions. The equations (9)-(12) are just a translation of  $(C)$  into the Lie algebra  $Sp(1) \times Sp(1)$  (compare [10]).

Moreover, from the structure equations of the bundles  $TM$  and  $NM$  with induced connections we have

$$(13) \quad 2i d\omega_{12} \doteq -K \bar{\varphi} \wedge \varphi,$$

$$(14) \quad 2i d\omega_{34} = -K_N \bar{\varphi} \wedge \varphi.$$

Hence together with (11) and the definition of the  $\psi_{\pm}$  we get

$$(15) \quad |k_{\pm}|^2 = c - K \pm K_N + H^2$$

where  $H := \|\eta\| = |h_{\pm}|$ .

In particular, the immersion is minimal if and only if  $\psi_+$  and  $\psi_-$  are  $(1, 0)$ -forms, i.e. multiples of  $\varphi$ . Note that for any conformal coordinate  $z: M \rightarrow \mathbb{C}$  preserving the orientation given by  $(e_1, e_2)$ , we have  $\varphi = \mu dz$  for some complex valued function  $\mu$ . Since  $ds^2 = \varphi \bar{\varphi} = |\mu|^2 dz d\bar{z}$ , so  $\lambda := |\mu|$  is the conformal factor.

The mean curvature vector field is parallel if and only if

$$0 = \langle D\eta, e_{\beta} \rangle = \langle D(\sum h_{\alpha} e_{\alpha}), e_{\beta} \rangle = dh_{\beta} + h_{\alpha} \omega_{\alpha\beta},$$

hence

$$(16) \quad dh_{\pm} = \mp i h_{\pm} \omega_{34}.$$

So by (9), this is equivalent to  $d(h_{\pm} \bar{\varphi}) = i\omega_{\mp} \wedge (h_{\pm} \bar{\varphi})$ . Or, in other words, the  $(1, 0)$ -form

$$\psi'_{\pm} := \psi_{\pm} - h_{\pm} \bar{\varphi} = k_{\pm} \varphi,$$

the  $(1, 0)$ -part of  $\psi_{\pm}$ , satisfies

$$(12') \quad d\psi'_{\pm} = i\omega_{\mp} \wedge \psi'_{\pm}.$$

It is easy to check that for an arbitrary immersion  $f: M \rightarrow Q_c^4$  of an oriented surface  $M$  the quartic form

$$\Phi := \psi'_+ \psi'_- \varphi^2 = k_+ k_- \varphi^4$$

is independent of the choice of the Darboux frame and hence globally defined. In fact, its real part is  $\Phi_R(x) = \|\alpha'(x, x)\|^2 - \|\alpha'(x, Jx)\|^2$  where  $J$  denotes the almost complex structure on  $M$ . If the immersion has parallel mean curvature vector of length  $H \geq 0$ , this form is holomorphic: if  $\varphi = \mu dz$  for some conformal coordinate  $z$ , putting  $p_{\pm} := \mu k_{\pm}$  we get  $\psi'_{\pm} = p_{\pm} dz$  and  $\Phi = p_+ p_- \mu^2 dz^2$ , and from (9) and (12'):

$$(d\mu + \mu i\omega_{12}) \wedge dz = 0, \quad (dp_{\pm} - p_{\pm} i\omega_{\pm}) \wedge dz = 0.$$

Therefore,  $d(p_+ p_- \mu^2) \wedge dz = 0$  which shows that  $p_+ p_- \mu^2$  is a holomorphic function. If  $H \neq 0$ , this quartic form splits as  $\Phi = \Phi_+ \Phi_-$  where  $\Phi_{\pm}$  are holomorphic quadratic forms: We may choose in this case  $e_3 = \eta / \|\eta\|$ , then  $e_3, e_4$  are parallel in the normal bundle. So  $\omega_{34}$  vanishes and hence the 2-forms  $\Phi_{\pm} := \psi_{\pm} \varphi$  are holomorphic quadratic forms, by a similar argument as above.

### 5. Proof of the theorems.

The link between the structure equations (9)-(12) and equations (1), (4), (7) is given by the following lemma which is essentially an easy special case of a theorem of Chern [5]. Let  $(M, ds^2)$  be a surface. A complex valued function  $p$  on  $M$  will be called of *holomorphic type* if locally  $p = p_0 p_1$  with  $p_0$  smooth and nowhere vanishing and  $p_1 \circ z^{-1}$  holomorphic for a suitable conformal coordinate  $z$  on  $M$ . Clearly then  $|p|$  is an *AVT* function.

LEMMA 2. Let  $p$  be a smooth complex valued function on  $M$ ,  $p \neq 0$ , and  $\omega$  a real valued 1-form on  $M$ . Let  $\psi := p dz$  for some conformal coordinate  $z$ . Then the equality

$$(*) \quad d\psi = i\omega \wedge \psi$$

is valid if and only if  $p$  is of holomorphic type and

$$(17) \quad \omega = 2 \operatorname{Im} (\partial_{\bar{z}}(\log p) d\bar{z}).$$

Moreover, then

$$(18) \quad d\omega = -\frac{1}{2i} \Delta \log |p| \bar{\varphi} \wedge \varphi.$$

REMARK. Note that  $\partial_{\bar{z}} \log p$  and  $\Delta \log |p|$  are well defined even at the zeros of  $p$ , if  $p$  is of holomorphic type.

PROOF (see [7]). Since  $\omega$  is real, we have  $\omega = b dz + \bar{b} d\bar{z}$  for some complex function  $b$ . Thus (\*) is equivalent to the differential equation for  $p$ :

$$(**) \quad \partial_{\bar{z}} p = i\bar{b}p.$$

Let  $u$  be a solution of the inhomogeneous Cauchy-Riemann equation  $\partial_{\bar{z}} u = i\bar{b}$  and put  $p_0 := e^u$ . Then  $p_0$  is a nowhere vanishing solution of (\*\*). If  $p$  is any other solution of (\*\*), then  $p_1 := p/p_0$  is holomorphic. So  $p$  is of holomorphic type and (\*\*) is equivalent to  $\bar{b} = (1/i)\partial_{\bar{z}}(\log p)$  and hence to (17). Differentiating (17), we get (18).

(a) *The local equations* (1), (4), (7). Now let  $(M, ds^2)$  be a surface and  $f: M \rightarrow Q_c^4$  an isometric immersion with parallel mean curvature vector field of length  $H \geq 0$ . Let  $e_1, \dots, e_4$  be a Darboux frame along  $f$ . Then we may apply Lemma 2 to the forms  $\psi_{\pm} = k_{\pm}\varphi = k_{\pm}\mu dz$ , replacing (\*) with (12'). So the function

$$a_{\pm} := |k_{\pm}| = (c + H^2 - K_{\pm}K_N)^{\frac{1}{2}}$$

is  $AVT$  (since  $|\mu| = \lambda > 0$ ), and if  $a_{\pm} \neq 0$ , we get from (18)

$$-2i d\omega_{\mp} = (\Delta \log a_{\pm} - K)\bar{\varphi} \wedge \varphi$$

since  $\Delta \log \lambda = -K$  (which also follows from (18), using (9) and (13)). On the other hand, we get from (13) and (14)

$$-2i d\omega_{\mp} = (K \mp K_N)\bar{\varphi} \wedge \varphi$$

and therefore,

$$(19) \quad \Delta \log a_{\pm} = 2K \mp K_N.$$

If  $H \neq 0$ , the form  $\omega_{34}$  is exact by (16) and therefore  $K_N = 0$ , by (14); so we get (1). If  $H = 0$ , we get (4). If  $a_+ a_- \equiv 0$  then either  $a_+ \equiv 0$  or  $a_- \equiv 0$  since  $a_+, a_-$  are both  $AVT$ . If  $a_+ = a_- = 0$ , the immersion is totally umbilic (i.e.  $\alpha' = 0$ ), hence totally geodesic if  $H = 0$ . If  $a_- = 0, a_+ \neq 0$ , we have  $H = 0$  (since otherwise  $a_+ = a_-$ ) and there-

fore  $K_N = c - K$  and  $a_+ = (2(c - K))^{\frac{1}{2}}$ , and similarly if  $a_+ = 0$ ,  $a_- \neq 0$ . This proves (7).

(b) *Uniqueness.* Let  $M$  be oriented,  $\hat{Q}_c^4$  simply connected and  $\tilde{f}, \bar{f}: M \rightarrow \hat{Q}_c^4$  two isometric immersions, both with the same normal curvature  $K_N$  and with parallel mean curvature vector field of length  $H \geq 0$ . Assume that  $f, \bar{f}$  induce the same holomorphic quartic form  $\Phi$  on  $M$ . If  $H \neq 0$ , assume further that also  $\Phi_+$  and  $\Phi_-$  are the same for both immersions. We claim that then  $\bar{f} = g \circ f$  for some oriented isometry  $g$  of  $\hat{Q}_c^4$ .

In fact, in case  $H \neq 0$ , the 1-forms  $\psi_+, \psi_-$  are the same for both immersions, so  $\omega_{\pm}$  are the same by (12') and hence the immersions are congruent. In case  $H = 0$ , let  $e_1, e_2, e_3, e_4$  and  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$  be local oriented Darboux frames along  $f$  and  $\bar{f}$  resp. (We consider  $TM$  as a subbundle of  $f^*T\hat{Q}$  and  $\bar{f}^*T\hat{Q}$ .) Then we have  $|k_{\pm}| = |\tilde{k}_{\pm}|$ ,  $k_+k_- = \tilde{k}_+\tilde{k}_-$ , and therefore

$$\tilde{\psi}_+ = e^{i\tau}\psi_+, \quad \tilde{\psi}_- = e^{-i\tau}\psi_-$$

for some real function  $\tau$ . Rotating the normal frame of  $\bar{f}$  by this angle  $\tau$ , we get  $\tilde{\psi}_{\pm} = \psi_{\pm}$  for this new frame. As before, we get  $\tilde{\omega}_{\pm} = \omega_{\pm}$  and so  $f$  and  $\bar{f}$  are congruent. In both cases, the isometry  $g$  of  $\hat{Q}_c^4$  taking  $f$  to  $\bar{f}$  preserves the orientation of both the tangent and the normal bundle, so it is oriented.

REMARK. In fact, it is not necessary to assume that  $\tilde{K}_N = K_N$ . We have  $k_+k_- = (c - K)^2 - K_N^2$ , and since  $\Phi = \tilde{\Phi}$ , we have  $\tilde{K}_N = \pm K_N$ . In case  $\tilde{K}_N = -K_N$ , the immersion  $f$  and  $\bar{f}$  differ by an orientation reversing isometry. Similar, in the case  $H \neq 0$ , it is only necessary to assume that  $\tilde{\Phi} = \Phi$  and  $\tilde{\psi}_3 = \psi_3$  where  $e_3, \tilde{e}_3$  are pointing in the direction of the mean curvature vector.

(c) *Existence.* Let  $(M, ds^2)$  be a simply connected surface,  $H \geq 0$  some number and  $K_N$  a function which is zero if  $H > 0$ , such that

$$a_{\pm} := (c + H^2 - K_{\pm}K_N)^{\frac{1}{2}}$$

is  $AVT$  satisfying

$$(17) \quad \Delta \log a_{\pm} = 2K \mp K_N$$

unless  $a_{\pm} \equiv 0$ . Let  $U \subset M$  be an open disk and  $z: U \rightarrow \mathbb{C}$  a conformal coordinate. Choose an oriented orthonormal basis  $e_1, e_2$  on  $U$ . Put  $\omega_i = \langle \cdot, e_i \rangle$  and  $\varphi = \omega_1 + i\omega_2$ . Then  $\varphi = \mu dz$  with  $|\mu| = \lambda$  the conformal factor. By (17) we have

$$\Delta \log \lambda^4 a_+ a_- = 0$$

(recall that  $\Delta \log \lambda = -K$ ). Since  $b := \lambda^4 a_+ a_-$  is  $AVT$ , there exists a holomorphic function  $g$  with  $b = |g|$  (see [7], 3.12). In fact we have  $b = b_0 |h|$  with  $h$  holomorphic and  $b_0 > 0$  and  $\Delta \log b_0 = 0$ . So  $\log b_0$  is the real part of a holomorphic function  $l$  and thus  $g = e^l \cdot h$  is holomorphic with  $|g| = b$ . In fact,  $g$  is uniquely determined up to a constant factor  $e^{i\tau}$ . Choose any such  $g$ . Let  $k_+, k_-$  be functions of holomorphic type satisfying  $|k_{\pm}| = a_{\pm}$  and  $k_+ k_- \mu^4 = g$  and put  $\psi_{\pm} = k_{\pm} \varphi + H \bar{\varphi}$ . Then (10) is automatically satisfied. To satisfy (12') and hence (12), according to Lemma 2 we have to put

$$\omega_{\mp} = 2 \operatorname{Im} (\partial_{\bar{z}} (\log p_{\pm}) d\bar{z})$$

whenever  $p_{\pm} := \mu k_{\pm}$  is not the zero function. Then we have

$$2i d\omega_{\mp} = - (\Delta \log a_{\pm} + \Delta \log \lambda) \bar{\varphi} \wedge \varphi = - (K \mp K_N) \bar{\varphi} \wedge \varphi.$$

On the other hand,

$$\bar{\psi}_{\pm} \wedge \psi_{\pm} = (a_{\pm}^2 - H^2) \bar{\varphi} \wedge \varphi = (c - K \pm K_N) \bar{\varphi} \wedge \varphi;$$

this proves Equation (11). Finitely, put

$$\tilde{\omega}_{12} := \frac{1}{2} (\omega_+ + \omega_-) = \operatorname{Im} (\partial_{\bar{z}} (\log p_+ p_-) d\bar{z}) = -2 \operatorname{Im} (\partial_{\bar{z}} (\log \mu)) d\bar{z}.$$

Since  $\varphi = \mu dz$ , we get by Lemma 2

$$d\varphi = i \tilde{\omega}_{12} \wedge \varphi$$

and hence  $\tilde{\omega}_{12} = \omega_{12}$ . Thus (9) is satisfied.

It remains to consider the case  $a_+ a_- \equiv 0$  which implies  $H = 0$ . If  $a_- \equiv 0$ , that means  $K_N + K = c$ , we put  $\psi_+, \omega_-$  as above and  $\psi_- = 0, \omega_+ = 2\omega_{12} - \omega_-$ . Then  $\omega_-$  still satisfies (11) and hence

$$2i d\omega_- = - (K - K_N) \bar{\varphi} \wedge \varphi$$

as before. Therefore,

$$2i d\omega_+ = 2i(2d\omega_{12} - d\omega_-) = - (K + K_N)\bar{\varphi} \wedge \varphi = - c\bar{\varphi} \wedge \varphi,$$

and hence  $\omega_+$  satisfies (11). Equation (12) is valid for  $\psi_+$ ,  $\omega_-$ , as before, and it is trivial for  $\psi_-$ ,  $\omega_+$ . (9) is satisfied by definition of  $\omega_{12}$  (as an intrinsic quantity).

If instead  $a_+ \equiv 0$ , we argue similarly replacing  $+$  with  $-$ . If both  $a_+$  and  $a_-$  are the zero functions, i.e.  $K = c$ ,  $K_N = 0$ , we put  $\psi_+ = \psi_- = 0$ ,  $\omega_+ = \omega_- = \omega_{12}$ , and Equations (9)-(12) are satisfied.

Thus, by the general existence theorem, we get an isometric immersion  $f: U \rightarrow \hat{Q}_c^4$  with parallel mean curvature vector of length  $H$  (since (12') is satisfied), together with a Darboux frame  $e_1, \dots, e_4$  along  $f$ . In fact, we recover any quartic form  $\Phi$  and in the case  $H \neq 0$  any two quadratic forms  $\Phi_+$ ,  $\Phi_-$ , holomorphic with given absolute values, by choosing the functions  $g$ ,  $k_+$ ,  $k_-$  suitably.

Now cover  $M$  by open coordinate balls  $U_t$ . Then we get immersions  $f_t: U_t \rightarrow \hat{Q}_c^4$ . Choose these so that the corresponding quartic or quadratic forms agree in the intersections  $U_t \cap U_s$ . By simple connectivity of  $M$ , this can be achieved. Then  $f_t|_{U_t \cap U_s} = g \circ f_s|_{U_t \cap U_s}$  for some proper motion  $g$  of  $\hat{Q}_c^4$ , by the uniqueness part (b). Thus  $f_t$  and  $g \circ f_s$  together define an immersion of  $U_t \cup U_s$ . Continuing this process, we get an immersion  $f: M \rightarrow \hat{Q}_c^4$  with the desired properties.

It remains to verify the remark following Theorem 1. If  $f: M \rightarrow Q_c^4$  is an immersion with parallel mean curvature vector of length  $H \neq 0$ , the holomorphic quadratic forms  $\Phi_+$  and  $\Phi_-$  have the same absolute values  $|k_+| = |k_-| = (c + H^2 - K)^{\frac{1}{2}}$ . So they differ by a constant factor of unit length: Say  $\varphi_+ = e^{2\delta i} \varphi_-$ . Hence  $\psi'_+ = e^{2\delta i} \psi'_-$ , and putting  $\psi'_\alpha := \psi_\alpha - h_\alpha \bar{\varphi}$  (see ch. 4), we get

$$\psi'_3 \sin \delta = \varphi'_4 \cos \delta.$$

In other words, the parallel vector  $\xi := -(\sin \delta)e_3 + (\cos \delta)e_4$  is umbilic, i.e.  $\langle \alpha', \xi \rangle = 0$  (compare [17]). It is easy to see ([17]), that  $f(M)$  lies in a totally umbilic hypersurface  $Q_c^3$  in  $Q_c^4$  with normal vector  $\xi$ . Thus  $\sigma := \pi/2 + \delta$  is the angle between the mean curvature vector  $\eta = He_3$  and  $\xi$ .

On the other hand, if  $f, \tilde{f}: M \rightarrow Q_c^4$  are isometric immersions with parallel mean curvature vector of length  $H \geq 0$  with  $|\tilde{\Phi}| = |\Phi|$  and

$\tilde{\psi}_+/\tilde{\psi}_- = \psi_+/\psi_-$ , then  $\tilde{\Phi} = e^{4\tau i} \Phi$  for some constant angle  $\tau$ . Hence  $\tilde{\psi}_\pm = e^{2\tau i} \psi_\pm$  and so  $\tilde{\alpha} = R_\tau^* \alpha$  where  $R_\tau$  denotes the rotation by  $\tau$  in the tangent plane.

### Appendix: Mean curvature vector of constant length.

We want to give an example of a surface in 4-space whose mean curvature vector has constant length but whose quartic form  $\Phi$  (see ch. 4) is not holomorphic (compare ch. 2).

Put  $e: \mathbf{R} \rightarrow \mathbf{C}$ ,  $e(t) = e^{it}$ . Consider the immersion  $f: I \times J \rightarrow \mathbf{C}^2 = \mathbf{R}^4$ ,  $f(s, t) = (e(s), g(s)e(t))$  where  $I, J$  are open intervals and  $g: I \rightarrow (0, \infty)$  some smooth function. Let  $f_s, f_t$  denote the partial derivatives. We have  $f_s \perp f_t$  and  $\|f_s\| = (1 + g'^2)^{\frac{1}{2}} =: h$ ,  $\|f_t\| = g$ . Let us put

$$\begin{aligned} e_1 &= \frac{1}{h} f_s = \frac{1}{h} (e', g'e), & e_2 &= \frac{1}{g} f_t = (0, e'), \\ e_3 &= (e, 0), & e_4 &= \frac{1}{h} (-g'e', e); \end{aligned}$$

this is a Darboux frame for the immersion  $f$ . For the 2nd fundamental form  $\alpha$  we get in terms of this frame:

$$\alpha(e_1, e_1) = -h^{-2}e_3 + h^{-3}g''e_4,$$

$$\alpha(e_2, e_2) = -\frac{1}{gh}e_4,$$

$$\alpha(e_1, e_2) = 0.$$

Therefore, the mean curvature vector  $\eta = \frac{1}{2}(\alpha(e_1, e_1) + \alpha(e_2, e_2))$  has constant length if and only if

$$(A1) \quad h^{-4} + \left( h^{-3}g'' - \frac{1}{gh} \right)^2 = c_1 = \text{const.}$$

On the other hand, the corresponding quartic form is

$$\Phi = \psi'_+ \psi'_- \varphi^2 = (k_3^2 + k_4^2) \varphi^4$$

with

$$k_3 = -\frac{1}{2}h^{-2}, \quad k_4 = \frac{1}{2}\left(h^{-3}g'' + \frac{1}{gh}\right)$$

and  $\varphi = \omega_1 + i\omega_2$  where  $\omega_i = \langle df, e_i \rangle$ . Now putting  $du = (\sigma/g)ds$  we have  $\varphi = g(du + i dt)$ , and so  $(u, t)$  is a conformal coordinate system for the metric on  $I \times J$  induced by  $f$ , with conformal factor  $g$ . Since its coefficient function is real,  $\Phi = (k_3^2 + k_4^2)g^4(du + i dt)^4$  is holomorphic if and only if  $(k_3^2 + k_4^2)g^4 = \text{const}$ , hence iff

$$(A2) \quad \left(h^{-4} + \left(h^{-3}g'' + \frac{1}{gh}\right)^2\right)g^4 = c_2 = \text{const}.$$

Since  $h^2 = 1 + g'^2$ , Equations (A1) and (A2) are 2nd order differential equations for  $g$ . Since they are independent, one finds a function  $g$  which solves (A1) but not (A2).

To be specific, let  $c_1 = 5$  and consider the solution  $g$  of

$$(A1') \quad g'' = g^{-1}(1 + g'^2) + (5(1 + g'^2)^3 - (1 + g'^2))^{\frac{1}{3}}$$

with initial values

$$g(0) = 1, \quad g'(0) = 0.$$

Then  $g$  solves also (A1), and it follows from (A1') that  $g$  is even with  $g''(0) = 3$ ,  $g^{(4)}(0) = 78$ . Instead, from (A2) we would get  $g^{(4)}(0) = 72$ , so  $g$  does not solve (A2).

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