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On Some Subgroups of Infinite Rank Butler Groups.

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1. Introduction.

Al groups in this note are abelian and torsion-free unless stated otherwise. Undefined notations are standard as in [F]. Some twenty years ago M. C. R. Butler [Bu] studied pure subgroups of finite rank completely decomposable groups. These groups were later called Butler groups by L. Lady. Following the lead of Bican and Salce [B-S] we call a group B a Butler group if $B\text{ext}(B, T) = 0$ for all torsion groups T. The functor $B\text{ext}(\cdot, \cdot)$ is the subfunctor of $\text{Ext}(\cdot, \cdot)$ of all balanced extensions as introduced by R. Hunter [Hu]. Butler groups of finite rank have been studied to some extend, e.g. [A1], [B]. Recently, Butler groups of infinite rank attracted some attention, see [A2], [B-S-S], [A-H] and [D-R]. D. Arnold raises in [A2] the question about Butler groups being closed with respect to pure countable subgroups. This question is of particular interest in the light of the results in [D-R]. All main results in [D-R] require the hypothesis « All pure finite rank subgroups of the Butler group B $(|B| = \aleph_1)$ are again Butler groups ». The purpose of this paper is to show that for Butler groups of rank $\aleph_1$ this hypothesis is redundant:

THEOREM 1. Each pure, countable subgroup of a Butler group of rank $\aleph_1$ is again a Butler group.

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Observe that a countable torsion-free group is a Butler group iff each pure finite rank subgroup is a Butler group, c.f. [B-S]. The main theorem in [D-R] now reads as follows:

THEOREM. Assume $V = L$ holds. The following are equivalent for a torsion-free group B of cardinality $\aleph_1$:

(*a*) $B\text{ext}(B, S) = 0$ for all torsion groups S.

(*b*) $B\text{ext}(B, T) = 0$ for any countable, Σ-cyclic torsion group T.

(*c*) B is a B_2-group, i.e. B has an ω_1-filtration $B = \bigcup_{\alpha<\omega_1} B_\alpha$ into pure, countable subgroups B_α with $B_0 = 0$ and $B_{\alpha+1} = B_\alpha + C_\alpha$, C_α a finite rank Butler group.

(*d*) B has an ω_1-filtration where each B_α is a (pure and countable) decent subgroup of B, c.f. [A-H] or [D-R].

(*e*) B has an ω_1-filtration such that each B_α has the T.E.P. in B. (We refer to [D-R] for the definition of the torsion extension property T.E.P.)

This theorem, as stated above, is undecidable in ZFC. But we would like to conjecture, that without (*b*) it's valid in ZFC.

A subgroup A of a group B is called separable in B, c.f. [H1], if for any $b \in B - A$ there is a countable sequence $\{a_n | n < \omega\} \subset A$, such that for any $a \in A$, the height sequences satisfy the inequality $|b + a| \leqslant |b + a_n|$ for some $n < \omega$. Our Theorem 1 is an immediate consequence of the more general

THEOREM 2. Let A be a pure and separable subgroup of the torsion-free group B with B/A countable. If K is a generalized regular subgroup of A, then there exists a generalized regular subgroup L of B with $L \cap A = K$.

The notion of a generalized regular subgroup was defined in [B] and used the characterize countable Butler groups: The countable group B is a Butler group iff all localizations B_p, p a prime, are completely decomposable and for each generalized regular subgroup K of B and each (pure) finite rank subgroup H of B, $(H/(H \cap K))_p = 0$ for almost all p, c.f. [A1], [B-S]. In order to show that Theorem 1 follows from Theorem 2, let B be a Butler groups of rank $\aleph_1$ and H a pure finite rank subgroup of B. Consider an $\aleph_1$-filtration $B = \bigcup_{\alpha<\omega_1} B_\alpha$ into pure countable subgroups B_α with $B_0 = H$. Let K_0 be a generalized regular subgroup of $H = B_0$. Since B_1 is countable, B_1/B_0 is countable and Theorem 2 shows the existence of a generalized regular subgroup K_1 of B_1 with $K_1 \cap B_0 = K_0$. By transfinite induction there are generalized regular subgroups K_α of B_α for all $\alpha < \omega_1$ with $K_{\alpha+1} \cap \cap B_\alpha = K_\alpha$. The subgroup $K = \bigcup_{\alpha<\omega_1} K_\alpha$ is a generalized regular subgroup of B. Since B is a Butler group, $H/K_0 \cong (H+K)/K \subset B/K$ and $(B/K)_p = 0$ for almost all primes p. Hence $(H/K_0)_p = 0$ for almost all p and the characterization of countable Butler groups mentioned above shows that H is a Butler group.

We give an example showing that the separability condition in Theorem 2 is indispensable. Therefore, our approach doesn't yield a complete answer to D. Arnold's question. We only get the following:

COROLLARY. A pure and countable subgroup of a Butler group B is Butler provided B satisfies the third axiom of countability [H2] with respect to separable subgroups.

2. Notations.

For an element x of a group G, $|x| = (|x|_p)_{p \text{ prime}}$ is the height sequence of x and we add a superscript $|x|^G$ to indicate in which group the height is computed. $|x|_p$ denotes the p-height. We say $|x| \leqslant |y|$ if $|x|_p \leqslant |y|_p$ for all primes p. Recall that a subgroup K of a torsion-free group A is a (full) generalized regular subgroup if A/K is torsion and for each pure rank one subgroup R of A the p-primary part $(R/(R \cap K))_p = 0$ for almost all primes p. We call the generalized regular subgroup K of A ω-regular, if for any pure finite rank subgroup H of A, $(H/(H \cap K))_p = 0$ for almost all primes p. In this paper we find it more useful to deal with an equivalent concept:

Let A be a torsion-free group and T torsion. We call a map $\varphi \in \in \mathrm{Hom}(A, T)$ *regular* if for any pure rank 1 subgroup R of A, $(\varphi(R))_p = 0$ for almost all primes p. Observe that φ is regular iff the kernel of φ is a generalized regular subgroup. We call φ ω-regular, if $(\varphi(H))_p = 0$ for almost all primes p whenever H is a pure finite rank subgroup of A. Observe that if A is completely decomposable or if A is the sum of finitely many pure rank 1 subgroups then each regular map is obviously ω-regular. If A is any group, $\bar{A}$ denotes the divisible hull of A. Let ω denote the first infinite ordinal, i.e. $\omega = \{0, 1, 2, \dots\}$.

3. Lifting regular maps.

If $\varphi \in \operatorname{Hom}(A, T)$ is regular, we say φ lifts to a regular $\bar{\varphi} \in \operatorname{Hom}(\bar{A}, \bar{T})$ if $\bar{\varphi}$ is regular and $\bar{\varphi}|A = \varphi$.

PROPOSITION 1. Let $\varphi: A \to T$ be a regular map from the torsion-free group A into the torsion group T. If φ lifts to a regular $\bar{\varphi}: \bar{A} \to \bar{T}$, then φ is ω-regular. The converse holds provided that A is countable.

PROOF. If φ lifts to a regular $\bar{\varphi}$, then $\bar{\varphi}$ is ω-regular since $\bar{A}$ is divisible and so is φ, being the restriction of $\bar{\varphi}$ to A. Now assume φ is ω-regular. If A has finite rank, there is a finite set P of primes and $(\varphi(A))_p = 0$ for $p \notin P$. Then φ extends to a map $\bar{\varphi} \in \operatorname{Hom}(\bar{A}, \bigoplus_{p \in P} \bar{T}_p)$. Since P is finite, $\bar{\varphi}$ is regular. Hence we may assume that A has infinite rank, i.e. $A = \bigcup_{i=1}^{\infty} A_i$, A_i pure of finite rank, $A_i \subset A_{i+1}$ for all i and $\operatorname{rk}(A_{i+1}/A_i) = 1$. Then $\bar{A} = \bigcup_{i=1}^{\infty} \bar{A}_i$ and $\bar{A}_{i+1} = \bar{A}_i \oplus e_i Q$, $e_i \in A_{i+1}$. Let $\varphi_n = \varphi|A_n$ and assume we constructed an ω-regular $\bar{\varphi}_n: \bar{A}_n \to \bar{T}$ already with $\bar{\varphi}_n|A_n = \varphi_n$. Let $\pi_0: \bar{A}_{n+1} \to \bar{A}_n$ and $\pi_1: \bar{A}_{n+1} \to e_n Q$ be the natural projections induced by the decomposition $\bar{A}_{n+1} = \bar{A}_n \oplus e_n Q$. We define a map $\varrho: \pi_1(A_{n+1}) \to \bar{T}$ by setting $\varrho(\pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \varphi_{n+1}(x)$ for all $x \in A_{n+1}$. We have to show that ϱ is well defined. Let $x \in A_{n+1}$ such that $\pi_1(x) = 0$, i.e. $x \in A_{n+1} \cap \bar{A}_n = A_n$ since A_n is pure in A_{n+1}. This implies $\bar{\varphi}_n(\pi_0(x)) = \bar{\varphi}_n(x) = \varphi_n(x) = \varphi_{n+1}(x)$. This shows $\varrho \in \operatorname{Hom}(\pi_1(A_{n+1}), \bar{T})$. Let R be the subgroup of Q with $e_n R = \pi_1(A_{n+1})$. Then $\varrho(e_n R) = \varrho(\pi_1(A_{n+1})) \subset \bar{\varphi}_n(\bar{A}_n) + \varphi_{n+1}(A_{n+1})$ and since $\bar{\varphi}_n$ and φ_{n+1} are both ω-regular, $(\varrho(e_n R))_p = 0$ for almost all p. Hence ϱ lifts to a regular $\bar{\varrho}: e_n Q \to \bar{T}$. Now consider the map $\bar{\varphi}_n - \bar{\varrho}: \bar{A}_n \to \bar{T}$ and let $x \in A_{n+1}$. Then

$$(\bar{\varphi}_n - \bar{\varrho})(x) = (\bar{\varphi}_n - \bar{\varrho})(\pi_0(x) + \pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \bar{\varrho}(\pi_1(x)) = \\ = \bar{\varphi}_n(\pi_0(x)) - \varrho(\pi_1(x)) = \varphi_{n+1}(x).$$

Thus $\bar{\varphi}_n - \bar{\varrho}$ is the desired (ω)-regular map extending φ_{n+1}. Induction completes the proof. □

PROPOSITION 2. Let A be a pure subgroup of B, C a finite rank Butler subgroup such that $B = A + C$. Let $\varphi: B \to T$ be regular and assume that $\varphi|A$ lifts to a regular $\psi: \bar{A} \to \bar{T}$. Then there exists $\Psi: \bar{B} \to \bar{T}$ regular with $\Psi|\bar{A} = \psi$ and $\Psi|B = \varphi$ if only $\varphi|C$ is ω-regular.

PROOF. We will use a similar argument as in the proof of the previous proposition. Let $\bar{B} = \bar{A} \oplus X$ and $\pi_0: \bar{B} \to \bar{A}$ and $\pi_1: \bar{B} \to X$ be the natural projections. Then $\pi_1(B) = \pi_1(C)$ is a Butler group of finite rank. Again, define $\varrho: \pi_1(B) \to \bar{T}$ by $\varrho(\pi_1(x)) = \psi(\pi_0(x)) - \varphi(x)$ for all $x \in B$. Then $\varrho(\pi_1(B)) = \varrho(\pi_1(C)) \subset \psi(\pi_0(C)) + \varphi(C)$. Since C has finite rank and ψ is regular, $\psi(\pi_0(C))_p = 0$ for almost all primes p and since $\varphi|C$ is a regular map from the finite rank Butler group C, $(\varphi(C))_p = 0$ for almost all primes p as well. Hence $\varrho(\pi_1(B))_p = 0$ for almost all p and since $\pi_1(B)$ is countable, we may use Proposition 1 to conclude that ϱ lifts to a regular $\bar{\varrho}: \overline{\pi_1(B)} = X \to \bar{T}$. Again $\Psi = \psi - \bar{\varrho}: \bar{B} \to \bar{T}$ is a regular map extending φ. □

COROLLARY. (*a*) A countable torsionfree group B is a Butler group iff B is locally completely decomposable and each regular map from B into a torsion group lifts to a regular map from the divisible hull of B into the divisible hull of the torsion group. (*b*) If B is a B_2-group then each regular map $\varphi: B \to T$ lifts to a regular $\bar{\varphi}: \bar{B} \to \bar{T}$.

PROOF. Part (*a*) is just a reformulation of a result in [B-S]. In order to show (*b*), let B be a B_2-group, i.e. $B = \bigcup_{\alpha<\lambda} B_\alpha$ is a union of a smooth chain of pure subgroups B_α with $B_0 = 0$ and $B_{\alpha+1} = B_\alpha + H_\alpha$ for a finite rank Butler subgroup of B. Let $\varphi: B \to T$ be a regular map. By transfinite induction we find regular maps $\psi_\alpha: \bar{B}_\alpha \to \bar{T}$ such that $\psi_{\alpha+1}|\bar{B}_\alpha = \psi_\alpha$ and $\psi_\alpha|B_\alpha = \varphi|B_\alpha$. Set $\psi_0 = 0$ and suppose ψ_α is defined already. The map $\varphi|H_\alpha$ is regular and H_α is a Butler group of finite rank. Thus $\varphi|H_\alpha$ is ω-regular and Proposition 2 provides a map $\psi_{\alpha+1}$ with the desired properties.

We would like to raise the following

QUESTION. If B is any Butler group, is (*b*) true? (This would answer D. Arnold's question mentioned in the introduction.)

We are now going to prove our main

THEOREM 2′. Let A be a pure separable subgroup of B with B/A countable. Then each regular $\varphi: A \to T$ lifts to a regular $f: B \to \bar{T}$.

PROOF. Since A is separable in B and a countable extension of a separable subgroup is separable, there is a chain $A = A_0 \subset A_1 \subset \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset B$ with $\mathrm{rk}\,(A_{i+1}/A_i) = 1$, $B = \cup A_i$ and each A_i is separable in A_{i+1}. Therefore we may assume that B/A has rank 1. Then $\bar{B} = \bar{A} \oplus xQ$ with $x \in B - A$. If S is a subset of B, let $\langle S\rangle^*$ be the pure subgroup of B generated by S and let R be the subgroup of Q with $xR = B \cap xQ$. For any $q \in R$, we have a sequence $\{a_{q,n}\colon n < \omega)$ such that for any $a \in A$ there is some n such that $|a + qx|^B \leqslant \leqslant |a_{q,n} + qx|^B$. Choose an enumeration of R and $R \times \omega$ such that

$$\{a_{q,n} + qx\colon q \in R,\, n < \omega\} = \{a_i + q_i x\colon i < \omega\}\,.$$

Observe that the map $(i \to q_i)$ is not one to one. Let $q_i = r_i/s_i$ with r_i, s_i relatively prime. We may assume $r_i \neq 0$ and set $y_i = a_i + q_i x$. Define $P_i = \{p$ prime: p doesn's divide neither r_i nor s_i, $|y_i|_p^B > > \min\{|a_i|_p^A, |x|_p^B\}\}$. Note that for $p \in P_i$, $|x|_p^B = |q_i x|_p^B$. For xS the image of the projection of B into xQ, we define a map $g\colon xS \to \bar{T}$. First we define $g(xR) = 0$. In order to continue, we'll define $g(p^{-j}x)$ for $j < \omega$ with $p^{-j} \notin R$. In order to define a map $g\colon xS \to \bar{T}$, we only have to make sure that $p\big(g(p^{-(j+1)}x)\big) = g(p^{-j}x)$ and avoid contradictory assignments. We'll define g in such a way that for all $i < \omega$, $g(p^{-n}x) = s_i r_i^{-1}\big(\bar{\varphi}(p^{-n}a_i)\big)_p$ for almost all primes p. Here $\bar{\varphi}\colon \bar{A} \to \bar{T}$ is a fixed map extending φ. For $p \in P_1$, set $l_{1,p} = |y_1|_p^B$ and observe $l_{1,p} > |x|_p^B = |a_1|_p^B = |a_1|_p^A$. Now define $g(p^{-l_{1,p}}x) = -\,(s_i r_i^{-1})\bar{\varphi}(a_i)_p$. Assume we defined $g(p^{-l_{1,p}}x)$ for all $i < n$, $l_{i,p} = |y_i|_p^B$ and all $p \in P_i$ such that $g(p^{-l_{1,p}}x) = g(p^{-l_{1,p}}x)_p = -\,(s_i r_i^{-1})\bar{\varphi}(a_i)_p$ for almost all $p \in \bigcup_{i=1}^{n-1} P_i$. Let $E = \{i \mid 1 \leqslant i < n,\; P_n \cap P_i$ is infinite$\}$, $i \in E$ and $p \in P_n \cap P_i$. This implies $l_{i,p} > \min\{|a_i|_p, |x|_p\}$ and $l_{n,p} > \min\{|a_n|_p, |x|_p\}$ and therefore $l_{n,p} > |a_n|_p = |x|_p = |a_i|_p$. Let $l_p = \min\{l_{i,p}, l_{n,p}\}$. Then y_n, y_i, $r_i s_n y_n$ and $r_n s_i y_i$ are all in $p^{l_p}B$. Hence

$$r_i s_n y_n - r_n s_i y_i = r_i s_n a_n - r_n s_i a_i \in p^{l_p}B \cap A = p^{l_p}A\,.$$

Since φ is regular, there is a subset $W_{i,n}$ of $P_i \cap P_n$ with $(P_i \cap P_n) - - W_{i,n}$ finite and $\varphi\big(p^{-l_p}(r_i s_n a_n - r_n s_i a_i)\big)_p = 0$ for all $p \in W_{i,n}$. Thus $(s_n r_n^{-1})\bar{\varphi}(p^{-l_p}a_n) = (s_i r_i^{-1})\bar{\varphi}(p^{-l_p}a_i)$ for all $p \in W_{i,n}$. We now consider two cases for a prime $p \in W_n$, where $W_n = \bigcup_{i \in E} W_{i,n}$.

Case 1. For all $i \in E$ with $p \in W_{i,n}$ we have $l_{i,p} \leqslant l_{n,p}$. Here we define $g(p^{-l_{n,p}}x)$ to be an element in $\overline{T}_p$ such that $p^{l_{n,p}-l_{j,p}}g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{j,p}}a_j)$ where $l_{j,p} = \max\{l_{ip,} | \, i \in E\}$.

Case 2. There is $i \in E$ such that $p \in W_{i,n}$ and $l_{n,p} < l_{i,p}$. Again, let $l_{j,p} = \max\{l_{i,p}\; i \in E\}$ and observe that

$$g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{j,p}}a_j)p^{l_{j,p}-n,p}$$

was already defined. If $p \in P_n - \bigcup_{i=1}^{n-1} P_i$, set

$$g(p^{-l_{n,p}}x) = -q_n^{-1}\bar{\varphi}(p^{-l_{n,p}}a_n)\,.$$

Our observations above show that these assignments define a map g' on a subgroup xS' of xS. Let g be any lifting of g' to xS. Our construction shows that our map g has the property that for any $i < \omega$, $(\bar{\varphi} \oplus g)(\langle y_i \rangle^*)_p = 0$ for almost all primes p. Let $f = (\bar{\varphi} \oplus g)|B$ and $y \in B - A$. Then there is a natural number $k \neq 0$ such that $ky = a + qx$ with $a \in A$, $q \in R$. For some $i < \omega$, we have $q = q_i$ and $|a + qx|^B \leqslant |a_i + qx|^B$. This implies $|a - a_i|^A \geqslant |a + qx|^B$ and for $e \leqslant |a + qx|_p$ we get

$$f(p^{-e}(a + qx))_p = \bar{\varphi}(p^{-e}a)_p + g(p^{-e}qx)_p = \bar{\varphi}(p^{-e}a_i)_p + g(p^{-e}qx)_p = f(p^{-e}(a_i + qx))_p = 0$$

for almost all p, since $p^{-e}(a - a_i) \in A$ and φ regular. This shows that $f(\langle y \rangle_*)_p = 0$ for almost all primes p, i.e. f is the desired map. □

We conclude this paper with the promised

EXAMPLE. There is a torsion-free group B and a pure subgroup A of B with $rk(B/A) = 1$ and a regular map $\varphi: A \to T = \bigoplus_{p \text{ prime}} Z(p^\infty)$ such that φ can not be lifted to a regular $\psi: B \to T$.

PROOF. Let $\{p_{n,i}: n, i < \omega\}$ be a list of all primes and $F = \bigoplus_{i<\omega} e_i Z$ a free group of countable rank. Call two maps $f, g: \omega \to \omega$ almost disjoint, if $\{i < \omega: f(i) = g(i)\}$ is finite. Zorn's Lemma implies that there is a maximal family Σ of almost disjoint maps from ω into ω.

We introduce independent elements a_f, $f \in \Sigma$ and define $A = < F$, a_f, $p_{n,f(n)}^{-1}(e_n - a_f)$: $f \in \Sigma$, $n \langle \omega \rangle$. It's not hard to see that A is homogeneous of type 0 and $p^{-1}(a_f - a_g) \in A$ iff $p_{n,g(n)} = p = p_{n,f(n)}$ for some $n < \omega$, i.e. $f(n) = g(n)$. Define $\varphi: A \to T$ by setting $\varphi(e_n) = \varphi(a_f) = 0$ for all $n < \omega$, $f \in \Sigma$ and $\varphi(p_{n,f(n)}^{-1}(e_n - a_f) = t_{n,f(n)}$, an element of exponent 1 in $Z(p_{n,f(n)}^{\infty})$. Since A is homogeneous of type 0, φ is regular.

We now define a rank 1 extension B of A by introducing a new independent element x and setting

$$B = \langle A, x, p_{n,f(n)}^{-1}(x + a_f), p_{n,i}^{-1}(x + e_n) \rangle : f \in \Sigma, n, i < \omega .$$

Routine verification shows that A is pure in B and B/A is rank 1 of type $(1, 1, 1, \ldots)$.

By way of contradiction assume that $\varphi: A \to T$ lifts to a regular map $\psi: B \to T$ and let $\Psi: \bar{B} \to T$ be any map with $\Psi|B = \psi$. If $n < \omega$, there is $i_0 = i_0(n)$ such that $\psi(p_{n,i}^{-1}(x + e_n)) = 0$ for all $i \geqslant i_0(n)$, i.e. $-\Psi(p_{n,i}^{-1}x) = \Psi(p_{n,i}^{-1}e_n)$. On the other hand, if $f \in \Sigma$, there is $n_1(f) < \omega$ such that $\psi(p_{n,f(n)}^{-1}(x + a_f)) = 0$ for all $n \geqslant n_1(f)$, now let $\sigma: \omega \to \omega$ be the map defined by $\sigma(n) = i_0(n)$. Then $-\Psi(p_{n,\sigma(n)}^{-1}x) = \Psi(p_{n,\sigma(n)}^{-1}e_n)$ for all n. Since Σ is a maximal almost disjoint family of maps, there is some $f \in \Sigma$ such that $W = \{n < \omega: n_1(f) < n, f(n) = \sigma(n)\}$ is not empty. Let $n \in W$. Then

$$\Psi(p_{n,\sigma(n)}^{-1}e_n) = -\Psi(p_{n,\sigma(n)}x) = -\Psi(p_{n,f(n)}^{-1}x) = \Psi(p_{n,f(n)}^{-1}a_f)$$

and $\sigma(n) = f(n)$. This implies

$$0 = \Psi(p_{n,f(n)}^{-1}(e_n - a_f)) = \psi(p_{n,f(n)}^{-1}(e_n - a_f)) = t_{n,f(n)},$$

a contradiction. □

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Added in proof. In a forthcoming paper (with P. Hill and K. M. Rangaswamy) we will show in *ZFC* that conditions (*a*) and (*c*) in our theorem are equivalent for any torsion-free group B of cardinality $\leqslant \aleph_1$. The same holds if $|B| \leqslant \aleph_\omega$ and $2^{\aleph_0} = \aleph_1$.

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