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Some Commutativity Results for Rings.

WALTER STREB (*)

SUMMARY - It is proved that certain rings satisfying variable identities of the form \([x^m, y^n, \ldots, y^n] = 0\) (in particular \([x^m, y^n, y^n] = 0\) with \(n\) bounded) must have nil commutator ideals.

In this paper we prove results based on questions of Herstein [2, p. 357] and generalizing results of Klein, Nada and Bell [3] and Klein and Nada [4].

Let \(R\) be an associative ring and \(Z\) respectively \(Z^+\) be the set of integers respectively positive integers. For \(a, b \in R\) define generalized commutators \([a, b]_k, \ k \in Z^+,\) as follows: \([a, b]_1 = [a, b] = ab - ba\) and for \(i \in Z^+, [a, b]_{i+1} = [[a, b]_i, b]. \) \(R\) is called a \(k\)-ring if for all \(a, b \in R\) there exists \(m = m(a, b), n = n(a, b) \in Z^+\) such that \([a^m, b^n]_k = 0\). \(R\) is called a \(n\)-bounded \(k\)-ring if the above \(n\) is fixed. Let \(Z\{X\}\) be the free \(Z\)-algebra generated by the noncommuting indeterminates \(x_1, x_2, x_3, \ldots\) [5; pp. 2-4]. Substitute \(r_i \in R\) for \(x_i\) in \(f \in Z\{X\}\) to get an element of \(R\). The additive subgroup of \(R\) generated by all these elements is denoted by \(f(R)\). Let \(f \in Z\{X\}\), and \(N \in Z^+. \) \(R\) is called a \(N\)-\(j\)-\(k\)-ring if for all \(a \in f(R)\) and \(b \in R\) there exists \(m = m(a, b), n = n(a, b) \in Z^+\) such that \(m < N\) and \([a^m, b^n]_k = 0.\) \(R\) is called left (right)-\(s\)-unital if \(a \in Ra (a \in aR)\) for all \(a \in R.\)

Let \(R_{\text{reg}}\) be the set of (left and right) regular elements of \(R, R_{\text{nil}}\) the set of nilpotent elements of \(R, R'\) the commutator ideal of \(R, C(R)\) the center of \(R\) and \(i \wedge j\) the greatest common divisor of \(i, j \in Z^+\).

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For $a \in R$ and $A, B \subseteq R$ let $C_R(a) = \{ b \in R : ba = ab \}$ and $[A, B]$ be the additive subgroup of $R$ generated by $\{ [a, b] : a \in A, b \in B \}$. We shall prove:

**Theorem.** Each of the following conditions implies $R' \subseteq R_{\text{nil}}$:

1. $R$ is a 2-ring and a $n$-bounded $k$-ring (in particular, $R$ is a $n$-bounded 2-ring).
2. $R$ is a $N$-f-2-ring and a $k$-ring.
3. $R$ is a 2-ring and a $N$-f-4-ring.
4. $R$ is a left-or right-$s$-unital $k$-ring.

This results generalize the following sufficient conditions: For all $a, b \in R$ there exists $m = m(a, b) \in \mathbb{Z}^+$ such that $[a^m, b]_a = 0$ [4; Theorem, p. 361]. Let $N \in \mathbb{Z}^+$. For all $a, b \in R$ there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $m < N$ and $[a^m, b^n]_a = 0$ [3; Theorem 1, p. 286]. $R$ is a $k$-ring with $1$ [3; Theorem 3, p. 288]. We first prove:

**Lemma.** Let $R$ be prime, torsionfree, $R = R_{\text{reg}} \cup R_{\text{nil}}$ and $C(R) = 0$.

(a) Let $0 \neq f \in \mathbb{Z}[X]$. Then there exists an ideal $I \neq 0$ of $R$ such that $[I, R] \subseteq f(R)$.

(b) Let $L \neq 0$ be a Lie ideal of $R$ and $N \in \mathbb{Z}^+$. For all $a \in R$ and $b \in R_{\text{reg}}$ suppose there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $m < N$ and $[a^m, b^n]_a = 0$. Then $c^2 = 0$ for all $c \in C_R(b) \cap R_{\text{nil}}$ and $b \in R_{\text{reg}}$ if $k = 4$ and $C_R(b) \cap R_{\text{nil}} = 0$ for all $b \in R_{\text{reg}}$ if $k = 2$.

(c) Let $R$ be a $n$-bounded $k$-ring. Then $C_R(b^i) \subseteq C_R(b^n)$ for all $b \in R_{\text{reg}}$ and $i \in \mathbb{Z}^+$.

**Proof.** (a) Using [5; pp. 6, 7] we get a multilinear polynomial $0 \neq g \in \mathbb{Z}[X]$ such that $g(R) \subseteq f(R)$. Since $g(R) \neq 0$ [5; Theorem 1.6.27, p. 47] and $[g(R), R] \subseteq g(R)$ the conclusion follows by [1; Theorem 6, p. 570].

(b) Let $k = 4$. Assume that there exists $b \in R_{\text{reg}}$, $c \in C_R(b)$ and $2 < i \in \mathbb{Z}^+$ such that $c^{i+1} = 0 \neq c$. We shall get a contradiction. For each $a \in L$ and $M \in \mathbb{Z}^+$ there exists a subset $\mathcal{M}$ of $\mathbb{Z}^+$ with $M$ elements and $m, n \in \mathbb{Z}^+$ with $m < N$ such that $[a^m, (b + ic)^n]_a = 0$ for all $i \in \mathcal{M}$. Using $c^{i+1} = 0$ and a Vandermonde argument analogous to [3; p. 287] we get homogeneous equations $g_i(a, b, c) = 0$, where $a$, $b$ and $c$ appear in each formal monomial exactly $m$, $4n - j$ and $j$-times. We use tacitly $b \in R_{\text{reg}}$ and $c^{i+1} = 0$. 
Since
\[ 0 = g_4(a, b, c) c^i = 4 \left( \binom{n}{1} \right) \left[ [a^m, b^n]_3, b^{n-1}c \right] c^i = 4nb^{n-1}c[a^m, b^n]_3c^i \]
we have \( c^i[a^m, b^n]_3c^i = 0 \) and \( c^{i-1}g_4(a, b, c)c^{i-1} = 0 \), hence
\[
0 = 4 \binom{n}{2} c^{i-1} \left[ [a^m, b^n]_3, b^{n-2}c^2 \right] c^{i-1} + \\
+ 6 \binom{n}{1} \binom{n}{1} c^{i-1} \left[ [a^m, b^n]_2, b^{n-1}c \right] c^{i-1} = -12n^2b^{n-1}c^i[a^m, b^n]_2c^ib^{n-1},
\]
therefore
\[
c^i[a^m, b^n]_c c^i = 0 \quad \text{and} \quad c^{i-2}g_4(a, b, c)c^{i-1} = 0.
\]
Analogously we get
\[
c^i[a^m, b^n]c^i = 0 \quad \text{and} \quad c^{i-2}g_4(a, b, c)c^{i-2} = 0
\]
and finally \( c^i[a^m, b^n]c^i = 0 \).

Choose \( m(a) \) in \( \mathbb{Z}^+ \) maximal with respect to \( c^ia^{m(a)}c^i = 0 \). Put \( M = \max \{ m(a) : a \in L \} \). Choose \( d \in L \) such that \( m(d) = M \). For each \( a \in L \) there exists \( M > m \in \mathbb{Z}^+ \) such that \( c^i(ia + d)^m c^i = 0 \) for infinitely many \( i \in \mathbb{Z}^+ \). Using a Vandermonde argument we get \( c^i[a^m, b^n]c^i = 0 = c^i[d^m]c^i \), hence \( m = M \). We have proved that \( c^i[a^m, b^n]c^i = 0 \) for all \( a \in L \). There exists an ideal \( I \neq 0 \) of \( R \) such that \( [I, I] \subseteq L \) [1; Theorem 6, p. 570]. Using (a) for \( R = I \) and \( f = [x_1, x_2]^M \) we get an ideal \( J \neq 0 \) of \( I \) such that \( [J, J] \subseteq f(I) \). Then \( K = IJJ \neq 0 \) is an ideal of \( R \) and \( 0 = c[K, cK]c^i = c^iKcKc^i \), hence \( c^i = 0 \), a contradiction.

Let \( k = 2 \). The condition for \( k = 4 \) is still satisfied, hence \( c^2 = 0 \) for all \( b \in R_{reg} \) and \( c \in C_R(b) \cap R_{nil} \). As above we get \( c = 0 \) using \( 0 = g_4(a, b, c) = 2n^2b^{n-1}c^m c^{ob^{n-1}} \).

(c) Let \( b \in R_{reg}, i, j \in \mathbb{Z}^+ \) and \( l = i \cap n \). We show (i)-(iv) step by step.

(i) \( C_R(b^i) \cap C_R(b^j) \subseteq C_R(b^{ij}) \).

We can assume that \( i < j \). For \( a \in C_R(b^i) \cap C_R(b^j) \) we have
\[
0 = [a, b^j] = [a, b^i]b^{i-j} + b^i[a, b^{i-j}] = b^i[a, b^{i-j}],
\]
hence \([a, b^i] = 0 = [a, b^{i-1}]\). By induction over \(i + j\) we get the conclusion.

(ii) Let \(a \in C_R(b^i)\) and \(a^2 = 0\). Then \(a \in C_R(b^i)\).

Let \(m \in \mathbb{Z}^+\) be such that \(0 = [(a + b^i)^m, b^n] = mb^{i(m-1)}[a, b^n]_k\).
Then \([a, b^n]_k = 0\). For \(c = [a, b^n]_{k-1}\) we have \([c, b^n] = 0 = [c, b^n]\), hence \([c, b^n] = 0\) by (i). For \(c = [a, b^i]\) we have \([c, b^n]_{k-1} = 0 = [c, b^n]\), hence \([a, b^i]_{k-1} = 0\) by induction over \(k\). For \(c = [a, b^i]_{k-2}\) and \(j = i/l\) we have \(0 = [c, b^i] = j(b^{i-1})[a, b^i]_k\) hence \([a, b^i]_{k-1} = 0\), therefore \(a \in C_R(b^i)\) by induction over \(k\).

By induction over the index of nilpotence of \(a\) we get

(iii) Let \(a \in C_R(b^i) \cap R_{nil}\). Then \(a \in C_R(b^i)\).

(iv) \(C_R(b^i) \subseteq C_R(b^i) \subseteq C_R(b^n)\).

If \(C_R(b^i) \subseteq R_{reg}\), then \(C_R(b^i)\) is commutative by [3; Lemma, p. 286], hence (iv). Otherwise let \(a \in C_R(b^i), a^2 = 0\) and \(c \in C_R(b^i) \cap R_{reg}\). Then \(ac \in C_R(b^i) \cap R_{nil}\), hence \(0 = [ac, b^i] = a[c, b^i]\) by (iii), therefore \([c, b^i] \in C_R(b^i) \cap R_{nil}\), hence \([c, b^i]_2 = 0\) by (iii), finally \(c \in C_R(b^i)\) as above.

PROOF OF THEOREM. (1)-(3) Let us assume that \(R' \not\subseteq R_{nil}\). We shall get a contradiction. By [2] we can assume, that \(R\) is prime, torsionfree, \(R = R_{reg} \cup R_{nil}, C(R) = 0\), \(R\) is a \(k\)-ring but not a \(k-1\)-ring and \(k > 1\).

(1) We show (i)-(iii) step by step.

(i) Let \(a \in R, b \in R_{reg}\) and \(m, i \in \mathbb{Z}^+\). Then \([a^m, b^i]_2 = 0\) implies \([a^m, b^n]_2 = 0\).

By (c) we have \(0 = [[a^m, b^i], b^n] = [[a^m, b^n], b^i]\), hence \([a^m, b^n]_2 = 0\).

(ii) For \(a, b \in R_{reg}\) there exists \(m = m(a, b) \in \mathbb{Z}^+\) such that \(n|m\) and \([a^i, b^j]_2 = [b^i, a^j]_2 = [a^i, b^j]_2 = 0\) for all \(i, j \in m\mathbb{Z}^+\).

By (i) there exists \(r, s \in \mathbb{Z}^+\) such that \([a^{nr}, b^n]_2 = 0 = [b^{ns}, a^n]_2\). For \(u = a^{nr}\) and \(v = b^{ns}\) we have \([u, v]_2 = 0 = [v, u]_2\). By (i) there exists \(1 < t \in \mathbb{Z}^+\) such that \(0 = [u^t, b^n]_2\). Hence \(0 = [u^t, v]_2 = i(t - 1)u^{t-2}[u, v]^2\), therefore \([u, v]^2 = 0\). We have \([u^t, v]_2 = [v^t, u]_2 = [u^t, v]_2 = 0\) for all \(i, j \in \mathbb{Z}^+\). Using \(m = nrs\) we get (ii).

(iii) Let \(a, b \in R_{reg}, c \in C_R(a)\) and \(c^2 = 0\). Then \([c, b^n]_2 = 0\).
Let $m = m(a, b)$ as in (ii). By (i) there exists $1 < l \in \mathbb{Z}^+$ such that $[(a^m + c^i), b^m]_2 = 0$. Analogous to [2; p. 355] we get (iii).

By an argument as in [2; pp. 355, 356] we get a Lie ideal $L \neq 0$ of $R$ such that $[a, b^n]_1 = 0$ for all $a \in L$ and $b \in R_{reg}$. By (b) we have $c^2 = 0$ for all $b \in R_{reg}$ and $c \in C_R(b) \cap R_{nil}$. We conclude the proof as in [4; p. 361].

(2) Since $R$ is not a $k$-ring there exists $a, b \in R$ such that $[a^i, b^j]_{k-1} \neq 0$ for all $i, j \in \mathbb{Z}^+$. There exists $m, n, 1 < l \in \mathbb{Z}^+$ such that $[a^m, b^n]_k = 0 = [a^{m^i}, b^n]_k$. Using the formula $[uv, w] = [u, w]v + u[v, w]$ we get $0 = [a^{m^i}, b^n]_{(k-1)} = [a^m, b^n]_{k-1}$. Hence there exists $b \in R_{reg}$ and $c \in C_R(b)$ such that $c \neq 0 = c^2$ in contradiction to (a) and (b).

(3) We have $c^2 = 0$ for all $b \in R_{reg}$ and $c \in C_R(b) \cap R_{nil}$ by (a) and (b) and can conclude the proof as in [4; p. 361].

(4) Let $R$ be left-s-unital. Assume that $R' \notin R_{nil}$. Choose a finite subset $A$ of $R$ and $e \in R$ such that $S' \notin S_{nil}$ for the subring $S$ of $R$ generated by $A$ and $ea = a$ for all $a \in A$ [6]. Let $T$ be the subring of $R$ generated by $A \cup \{e\}$ and $I$ the ideal of $T$ generated by $\{ae - a : a \in A \cup \{e\}\}$. Since $IA = 0$ $T/I$ has no nil commutator ideal. But $T/I$ is a $k$-ring with 1 in contradiction to [2; Theorem 3, p. 288].

**Remark.** For $a, b \in R$ define $a \circ b = ab + ba$. Let $a, b \in R$, $m, n_i \in \mathbb{Z}^+$ and $\ast_i \in \{[ , ], \circ\}$ such that $\left( ((a^m \ast_1 b^{n_1}) \ast_2 b^{n_2}) \ldots \right) \ast_k b^{n_k} = 0$. Then using the formula $[u, v^2] = [u, v]v = [u \circ v, v]$ we get $[a^m, b^n]_k = 0$ for $n = 2I_n$. Thus the use of $\circ$ as well as $[,]$ provides no real generalisation.

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