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A p-Stable Action of the Automorphism Group of a Group.

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SUMMARY - The following Theorem is proved: « Let G be a group and Aut (G) its automorphism group. If Aut (G) is not p-stable over G, p an odd prime, then Aut (G) involves $SL(2, p)$ ». This Theorem is a generalization of a classic result on stability ([4], Th. 3.8.3).

Introduction. Notation.

In this note all groups will be finite. p will denote a prime number, and $O_p(G \bmod N)$ the inverse image in G of the p-radical of G/N. If we assume that $A \leqslant \mathrm{Aut}(G)$ and H is a subgroup of G which is fixed by A, then, whenever $a \in A$, we denote $[H, a] = \langle [x, a] = x^{-1}x^a / x \in H \rangle$, $[H, a, a] = [[H, a], a]$, $[H, A] = \langle [H, a] / a \in A \rangle$, and $[H, A, A] = [[H, A], A]$. The remainder of the notation is standard, and is taken mainly from [4]. In particular, $\Pi(G)$ is the set of all primes which divide the order of the group G.

The first concept concerning stability is the following:

DEFINITION A ([4], 3.8). Let G be a group with no nontrivial normal p-subgroups, p odd. A faithful representation ψ of G on a vector space V over $GF(p^n)$ will be called p-stable provided no p-element of $(G)\psi$ has a quadratic minimal polynomial on V. Moreover, G is said to be p-stable if all such faithful representations of G are p-stable. (We will say G is p-stable linear to distinguish it from later definitions of p-stability).

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And the main result on p-stability linear:

THEOREM A ([4], 3.8.3). Let G be a group with no nontrivial normal p-subgroups, p odd. If G is not p-stable linear, then G involves $SL(2,p)$.

Later we can find other definitions of p-stable groups, ([3], [5]) π-stable groups, π a set of primes, ([1]), and in general, $\mathcal{F}$-stable groups, where $\mathcal{F}$ is either a saturated Formation, ([2]), or a Fitting class, ([8]). As well, we find the close relationship between the stability of a group G, and the fact that G does not involve the special affin groups $SA(2,p)$, for adequated primes p, ([3], [5], [2], [7]).

The aim of this paper is to study the structure of the automorphisms group of a group by using these ideas.

We give the following Definition:

DEFINITION 1. Let G be a group. We say that the automorphisms group of G, Aut(G), is p-estable over G if, whenever A is a p-subgroup of Aut(G) and B is a p-subgroup of G, such that B is fixed by A, (that is, $A \leqslant N_{\mathrm{Aut}(G)}(B)$), and $[B, A, A] = 1$, then

$$A \leqslant O_p\big(N_{\mathrm{Aut}(G)}(B) \bmod C_{\mathrm{Aut}(G)}(B)\big).$$

We obtain the following result:

THEOREM 1. Let G be a group. If Aut(G) is not p-stable over G, p odd, then Aut(G) involves the special linear group $SL(2,p)$.

First, we give a preliminary result:

LEMMA. For a group G, the following are equivalent:

i) Aut(G) is p-stable over G.

ii) Whenever B is a p-subgroup of G, and x is a p-automorphism of G which fixes B, and such that $[B, x, x] = 1$, then

$$x \in O_p\big(N_{\mathrm{Aut}(G)}(B) \bmod C_{\mathrm{Aut}(G)}(B)\big).$$

iii) Whenever B is a p-subgroup of G, and x is an automorphism of G which fixes B, and such that $[B, x, x] = 1$, then

$$x \in O_p\big(N_{\mathrm{Aut}(G)}(B) \bmod C_{\mathrm{Aut}(G)}(B)\big).$$

PROOF. i) $\to$ ii) Let B and x be as in (ii). The conclusion follows easily by taking $A = \langle x \rangle$, and using (i).

ii) $\to$ iii) Let B and x be as in (iii). Let $\Pi(\langle x \rangle) = \{p_1, \dots, p_n\}$. Then $\langle x \rangle = \langle x_1 \rangle \times \dots \times \langle x_n \rangle$, where $\langle x_i \rangle$ is a p_i-group, for each $i = 1, \dots, n$. Obviously, each x_i, $i = 1, \dots, n$, fixes B and satisfies $[B, \langle x_i \rangle, \langle x_i \rangle] = 1$.

If $p_i \neq p$, then $x_i \in C_{\mathrm{Aut}(G)}(B)$. Otherwise, we have

$$x_i \in O_p\big(_{\mathrm{Aut}(G)}(B) \bmod C_{\mathrm{Aut}(G)}(B)\big)$$

by using (ii). So,

$$x \in O_p\big(N_{\mathrm{Aut}(G)}(B) \bmod C_{\mathrm{Aut}(G)}(B)\big)\,.$$

iii) $\to$ i) Let A and B be as in Def. 1. For every $x \in A$, B and x satisfy the assumptions in (iii). Then, the conclusion is clear.

PROOF OF THEOREM 1. Because of the previous Lemma, if Aut (G) is not p-stable over G, there exist a p-subgroup P of G, and a p-element x in $N_{\mathrm{Aut}(G)}(P)$ such that $[P, x, x] = 1$, and

$$x \notin O_p\big(N_{\mathrm{Aut}(G)}(P) \bmod C_{\mathrm{Aut}(G)}(P)\big)\,.$$

But, from $\big([5]$, Th. IX.7.8$\big)$, this is equivalent to the existence of an element $g \in N_{\mathrm{Aut}(G)}(P)$ such that $\langle x, x^g \rangle C_{\mathrm{Aut}(G)}(P)/C_{\mathrm{Aut}(G)}(P)$ is not a p-group.

Let $T = \langle x, x^g \rangle$. Let us refine the T-series $P \rhd 1$ to a T-chief series $P = P_0 \rhd P_1 \rhd \dots \rhd P_r = 1$ (*)

For each $i = 0, \dots, r-1$, let $\overline{P}_i = P_i/P_{i+1}$: So, it is clear that each $\overline{P}_i$, $i = 0, \dots, r-1$, is an abelian p-element group, that is, it is a vector space over $GF(p)$, and each $TC_{\mathrm{Aut}(G)}(\overline{P}_i)/C_{\mathrm{Aut}(G)}(\overline{P}_i) = \overline{T}_i$, $i = 0, \dots, r-1$, acts faithfully and irreducibly on $\overline{P}_i$: Therefore, since $\big([4]$, Th. 3.1.3$\big)$, it follows that $O_p(\overline{T}_i) = 1$.

If $\overline{T}_i$ was a p-group for every $i = 0, \dots, r-1$, that is, $\overline{T}_i = 1$, then $T \leqslant \bigcap_{i=0}^{r-1} C_{\mathrm{Aut}(G)}(\overline{P}_i)$. Whence T would stabilize the series (*) of P, and we would get a contradiction because $TC_{\mathrm{Aut}(G)}(P)/C_{\mathrm{Aut}(G)}(P)$ is not a p-group.

Therefore, there exists an $i \in \{0, \dots, r-1\}$ such that $\overline{T}_i \neq 1$, that is, $\overline{T}_i$ is not p-stable linear, (Def. A). Now, because of Th. A, $\overline{T}_i$ involves $SL(2, p)$, and clearly Aut (G) involves $SL(2, p)$.

Using an argument similar to ([4], Th. 3.8.4), and Th. 1, we get the following:

COROLLARY. Let G be a group. If one of the following conditions is verified in Aut(G):

i) Aut(G) is of odd order.

ii) A Sylow 2-subgroup of Aut(G) is abelian.

iii) A Sylow 2-subgroup of Aut(G) is dihedral.

iv) Aut(G) is isomorphic to $L_2(q)$.

v) Aut(G) is soluble and either $p \geqslant 5$ or $p = 3$ and $SL(2, 3)$ is not involved in Aut(G).

Then, Aut(G) is p-stable over G.

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REFERENCES

[1] Z. ARAD, *A characteristic subgroup of π-stable groups*, Canad. J. Math., **26** (1974), pp. 1509-1514.

[2] L. M. EZQUERRO, *$\mathfrak{F}$-stability of finite groups*, J. Alg., **92**, N. 2 (1985), pp. 395-399.

[3] G. GLAUBERMAN, *A characteristic subgroup of a p-stable group*, Canad. J. Math., **20** (1968), pp. 1101-1135.

[4] D. GORENSTEIN, *Finite Groups*, Harper & Row, New York, 1968.

[5] B. HUPPERT - N. BLACKBURN, *Finite Groups II*, Springer-Verlag, Berlin, 1982.

[6] M. D. PÉREZ-RAMOS, *Estabilidad con respecto de Nilpotentes, de Cuasinilpotentes y de clases de Fitting en general*, Tesis Doctoral, U. de Valencia, 1986.

[7] M. D. PÉREZ-RAMOS, *Stability with respect to a Fitting class*, preprint.

[8] M. D. PÉREZ-RAMOS, *A characteristic subgroup of $\mathcal{N}$-stable groups*, Isr. J. Math., **54**, N. 1 (1986), pp. 51-59.

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