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On Some Subgroups of Infinite Rank Butler Groups.

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1. Introduction.

Al groups in this note are abelian and torsion-free unless stated otherwise. Undefined notations are standard as in [F]. Some twenty years ago M. C. R. Butler [Bu] studied pure subgroups of finite rank completely decomposable groups. These groups were later called Butler groups by L. Lady. Following the lead of Bican and Salce [B-S] we call a group B a Butler group if $B\text{ext}(B, T) = 0$ for all torsion groups T. The functor $B\text{ext}(\cdot,\cdot)$ is the subfunctor of $\text{Ext}(\cdot,\cdot)$ of all balanced extensions as introduced by R. Hunter [Hu]. Butler groups of finite rank have been studied to some extend, e.g. [A1], [B]. Recently, Butler groups of infinite rank attracted some attention, see [A2], [B-S-S], [A-H] and [D-R]. D. Arnold raises in [A2] the question about Butler groups being closed with respect to pure countable subgroups. This question is of particular interest in the light of the results in [D-R]. All main results in [D-R] require the hypothesis « All pure finite rank subgroups of the Butler group B $(|B| = \aleph_1)$ are again Butler groups ». The purpose of this paper is to show that for Butler groups of rank $\aleph_1$ this hypothesis is redundant:

THEOREM 1. Each pure, countable subgroup of a Butler group of rank $\aleph_1$ is again a Butler group.

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Observe that a countable torsion-free group is a Butler group iff each pure finite rank subgroup is a Butler group, c.f. [B-S]. The main theorem in [D-R] now reads as follows:

THEOREM. Assume $V = L$ holds. The following are equivalent for a torsion-free group B of cardinality $\aleph_1$:

(a) $B\text{ext}(B, S) = 0$ for all torsion groups S.

(b) $B\text{ext}(B, T) = 0$ for any countable, Σ-cyclic torsion group T.

(c) B is a B_2-group, i.e. B has an ω_1-filtration $B = \bigcup_{\alpha<\omega_1} B_\alpha$ into pure, countable subgroups B_α with $B_0 = 0$ and $B_{\alpha+1} = B_\alpha + C_\alpha$, C_α a finite rank Butler group.

(d) B has an ω_1-filtration where each B_α is a (pure and countable) decent subgroup of B, c.f. [A-H] or [D-R].

(e) B has an ω_1-filtration such that each B_α has the T.E.P. in B. (We refer to [D-R] for the definition of the torsion extension property T.E.P.)

This theorem, as stated above, is undecidable in ZFC. But we would like to conjecture, that without (b) it's valid in ZFC.

A subgroup A of a group B is called separable in B, c.f. [H1], if for any $b \in B - A$ there is a countable sequence $\{a_n | n < \omega\} \subset A$, such that for any $a \in A$, the height sequences satisfy the inequality $|b + a| \leqslant |b + a_n|$ for some $n < \omega$. Our Theorem 1 is an immediate consequence of the more general

THEOREM 2. Let A be a pure and separable subgroup of the torsion-free group B with B/A countable. If K is a generalized regular subgroup of A, then there exists a generalized regular subgroup L of B with $L \cap A = K$.

The notion of a generalized regular subgroup was defined in [B] and used the characterize countable Butler groups: The countable group B is a Butler group iff all localizations B_p, p a prime, are completely decomposable and for each generalized regular subgroup K of B and each (pure) finite rank subgroup H of B, $(H/(H \cap K))_p = 0$ for almost all p, c.f. [A1], [B-S]. In order to show that Theorem 1 follows from Theorem 2, let B be a Butler groups of rank $\aleph_1$ and H a pure finite rank subgroup of B. Consider an $\aleph_1$-filtration $B = \bigcup_{\alpha<\omega_1} B_\alpha$ into pure countable subgroups B_α with $B_0 = H$. Let K_0 be a generalized regular subgroup of $H = B_0$. Since B_1 is countable, B_1/B_0 is countable and Theorem 2 shows the existence of a generalized regular subgroup K_1 of B_1 with $K_1 \cap B_0 = K_0$. By transfinite induction there are generalized regular subgroups K_α of B_α for all $\alpha < \omega_1$ with $K_{\alpha+1} \cap \cap B_\alpha = K_\alpha$. The subgroup $K = \bigcup_{\alpha<\omega_1} K_\alpha$ is a generalized regular subgroup of B. Since B is a Butler group, $H/K_0 \cong (H+K)/K \subset B/K$ and $(B/K)_p = 0$ for almost all primes p. Hence $(H/K_0)_p = 0$ for almost all p and the characterization of countable Butler groups mentioned above shows that H is a Butler group.

We give an example showing that the separability condition in Theorem 2 is indispensable. Therefore, our approach doesn't yield a complete answer to D. Arnold's question. We only get the following:

COROLLARY. A pure and countable subgroup of a Butler group B is Butler provided B satisfies the third axiom of countability [H2] with respect to separable subgroups.

2. Notations.

For an element x of a group G, $|x| = (|x|_p)_{p \text{ prime}}$ is the height sequence of x and we add a superscript $|x|^G$ to indicate in which group the height is computed. $|x|_p$ denotes the p-height. We say $|x| \leqslant |y|$ if $|x|_p \leqslant |y|_p$ for all primes p. Recall that a subgroup K of a torsion-free group A is a (full) generalized regular subgroup if A/K is torsion and for each pure rank one subgroup R of A the p-primary part $(R/(R \cap K))_p = 0$ for almost all primes p. We call the generalized regular subgroup K of A ω-regular, if for any pure finite rank subgroup H of A, $(H/(H \cap K))_p = 0$ for almost all primes p. In this paper we find it more useful to deal with an equivalent concept:

Let A be a torsion-free group and T torsion. We call a map $\varphi \in \in \mathrm{Hom}(A, T)$ *regular* if for any pure rank 1 subgroup R of A, $(\varphi(R))_p = 0$ for almost all primes p. Observe that φ is regular iff the kernel of φ is a generalized regular subgroup. We call φ ω-regular, if $(\varphi(H))_p = 0$ for almost all primes p whenever H is a pure finite rank subgroup of A. Observe that if A is completely decomposable or if A is the sum of finitely many pure rank 1 subgroups then each regular map is obviously ω-regular. If A is any group, $\bar{A}$ denotes the divisible hull of A. Let ω denote the first infinite ordinal, i.e. $\omega = \{0, 1, 2, \ldots\}$.

3. Lifting regular maps.

If $\varphi \in \operatorname{Hom}(A, T)$ is regular, we say φ lifts to a regular $\bar{\varphi} \in \operatorname{Hom}(\bar{A}, \bar{T})$ if $\bar{\varphi}$ is regular and $\bar{\varphi}|A = \varphi$.

PROPOSITION 1. Let $\varphi\colon A \to T$ be a regular map from the torsion-free group A into the torsion group T. If φ lifts to a regular $\bar{\varphi}\colon \bar{A} \to \bar{T}$, then φ is ω-regular. The converse holds provided that A is countable.

PROOF. If φ lifts to a regular $\bar{\varphi}$, then $\bar{\varphi}$ is ω-regular since $\bar{A}$ is divisible and so is φ, being the restriction of $\bar{\varphi}$ to A. Now assume φ is ω-regular. If A has finite rank, there is a finite set P of primes and $(\varphi(A))_p = 0$ for $p \notin P$. Then φ extends to a map $\bar{\varphi} \in \operatorname{Hom}(\bar{A}, \bigoplus_{p \in P} \bar{T}_p)$. Since P is finite, $\bar{\varphi}$ is regular. Hence we may assume that A has infinite rank, i.e. $A = \bigcup_{i=1}^{\infty} A_i$, A_i pure of finite rank, $A_i \subset A_{i+1}$ for all i and $\operatorname{rk}(A_{i+1}/A_i) = 1$. Then $\bar{A} = \bigcup_{i=1}^{\infty} \bar{A}_i$ and $\bar{A}_{i+1} = \bar{A}_i \oplus e_i Q$, $e_i \in A_{i+1}$. Let $\varphi_n = \varphi|A_n$ and assume we constructed an ω-regular $\bar{\varphi}_n\colon \bar{A}_n \to \bar{T}$ already with $\bar{\varphi}_n|A_n = \varphi_n$. Let $\pi_0\colon \bar{A}_{n+1} \to \bar{A}_n$ and $\pi_1\colon \bar{A}_{n+1} \to e_n Q$ be the natural projections induced by the decomposition $\bar{A}_{n+1} = \bar{A}_n \oplus e_n Q$. We define a map $\varrho\colon \pi_1(A_{n+1}) \to \bar{T}$ by setting $\varrho(\pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \varphi_{n+1}(x)$ for all $x \in A_{n+1}$. We have to show that ϱ is well defined. Let $x \in A_{n+1}$ such that $\pi_1(x) = 0$, i.e. $x \in A_{n+1} \cap \bar{A}_n = A_n$ since A_n is pure in A_{n+1}. This implies $\bar{\varphi}_n(\pi_0(x)) = \bar{\varphi}_n(x) = \varphi_n(x) = \varphi_{n+1}(x)$. This shows $\varrho \in \operatorname{Hom}(\pi_1(A_{n+1}), \bar{T})$. Let R be the subgroup of Q with $e_n R = \pi_1(A_{n+1})$. Then $\varrho(e_n R) = \varrho(\pi_1(A_{n+1})) \subset \bar{\varphi}_n(\bar{A}_n) + \varphi_{n+1}(A_{n+1})$ and since $\bar{\varphi}_n$ and φ_{n+1} are both ω-regular, $(\varrho(e_n R))_p = 0$ for almost all p. Hence ϱ lifts to a regular $\bar{\varrho}\colon e_n Q \to \bar{T}$. Now consider the map $\bar{\varphi}_n - \bar{\varrho}\colon \bar{A}_n \to \bar{T}$ and let $x \in A_{n+1}$. Then

$$(\bar{\varphi}_n - \bar{\varrho})(x) = (\bar{\varphi}_n - \bar{\varrho})(\pi_0(x) + \pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \bar{\varrho}(\pi_1(x)) = \\ = \bar{\varphi}_n(\pi_0(x)) - \varrho(\pi_1(x)) = \varphi_{n+1}(x) .$$

Thus $\bar{\varphi}_n - \bar{\varrho}$ is the desired (ω)-regular map extending φ_{n+1}. Induction completes the proof. □

PROPOSITION 2. Let A be a pure subgroup of B, C a finite rank Butler subgroup such that $B = A + C$. Let $\varphi\colon B \to T$ be regular and assume that $\varphi|A$ lifts to a regular $\psi\colon \bar{A} \to \bar{T}$. Then there exists $\Psi\colon \bar{B} \to \bar{T}$ regular with $\Psi|\bar{A} = \psi$ and $\Psi|B = \varphi$ if only $\varphi|C$ is ω-regular.

Proof. We will use a similar argument as in the proof of the previous proposition. Let $\bar{B} = \bar{A} \oplus X$ and $\pi_0\colon \bar{B} \to \bar{A}$ and $\pi_1\colon \bar{B} \to X$ be the natural projections. Then $\pi_1(B) = \pi_1(C)$ is a Butler group of finite rank. Again, define $\varrho\colon \pi_1(B) \to \bar{T}$ by $\varrho(\pi_1(x)) = \psi(\pi_0(x)) - \varphi(x)$ for all $x \in B$. Then $\varrho(\pi_1(B)) = \varrho(\pi_1(C)) \subset \psi(\pi_0(C)) + \varphi(C)$. Since C has finite rank and ψ is regular, $\psi(\pi_0(C))_p = 0$ for almost all primes p and since $\varphi|C$ is a regular map from the finite rank Butler group C, $(\varphi(C))_p = 0$ for almost all primes p as well. Hence $\varrho(\pi_1(B))_p = 0$ for almost all p and since $\pi_1(B)$ is countable, we may use Proposition 1 to conclude that ϱ lifts to a regular $\bar{\varrho}\colon \overline{\pi_1(B)} = X \to \bar{T}$. Again $\Psi = \psi - \bar{\varrho}\colon \bar{B} \to \bar{T}$ is a regular map extending φ. □

Corollary. (*a*) A countable torsionfree group B is a Butler group iff B is locally completely decomposable and each regular map from B into a torsion group lifts to a regular map from the divisible hull of B into the divisible hull of the torsion group. (*b*) If B is a B_2-group then each regular map $\varphi\colon B \to T$ lifts to a regular $\bar{\varphi}\colon \bar{B} \to \bar{T}$.

Proof. Part (*a*) is just a reformulation of a result in [B-S]. In order to show (*b*), let B be a B_2-group, i.e. $B = \bigcup_{\alpha<\lambda} B_\alpha$ is a union of a smooth chain of pure subgroups B_α with $B_0 = 0$ and $B_{\alpha+1} = B_\alpha + H_\alpha$ for a finite rank Butler subgroup of B. Let $\varphi\colon B \to T$ be a regular map. By transfinite induction we find regular maps $\psi_\alpha\colon \bar{B}_\alpha \to \bar{T}$ such that $\psi_{\alpha+1}|\bar{B}_\alpha = \psi_\alpha$ and $\psi_\alpha|B_\alpha = \varphi|B_\alpha$. Set $\psi_0 = 0$ and suppose ψ_α is defined already. The map $\varphi|H_\alpha$ is regular and H_α is a Butler group of finite rank. Thus $\varphi|H_\alpha$ is ω-regular and Proposition 2 provides a map $\psi_{\alpha+1}$ with the desired properties.

We would like to raise the following

Question. If B is any Butler group, is (*b*) true? (This would answer D. Arnold's question mentioned in the introduction.)

We are now going to prove our main

Theorem 2′. Let A be a pure separable subgroup of B with B/A countable. Then each regular $\varphi\colon A \to T$ lifts to a regular $f\colon B \to \bar{T}$.

PROOF. Since A is separable in B and a countable extension of a separable subgroup is separable, there is a chain $A = A_0 \subset A_1 \subset \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset B$ with $\operatorname{rk}(A_{i+1}/A_i) = 1$, $B = \bigcup A_i$ and each A_i is separable in A_{i+1}. Therefore we may assume that B/A has rank 1. Then $\bar{B} = \bar{A} \oplus xQ$ with $x \in B - A$. If S is a subset of B, let $\langle S \rangle^*$ be the pure subgroup of B generated by S and let R be the subgroup of Q with $xR = B \cap xQ$. For any $q \in R$, we have a sequence $\{a_{q,n}$: $n < \omega)$ such that for any $a \in A$ there is some n such that $|a + qx|^B \leqslant \leqslant |a_{q,n} + qx|^B$. Choose an enumeration of R and $R \times \omega$ such that

$$\{a_{q,n} + qx\colon q \in R,\ n < \omega\} = \{a_i + q_i x\colon i < \omega\}\,.$$

Observe that the map $(i \to q_i)$ is not one to one. Let $q_i = r_i/s_i$ with r_i, s_i relatively prime. We may assume $r_i \neq 0$ and set $y_i = a_i + q_i x$. Define $P_i = \{p$ prime: p doesn's divide neither r_i nor s_i, $|y_i|_p^B > > \min\{|a_i|_p^A, |x|_p^B\}\}$. Note that for $p \in P_i$, $|x|_p^B = |q_i x|_p^B$. For xS the image of the projection of B into xQ, we define a map $g\colon xS \to \bar{T}$. First we define $g(xR) = 0$. In order to continue, we'll define $g(p^{-j}x)$ for $j < \omega$ with $p^{-j} \notin R$. In order to define a map $g\colon xS \to \bar{T}$, we only have to make sure that $p\big(g(p^{-(j+1)}x)\big) = g(p^{-j}x)$ and avoid contradictory assignments. We'll define g in such a way that for all $i < \omega$, $g(p^{-n}x) = s_i r_i^{-1}\big(\bar{\varphi}(p^{-n}a_i)\big)_p$ for almost all primes p. Here $\bar{\varphi}\colon \bar{A} \to \bar{T}$ is a fixed map extending φ. For $'p \in P_1$, set $l_{1,p} = |y_1|_p^B$ and observe $l_{1,p} > |x|_p^B = |a_1|_p^B = |a_1|_p^A$. Now define $g(p^{-l_{1,p}}x) = -(s_i r_i^{-1})\bar{\varphi}(a_i)_p$. Assume we defined $g(p^{-l_{1,p}}x)$ for all $i < n$, $l_{i,p} = |y_i|_p^B$ and all $'p \in P_i$ such that $g(p^{-l_{1,p}}x) = g(p^{-l_{1,p}}x)_p = -(s_i r_i^{-1})\bar{\varphi}(a_i)_p$ for almost all $p \in \bigcup_{i=1}^{n-1} P_i$. Let $E = \{i | 1 \leqslant i < n,\ P_n \cap P_i$ is infinite$\}$, $i \in E$ and $p \in P_n \cap P_i$. This implies $l_{i,p} > \min\{|a_i|_p, |x|_p\}$ and $l_{n,p} > \min\{|a_n|_p, |x|_p\}$ and therefore $l_{n,p} > |a_n|_p = |x|_p = |a_i|_p$. Let $l_p = \min\{l_{i,p}, l_{n,p}\}$. Then y_n, y_i, $r_i s_n y_n$ and $r_n s_i y_i$ are all in $p^{l_p}B$. Hence

$$r_i s_n y_n - r_n s_i y_i = r_i s_n a_n - r_n s_i a_i \in p^{l_p}B \cap A = p^{l_p}A\,.$$

Since φ is regular, there is a subset $W_{i,n}$ of $P_i \cap P_n$ with $(P_i \cap P_n) - - W_{i,n}$ finite and $\varphi\big(p^{-l_p}(r_i s_n a_n - r_n s_i a_i)\big)_p = 0$ for all $p \in W_{i,n}$. Thus $(s_n r_n^{-1})\bar{\varphi}(p^{-l_p}a_n) = (s_i r_i^{-1})\bar{\varphi}(p^{-l_p}a_i)$ for all $p \in W_{i,n}$. We now consider two cases for a prime $p \in W_n$, where $W_n = \bigcup_{i \in E} W_{i,n}$.

Case 1. For all $i \in E$ with $p \in W_{i,n}$ we have $l_{i,p} \leqslant l_{n,p}$. Here we define $g(p^{-l_{n,p}}x)$ to be an element in $\bar{T}_p$ such that $p^{l_{n,p}-l_{j,p}}g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{j,p}}a_j)$ where $l_{j,p} = \max\{l_{ip,} | i \in E\}$.

Case 2. There is $i \in E$ such that $p \in W_{i,n}$ and $l_{n,p} < l_{i,p}$. Again, let $l_{j,p} = \max\{l_{i,p}\ i \in E\}$ and observe that

$$g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{j,p}}a_j)p^{l_{j,p}-n,p}$$

was already defined. If $p \in P_n - \bigcup_{i=1}^{n-1} P_i$, set

$$g(p^{-l_{n,p}}x) = -q_n^{-1}\bar{\varphi}(p^{-l_{n,p}}a_n)\,.$$

Our observations above show that these assignments define a map g' on a subgroup xS' of xS. Let g be any lifting of g' to xS. Our construction shows that our map g has the property that for any $i < \omega$, $(\bar{\varphi} \oplus g)(\langle y_i \rangle^*)_p = 0$ for almost all primes p. Let $f = (\bar{\varphi} \oplus g)|B$ and $y \in B - A$. Then there is a natural number $k \neq 0$ such that $ky = a + qx$ with $a \in A$, $q \in R$. For some $i < \omega$, we have $q = q_i$ and $|a + qx|^B \leqslant |a_i + qx|^B$. This implies $|a - a_i|^A \geqslant |a + qx|^B$ and for $e \leqslant |a + qx|_p$ we get

$$f(p^{-e}(a + qx))_p = \bar{\varphi}(p^{-e}a)_p + g(p^{-e}qx)_p = \bar{\varphi}(p^{-e}a_i)_p + \\ + g(p^{-e}qx)_p = f(p^{-e}(a_i + qx))_p = 0$$

for almost all p, since $p^{-e}(a - a_i) \in A$ and φ regular. This shows that $f(\langle y \rangle_*)_p = 0$ for almost all primes p, i.e. f is the desired map. □

We conclude this paper with the promised

EXAMPLE. There is a torsion-free group B and a pure subgroup A of B with $rk(B/A) = 1$ and a regular map $\varphi: A \to T = \bigoplus_{p \text{ prime}} Z(p^\infty)$ such that φ can not be lifted to a regular $\psi: B \to T$.

PROOF. Let $\{p_{n,i}: n, i < \omega\}$ be a list of all primes and $F = \bigoplus_{i<\omega} e_i Z$ a free group of countable rank. Call two maps $f, g: \omega \to \omega$ almost disjoint, if $\{i < \omega: f(i) = g(i)\}$ is finite. Zorn's Lemma implies that there is a maximal family Σ of almost disjoint maps from ω into ω.

We introduce independent elements a_f, $f \in \Sigma$ and define $A = < F$, a_f, $p_{n,f(n)}^{-1}(e_n - a_f)$: $f \in \Sigma$, $n\langle\omega\rangle$. It's not hard to see that A is homogeneous of type 0 and $p^{-1}(a_f - a_g) \in A$ iff $p_{n,g(n)} = p = p_{n,f(n)}$ for some $n < \omega$, i.e. $f(n) = g(n)$. Define $\varphi: A \to T$ by setting $\varphi(e_n) = \varphi(a_f) = 0$ for all $n < \omega$, $f \in \Sigma$ and $\varphi(p_{n,f(n)}^{-1}(e_n - a_f) = t_{n,f(n)}$, an element of exponent 1 in $Z(p_{n,f(n)}^{\infty})$. Since A is homogeneous of type 0, φ is regular.

We now define a rank 1 extension B of A by introducing a new independent element x and setting

$$B = \langle A, x, p_{n,f(n)}^{-1}(x + a_f), p_{n,i}^{-1}(x + e_n)\rangle: f \in \Sigma, n, i < \omega .$$

Routine verification shows that A is pure in B and B/A is rank 1 of type $(1, 1, 1, \dots)$.

By way of contradiction assume that $\varphi: A \to T$ lifts to a regular map $\psi: B \to T$ and let $\Psi: \bar{B} \to T$ be any map with $\Psi|B = \psi$. If $n < \omega$, there is $i_0 = i_0(n)$ such that $\psi(p_{n,i}^{-1}(x + e_n)) = 0$ for all $i \geqslant i_0(n)$, i.e. $-\Psi(p_{n,i}^{-1}x) = \Psi(p_{n,i}^{-1}e_n)$. On the other hand, if $f \in \Sigma$, there is $n_1(f) < \omega$ such that $\psi(p_{n,f(n)}^{-1}(x + a_f)) = 0$ for all $n \geqslant n_1(f)$, now let $\sigma: \omega \to \omega$ be the map defined by $\sigma(n) = i_0(n)$. Then $-\Psi(p_{n,\sigma(n)}^{-1}x) = \Psi(p_{n,\sigma(n)}^{-1}e_n)$ for all n. Since Σ is a maximal almost disjoint family of maps, there is some $f \in \Sigma$ such that $W = \{n < \omega: n_1(f) < n, f(n) = \sigma(n)\}$ is not empty. Let $n \in W$. Then

$$\Psi(p_{n,\sigma(n)}^{-1}e_n) = -\Psi(p_{n,\sigma(n)}x) = -\Psi(p_{n,f(n)}^{-1}x) = \Psi(p_{n,f(n)}^{-1}a_f)$$

and $\sigma(n) = f(n)$. This implies

$$0 = \Psi(p_{n,f(n)}^{-1}(e_n - a_f)) = \psi(p_{n,f(n)}^{-1}(e_n - a_f)) = t_{n,f(n)},$$

a contradiction. □

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Added in proof. In a forthcoming paper (with P. Hill and K. M. Rangaswamy) we will show in *ZFC* that conditions (*a*) and (*c*) in our theorem are equivalent for any torsion-free group B of cardinality $\leqslant \aleph_1$. The same holds if $|B| \leqslant \aleph_\omega$ and $2^{\aleph_0} = \aleph_1$.

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