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A Unified Approach to Abstract Linear Nonautonomous Parabolic Equations.

PAOLO ACQUISTAPACE - BRUNELLO TERRENI (*)

SUMMARY - We consider the linear nonautonomous Cauchy problem of parabolic type in a Banach space E. Our assumptions provide a unified treatment which applies to many situations where the domains of the operators may change with t. We study, existence, uniqueness and maximal regularity of strict and classical solutions, by means of a representation formula which does not make use of fundamental solutions. Comparisons with the available literature are also given.

0. Introduction.

Let E be a Banach space. We are concerned with the linear parabolic Cauchy problem

$$\begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [0, T] \\ u(0) = x \end{cases} \tag{0.1}$$

where $x \in E$ and $f\colon [0, T] \to E$ are prescribed data and $\{A(t)\}_{t\in[0,T]}$ is a family of closed linear operators in E which are generators of analytic semigroups, and whose domain $D_{A(t)}$ may change with t and be not dense in E: thus the semigroups $\{\exp[sA(t)]\}_{s\geqslant 0}$ may be not strongly continuous at $s = 0$.

We consider strict and classical solutions (see Definition 1.5 below), i.e. continuously differentiable solutions of Problem (0.1), studying

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their existence, uniqueness and maximal regularity: this means that for a solution u the functions u' and $A(\cdot)u(\cdot)$ turn out to be as smooth as the right member f is, provided (possibly) a suitable compatibility condition involving the vectors x and $f(0)$ holds. We prove maximal regularity results both in time and in space: time regularity means Hölder continuity with values in E, whereas space regularity means continuity with values in E and boundedness with values in suitable subspaces of E. Thus the regularity properties for the solution of (0.1) will turn out to be exactly the same as those known for the solutions of the autonomous version of (0.1) (see Sinestrari [16]).

Several authors have studied Problem (0.1) in the parabolic case under different assumptions. In this paper we try to provide a unified treatment of the subject: our hypotheses are generally weaker than those known in the literature. A detailed comparison with such different kinds of assumptions is made in Section 7, which we also refer to for the related references.

Our proof does not require the construction of the fundamental solution of (0.1). We find a suitable representation formula for $A(\cdot)u(\cdot)$, where u is a solution of (0.1) (assumed to exist); next, by an approximation procedure we are able to show that if the data x, f are smooth enough, then our formula indeed yields the unique solution of (0.1). From a technical point of view, this argument is a refinement of the one used in Acquistapace-Terreni [5], where we only studied strict solutions, assuming in addition the constancy, for some $\varrho \in]0, 1[$, of the interpolation spaces $(D_{A(t)}, E)_{1-\varrho,\infty}$ with respect to t. Here we study also classical solutions and try to get very precise results: for this reason we are forced to introduce some special Banach spaces, the so-called Z and Z^* spaces (see Definition 1.4 below), consisting of functions which are singular at $t = 0$, and being endowed with certain weighted uniform or Hölder norms. The use of such spaces allows us to obtain very sharp and concise existence and regularity results: of course a price must be paid in terms of tedious technicalities.

Here (as well as in [5]) our method is inspired by the more abstract theory of sums of noncommuting linear operators due to Da Prato-Grisvard [8]. However our technique is different, since it is based on a modified version of that theory, recently performed by Labbas-Terreni [14]; in the latter paper a heuristic derivation of the analogue of our representation formula in this more general setting can also be found.

It is to be noted that when the abstract theory of [14] is specialized to parabolic evolution equations, it just covers the case of strict solutions of (0.1) with homogeneous initial datum (i.e. $x = 0$): thus for the case of classical solutions, or of strict ones with $x \neq 0$, our theorems cannot be deduced by that theory. However even in the former case our results are more precise than the corresponding ones which can be obtained by [14].

Let us describe now the subject of the next sections. Section 1 contains some notations and assumptions, as well as some preliminary results; we also derive (just formally) our representation formula. The properties of each component (operators and functions) of this formula are analyzed in Section 2, whereas the properties of the function in itself, as defined by the formula, are summarized in Section 3, where we also prove uniqueness. Section 4 is devoted to the study of certain problems which approximate Problem (0.1) and are useful in the proof of existence; in Section 5 we prove the convergence as $n \to \infty$ of the solutions u_n of the approximating problems. In Section 6 we present our existence and maximal regularity theorems for strict and classical solutions; finally in Section 7 we compare our assumptions and results with those available in the literature.

We finish this section by noting that the results of the present paper were partially announced in Acquistapace-Terreni [6].

1. Notations, assumptions and preliminaries.

Let E be a Banach space, fix $T > 0$ and let $Y \hookrightarrow E$ be another Banach space. We will use the following Banach function spaces:

(*a*) $B(Y) = \{f\colon]0, T] \to Y$ strongly measurable and bounded$\}$, and $C(Y) = C([0, T], Y)$, $C^\alpha(Y) = C^\alpha([0, T], Y)$ $(\alpha \in]0, 1[)$, $C^1(Y) = C^1([0, T], Y)$, $L^1(Y) = L^1(0, T, Y)$ with their usual norms;

(*b*) for any $\mu \in [0, \infty[$, $B_\mu(Y) = \{f\colon]0, T] \to Y\colon t \to t^\mu f(t) \in B(Y)\}$, $C_\mu(Y) = \{f \in B_\mu(Y)$ continuous$\}$ with their obvious norms (thus, in particular, $B_0(Y) = B(Y)$ but $C_0(Y) \hookleftarrow C(Y)$;

(*c*) for $\mu \in [0, \infty[$,

$$I_\mu(Y) = \left\{f \in B_\mu(Y)\colon \exists Y - \lim_{a \to 0^+} \int_a^T f(s)\, ds\right\} \tag{1.1}$$

with norm given by

$$\|f\|_{I_\mu(Y)} = \|f\|_{B_\mu(Y)} + [f]_{*,Y}, \tag{1.2}$$

$$[f]_{*,Y} = \sup\{\|\int_a^b f(s)\,ds\|_Y : 0 < a \leqslant b \leqslant T\}. \tag{1.3}$$

It is clear that (1.2) defines a norm in $I_\mu(Y)$; in Lemma 1.7 below we will show that $I_\mu(Y)$ is indeed a Banach space. If $f \in I_\mu(Y)$, $\mu \geqslant 0$, we set by definition (with abuse of notation):

$$\int_0^b f(s)\,ds = Y - \lim_{a\to 0^+} \int_a^b f(s)\,ds \qquad \forall b \in]0, T]. \tag{1.4}$$

We will also use the function spaces $B_+(Y) = \{f:]0, T] \to Y$ strongly measurable: $f|_{[a,T]}$ is bounded in Y $\forall a \in]0, T[\}$, and $C_+(Y)$, $C_+^1(Y)$, $C_+^\alpha(Y)$ which are defined similarly.

Let $A: D_A \subseteq E \to E$ be a closed linear operator, generating a bounded analytic semigroup $\{\exp[sA]\}_{s\geqslant 0}$ (not necessarily strongly continuous at $s = 0$). Then in particular $D_A \hookrightarrow E$ when it is endowed with the graph norm. We consider the real interpolation spaces $(D_A, E)_{\beta,\infty}$ $(\beta \in]0, 1[)$ introduced by Lions-Peetre [15].

Definition 1.1. We set for $\beta \in]0, 1[$

$$D_A(\beta, \infty) = (D_A, E)_{1-\beta,\infty}.$$

It is plain that

$$D_A \hookrightarrow D_A(\beta, \infty) \hookrightarrow D_A(\alpha, \infty) \hookrightarrow \bar{D}_A \qquad \forall 0 < \alpha \leqslant \beta < 1. \tag{1.5}$$

The following characterizations hold (see Butzer-Berens [7] for the dense-domain case, Sinestrari [16] for the general case):

$$D_A(\beta, \infty) = \tag{1.6$_1$}$$
$$= \{x \in E: [x]_{1,\beta} = \sup\{s^{-\beta}\|(\exp[sA] - 1)x\|_E : s > 0\} < \infty\},$$

$$D_A(\beta, \infty) = \tag{1.6$_2$}$$
$$= \{x \in E: [x]_{2,\beta} = \sup\{s^{1-\beta}\|A \exp[sA]x\|_E : s > 0\} < \infty\},$$

$$(1.6)_3 \qquad D_A(\beta, \infty) = \\ = \{x \in E\colon [x]_{3,\beta} = \sup\{|\lambda|^\beta \|AR(\lambda, A)x\|_E\colon \lambda \in \varrho(A)\} < \infty\};$$

moreover the corresponding norms

$$(1.7)_i \qquad \|x\|_{i,\beta} = \|x\|_E = + [x]_{i,\beta}, \quad i = 1, 2, 3$$

are all equivalent to the usual norm of $D_A(\beta, \infty)$ as an interpolation space. Thus we will denote by $\|x\|_{D_A(\beta,\infty)}$, $[x]_{D_A(\beta,\infty)}$ any of the norms $(1.7)_i$ and seminorms defined in $(1.6)_i$. When $\beta = 0$ or $\beta = 1$ the characterizations $(1.6)_i$ still make sense, and one gets $D_A(0, \infty) = E$ and $D_A(1, \infty) \hookleftarrow D_A$ (without equality in general). However we will use the following convention:

CONVENTION 1.2. We set $D_A(0, \infty) = E$, $D_A(1, \infty) = D_A$.

Let us list now our assumptions. From now on, Hypotheses I and II below will be assumed throughout.

Hypothesis I. For each $t \in [0, T]$, $A(t)\colon D_{A(t)} \subseteq E \to E$ is a closed linear operator and there exist $\theta_0 \in]\pi/2, \pi[$, $M > 0$ such that

(i) $\varrho(A(t)) \supseteq S_{\theta_0} = \{z \in \mathbb{C}\colon |\arg z| \leqslant \theta_0\} \cup \{0\}$ $\forall t \in [0, T]$,

(ii) $\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leqslant M/(1 + |\lambda|)$, $\forall \lambda \in S_{\theta_0}$, $\forall t \in [0, T]$.

Hypothesis II. There exist $B > 0$, $k \in \mathbb{N}^+$, $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ with $0 \leqslant \beta_i < \alpha_i \leqslant 2$, such that

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(E)} \leqslant B \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1} \\ \forall \lambda \in S_{\theta_0} - \{0\}, \quad \forall 0 \leqslant s \leqslant t \leqslant T.$$

We also assume (which is not restrictive)

$$(1.8) \qquad \delta = \min\{\alpha_i - \beta_i\colon 1 \leqslant i \leqslant k\} \in]0, 1[.$$

REMARK 1.3. By Hypothesis I, each operator $A(t)$ generates a bounded analytic semigroup $\{\exp[sA(t)]\}_{s \geqslant 0}$; the domains $D_{A(t)}$ may change with t and be not dense in E, so that the semigroups may be not strongly continuous at 0 (but this is necessarily true if E is locally sequentially weakly compact: see Kato [10]). However the resolvent sets $\varrho(A(t))$ are assumed to contain a common sector S_{θ_0}. Hypothesis II

provides some regularity in the dependence on t of the operators $A(t)$. Further comments on Hypotheses I, II, as well as comparisons with the assumptions of other papers will be made in Section 7.

In the next sections we will need some other Banach function spaces. Namely, for $\beta \in]0, 1]$ we set (with a slight abuse of notation):

$$B(D_A(\beta, \infty)) = \{f \in B(E) \colon f(t) \in D_{A(t)}(\beta, \infty) \ \forall t \in [0, T], \text{ and}$$
$$[f]_{B(D_A(\beta,\infty))} = \sup \{[f(t)]_{D_{A(t)}(\beta,\infty)} \colon t \in [0, T]\} < \infty\} ,$$
$$\|f\|_{B(D_A(\beta,\infty))} = \|f\|_{B(E)} + [f]_{B(D_A(\beta,\infty))};$$

similarly we define the Banach spaces $C(D_A(\beta, \infty))$, $C^\alpha(D_A(\beta, \infty))$ and $B_\mu(D_A(\beta, \infty))$ $(\alpha \in]0, 1[, \mu \in [0, \infty[)$.

We also define the function spaces $B_+(D_A(\beta, \infty))$, $C_+(D_A(\beta, \infty))$, $C_+^\alpha(D_A(\beta, \infty))$ $(\alpha \in]0, 1[)$ similarly to what we did before (compare with the paragraph after (1.4)).

Finally, we have:

DEFINITION 1.4. (i) Let $\beta \in]0, 1[$, $\mu \in [-\beta, \infty[$. We set:

$$Z_{\mu,\beta}(E) = \begin{cases} \{f \in C^{|\mu|}(E) \cap C_+^\beta(E) \colon [f]_{\mu,\beta} < \infty\} & \text{if } \mu \in [-\beta, 0[\\ \{f \in B_\mu(E) \cap C_+^\beta(E) \colon [f]_{\mu,\beta} < \infty\} & \text{if } \mu \in [0, \infty[, \end{cases} \tag{1.9}$$

$$Z_\mu(D_A(\beta, \infty)) = \tag{1.10}$$
$$= \begin{cases} \{f \in B(D_A(|\mu|, \infty)) \cap B_+(D_A(\beta, \infty)) \colon \{f\}_{\mu,\beta} < \infty\} & \text{if } \mu \in [-\beta, 0[\\ \{f \in B_\mu(E) \cap B_+(D_A(\beta, \infty)) \colon \{f\}_{\mu,\beta} < \infty\} & \text{if } \mu \in [0, \infty[, \end{cases}$$

where

$$[f]_{\mu,\beta} = \sup \left\{ t^{\mu+\beta} \sup_{t/2 \leqslant r \leqslant s \leqslant t} \frac{\|f(s) - f(r)\|_E}{(s-r)^\beta} \colon t \in]0, T] \right\} \tag{1.11}$$

$$\{f\}_{\mu,\beta} = \sup \{t^{\mu+\beta} \sup_{t/2 \leqslant s \leqslant t} [f(s)]_{D_{A(s)}(\beta,\infty)} \colon t \in]0, T]\} . \tag{1.12}$$

The norms of $Z_{\mu,\beta}(E)$ and $Z_\mu(D_A(\beta, \infty))$ are given by

$$\|f\|_{Z_{\mu,\beta}(E)} = \begin{cases} \|f\|_{C^{|\mu|}(E)} + [f]_{\mu,\beta} & \text{if } \mu \in [-\beta, 0[\\ \|f\|_{B_\mu(E)} + [f]_{\mu,\beta} & \text{if } \mu \in [0, \infty[, \end{cases}$$

$$\|f\|_{Z_\mu(D_A(\beta,\infty))} = \begin{cases} \|f\|_{B(D_A(|\mu|,\infty))} + \{f\}_{\mu,\beta} & \text{if } \mu \in [-\beta, 0[\\ \|f\|_{B_\mu(E)} + \{f\}_{\mu,\beta} & \text{if } \mu \in [0, \infty[. \end{cases}$$

(ii) Let $\beta \in]0, 1[$, $\mu \in [1, \infty[$. We set

$$Z^*_{\mu,\beta}(E) = Z_{\mu,\beta}(E) \cap I_\mu(E)\,, \tag{1.13}$$

$$Z^*_\mu(D_A(\beta, \infty)) = Z_\mu(D_A(\beta, \infty)) \cap I_\mu(E)\,; \tag{1.14}$$

the norms of $Z^*_{\mu,\beta}(E)$ and $Z^*_\mu(D_A(\beta, \infty))$ are given by

$$\|f\|_{Z^*_{\mu,\beta}(E)} = \|f\|_{Z_{\mu,\beta}(E)} + [f]_{*,E}\,,$$

$$\|f\|_{Z^*_\mu(D_A(\beta,\infty))} = \|f\|_{Z_\mu(D_A(\beta,\infty))} + [f]_{*,E}\,.$$

It is easy to see that the above spaces are alla Banach spaces (the last two in view of Lemma 1.7 below).

EXAMPLE 1.5. Fix $y \in E$ and set

$$f(t) = t^{\beta-\mu}|\sin t^{-1}|^\beta \cdot y\,, \qquad g(t) = t^{-\mu}\cdot \exp[tA(t)]y\,,$$

$$h(t) = \frac{d}{dt}(t^{1+\mu}\sin t^{-2\mu})\cdot y:$$

by using $(1.6)_2$ it is readily seen that $g \in Z_\mu(D_A(\beta, \infty))$, whereas a tedious but elementary calculation shows that $f \in Z_{\mu,\beta}(E)$; finally, it is easy to see that $h \in I_\mu(E)$.

Let us define now strict and classical solutions of Problem (0.1).

DEFINITION 1.6. We say that $u \in C(E)$ is a *strict* (resp. *classical*) *solution* of (0.1) if $u \in C^1(E) \cap C(D_A)$ (resp. $u \in C^1_+(E) \cap C_+(D_A)$) and

$$u(0) = x\,, \qquad u' - Au \equiv f \quad \text{in } [0, T] \text{ (resp. in }]0, T]).$$

For the sake of simplicity, here and from now on we write (improperly) Au and $R(\lambda, A)u$ instead of $A(\cdot)u(\cdot)$, $R(\lambda, A(\cdot))u(\cdot)$.

We start with some preliminary results.

LEMMA 1.7. *Let $Y \hookrightarrow E$ be a Banach space. For each $\mu \in [0, \infty[$ the space $I_\mu(Y)$ defined by* (1.1) *is a Banach space with respect to the norm* (1.2)-(1.3).

PROOF. Let $\{f_\mu\}$ be a Cauchy sequence in $I_\mu(Y)$; then it converges in $B_\mu(Y)$ to some $f \in B_\mu(Y)$, and we have to show that

$$\exists Y - \lim_{a \to 0^+} \int_a^T f(s)\, ds\,. \tag{1.15}$$

For each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbf{N}$ such that

$$\sup_{0<a\leqslant b\leqslant T} \Big\| \int_a^b [f_n(s) - f_m(s)]\, ds \Big\|_Y \leqslant \varepsilon \qquad \forall n, m \geqslant n_\varepsilon;$$

as $n \to \infty$ we deduce that

$$\sup_{0<a\leqslant b\leqslant T} \Big\| \int_a^b [f(s) - f_m(s)]\, ds \Big\|_Y \leqslant \varepsilon \qquad \forall m \geqslant n_\varepsilon\,. \tag{1.16}$$

Now take $m = n_\varepsilon$: Since $a \to \int_a^T f_{n_\varepsilon}(s)\, ds$ converges in Y as $a \to 0^+$, there exists $\delta_\varepsilon > 0$ such that

$$\Big\| \int_a^b f_{n_\varepsilon}(s)\, ds \Big\|_Y \leqslant \varepsilon \qquad \forall 0 < a \leqslant b \leqslant \delta_\varepsilon\,.$$

Hence

$$\Big\| \int_a^b f(s)\, ds \Big\| \leqslant \Big\| \int_a^b [f(s) - f_{n_\varepsilon}(s)]\, ds \Big\|_Y + \Big\| \int_a^b f_{n_\varepsilon}(s)\, ds \Big\| \leqslant 2\varepsilon \qquad \forall 0 < a \leqslant b \leqslant \delta_\varepsilon$$

and (1.15) is proved. Finally, by (1.16), $f_n \to f$ in $I_\mu(Y)$. ■

Let us recall now some properties of the operators $A(t)$. We recall the well-known representation of the semigroup $\{\exp[sA(t)]\}_{s\geqslant 0}$

$$A(t)^m \exp[sA(t)] = \frac{1}{2\pi i} \int_\gamma \exp[s\lambda]\, \lambda^m R(\lambda, A(t))\, d\lambda\,, \tag{1.17}$$

$$m \in \mathbf{N}\,, \quad s > 0\,, \quad t \in [0, T]\,,$$

where $\gamma = \gamma^- \cup \gamma_0 \cup \gamma^+$ and

$$\gamma_0 = \{z \in \mathbf{C}\colon |z| = 1, |\arg z| \leqslant \theta\}\,, \qquad \gamma^\pm = \{z \in \mathbf{C}\colon |z| \geqslant 1, \arg z = \pm\theta\}$$

(with fixed $\theta \in]\pi/2, \theta_0[$). As a consequence we have:

Lemma 1.8. *Fix $t \in [0, T]$, $\beta \in]0, 1[$. Then:*

(i) $$\|A(t)^m \exp[sA(t)]\|_{\mathfrak{L}(D_{A(t)}(\sigma,\infty), D_{A(t)}(\varrho,\infty))} \leqslant \frac{c(\sigma, \varrho, m)}{s^{(m+\varrho-\sigma)\vee 0}}$$
$$\forall s > 0\,, \quad \forall \sigma, \varrho \in [0, 1]\,, \quad \forall m \in \mathbf{N};$$

(ii) $s \to \exp[sA(t)]x \in C(E)$ *if and only if* $x \in \overline{D_{A(t)}}$;

(iii) $s \to \exp[sA(t)]x \in C^\beta(E)$ *if and only if* $x \in D_{A(t)}(\beta, \infty)$;

(iv) $s \to \exp[sA(t)]x \in B(D_{A(t)}(\beta, \infty))$ *if and only if* $x \in D_{A(t)}(\beta, \infty)$.

Proof. (i)-(ii) See [16, Propositions 1.13-1.14 and 1.2 (i)].

(iii) Easy consequence of $(1.6)_1$.

(iv) Easy consequence of (i) and $(1.6)_2$. ■

In the preceding lemma and from now on, we denote by C any absolute constant occurring in our estimates; dependence on known quantities, if any, will be specified when necessary.

Lemma 1.9. *Let $\beta \in [0, 1]$ and $0 \leqslant r \leqslant s \leqslant t \leqslant T$. Then:*

(i) $$R(\lambda, A(t)) - R(\lambda, A(s)) = A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\cdot$$
$$\cdot[-\lambda A(s)R(\lambda, A(s))[A(s)^{-1} - A(r)^{-1}] + 1]A(r)R(\lambda, A(r)) \quad \forall \lambda \in S_{\theta_0};$$

(ii) $$\|R(\lambda, A(t)) - R(\lambda, A(s))\|_{\mathfrak{L}(D_{A(r)}(\beta,\infty), E)} \leqslant$$
$$\leqslant c(\beta) \sum_{j=1}^{k} (t-s)^{\alpha_j} |\lambda|^{\beta_j - 1 - \beta} \Big[1 + \sum_{i=1}^{k} (s-r)^{\alpha_i} |\lambda|^{\beta_i}\Big] \qquad \forall \lambda \in S_{\theta_0} - \{0\}\,.$$

Proof. – (i) It is a straightforward computation.

(ii) It follows by (i), Hypothesis II and $(1.6)_3$. ■

Lemma 1.10. *Let $\beta \in [0, 1]$ and $0 \leqslant r \leqslant s \leqslant t \leqslant T$. Then for $\xi > 0$ and $m \in \mathbf{N}$ we have:*

(i) $$\|A(t)^m \exp[\xi A(t)] - A(s)^m \exp[\xi A(s)]\|_{\mathfrak{L}(D_{A(r)}(\beta,\infty), E)} \leqslant$$
$$\leqslant c(\beta, m) \sum_{i=1}^{k} \frac{(t-s)^{\alpha_i}}{\xi^{m+\beta_i-\beta}} \Big[1 + \sum_{j=1}^{k} \frac{(s-r)^{\alpha_j}}{\xi^{\beta_j}}\Big];$$

(ii) $\|A(t)^m \exp[\xi A(t)]\|_{\mathcal{L}(D_{A(r)}(\beta,\infty),E)} \leqslant c(\beta, m)\left[\sum_{i=1}^{k} \frac{(t-r)^{\alpha_i}}{\xi^{m+\beta_i-\beta}} + \frac{1}{\xi^{(m-\beta)\vee 0}}\right].$

PROOF. (i) Easy consequence of (1.17) and Lemma 1.9 (ii).

(ii) If $m = 0$ the result is obvious; if $\beta = 0$ it is Lemma 1.8 (i). Otherwise it follows easily by (i) and Lemma 1.8 (i). ■

LEMMA 1.11. *Let $\beta \in [0, 1]$ and $0 \leqslant r \leqslant s \leqslant \sigma \leqslant t \leqslant T$. Then for $\xi > 0$ and $m \in \mathbb{N}^+$ we have:*

(i) $$\|A(t)^m \exp[\xi A(t)][A(s)^{-1} - A(r)^{-1}]\|_{\mathcal{L}(E, D_{A(t)}(\beta,\infty))} \leqslant$$
$$\leqslant c(\beta, m) \sum_{i=1}^{k} \frac{(s-r)^{\alpha_i}}{\xi^{m+\beta_i-1+\beta}}\left[1 + \sum_{j=1}^{k} \frac{(t-s)^{\alpha_j}}{\xi^{\beta_j}}\right];$$

(ii) $$\|[A(t)^m \exp[\xi A(t)] - A(\sigma)^m \exp[\xi A(\sigma)]][A(s)^{-1} - A(r)^{-1}]\|_{\mathcal{L}(E)} \leqslant$$
$$\leqslant c(m) \sum_{i,j=1}^{k} \frac{(t-\sigma)^{\alpha_i}(s-r)^{\alpha_j}}{\xi^{m+\beta_i+\beta_j-1}}\left[1 + \sum_{h=1}^{k} \frac{(\sigma-s)^{\alpha_h}}{\xi^{\beta_h}}\right].$$

PROOF. Part (ii) follows easily by Lemma 1.9 (i) and Hypothesis II. To prove part (i) write for each $\eta > 0$:

$$\eta^{1-\beta} A(t) \exp[\eta A(t)][A(t)^m \exp[\xi A(t)][A(s)^{-1} - A(r)^{-1}]] =$$
$$= \eta^{1-\beta}[A(t)^{m+1} \exp[(\eta + \xi)A(t)] - A(s)^{m+1} \exp[(\eta + \xi)A(s)]] \cdot$$
$$\cdot [A(s)^{-1} - A(r)^{-1}] + \eta^{1-\beta} A(s)^{m+1} \exp[(\eta + \xi)A(s)][A(s)^{-1} - A(r)^{-1}];$$

the result follows by part (ii), Hypothesis II and $(1.6)_2$. ■

We finish this section with a heuristic derivation of the representation formula for the strict solution u of Problem (0.1). Fix $t \in]0, T]$ and set

$$v(s) = \exp[(t-s)A(t)]u(s), \qquad s \in [0, t].$$

Then for each $s \in [0, t[$

$$v'(s) = -A(t) \exp[(t-s)A(t)]u(s) +$$
$$+ \exp[(t-s)A(t)][A(s)u(s) + f(s)] =$$
$$= A(t) \exp[(t-s)A(t)][A(t)^{-1} - A(s)^{-1}]A(s)u(s) +$$
$$+ \exp[(t-s)A(t)]f(s).$$

Integrate over $]0, t[$ and apply $A(t)$ to both sides: the result is

$$A(t)u(t) = \int_0^t Q(t, s)A(s)u(s)\,ds + L(f, x)(t)\,, \tag{1.18}$$

where we have set

$$Q(t, s) = A(t)^2 \exp[(t-s)A(t)][A(t)^{-1} - A(s)^{-1}]\,, \tag{1.19}$$

$$L(f, x)(t) = A(t)\int_0^t \exp[(t-s)A(t)]f(s)\,ds + A(t)\exp[tA(t)]x\,, \qquad t \in]0, T]\,. \tag{1.20}$$

The integral equation (1.18) in the unknown Au is of Volterra type, with integrable kernel (see Lemma 2.3 (i) below); thus if we set

$$Qg(t) = \int_0^t Q(t, s)g(s)\,ds\,, \qquad t \in [0, T]\,, \tag{1.21}$$

we can invert (1.18), obtaining for u the representation formula

$$u = A^{-1}(1-Q)^{-1}L(f, x)\,. \tag{1.22}$$

This argument is just formal: in particular, for $f \in C(E)$ and $x \in D_{A(0)}$ the function (1.20) is not meaningful in general; however we will see in Section 3 that under suitable assumptions and the data x, f the function $L(f, x)$ makes sense and the above argument works, leading to the representation formula (1.22) fro strict and classical solutions of (0.1).

2. Technicalities, I.

We analyze here the properties of the operator Q and the function $L(f, x)$, respectively defined by (1.21) and (1.20). As already remarked, Hypothesis I and II are always assumed; in particular the number δ is defined by (1.8). We also recall that the spaces $Z_{\mu,\beta}(E)$, $Z_\mu(D_A(\beta, \infty))$, $Z^*_{\mu,\beta}(E)$, $Z^*_\mu(D_A(\beta, \infty))$ were introduced in Definition 1.4.

(*a*) *The function* $L(f, x)$

PROPOSITION 2.1. *Fix* $\beta \in]0, \delta]$. *If* $x \in D_{A(0)}$, $f \in C^\beta(E)$, *then*:

(i) $L(f, x) \in C^\beta(E)$ *if and only if* $A(0)x + f(0) \in D_{A(0)}(\beta, \infty)$;

(ii) $L(f, x) + f \in B(D_A(\beta, \infty))$ *if and only if* $A(0)x + f(0) \in$ $\in D_{A(0)}(\beta, \infty)$.

If $x \in D_{A(0)}$, $f \in C(E) \cap Z_{0,\beta}(E)$, *then*:

(iii) $L(f, x) \in C(E)$ *if and only if* $A(0)x + f(0) \in \overline{D_{A(0)}}$.

Fix also $\mu \in [0, 1[$. *If* $x \in D_{A(0)}(1 - \mu, \infty)$, $f \in Z_{\mu,\beta}(E)$, *then*:

(iv) $L(f, x) \in Z_{\mu,\beta}(E)$;

(v) $L(f, x) + f \in Z_\mu(D_A(\beta, \infty))$.

Finally fix $\mu \in [1, 1 + \beta[$. *If* $x \in \overline{D_{A(0)}}$, $f \in Z_{\mu,\beta}(E) \cap L^1(E)$, *then*:

(vi) $L(f, x) \in Z^*_{\mu,\beta}(E)$;

(vii) $L(f, x) + f \in Z^*_\mu(D_A(\beta, \infty))$.

PROOF. – For each statement we have to split conveniently $L(f, x)(t)$ into several terms, and to estimate each one separately.

(i) We write

$$(2.1) \qquad L(f, x)(t) = \int_0^t A(t) \exp[(t - s)A(t)][f(s) - f(t)]\, ds + \\ + [\exp[tA(t)] - 1] f(t) + A(t) \exp[tA(t)]x;$$

hence if $0 \leqslant r \leqslant t \leqslant T$

$$(2.2) \qquad L(f, x)(t) - L(f, x)(r) = \int_r^t A(t) \exp[(t - s)A(t)][f(s) - f(t)]\, ds +$$

$$+ \int_0^r [A(t) \exp[(t - s)A(t)] - A(r) \exp[(t - s)A(r)]]\, [f(s) - f(t)]\, ds +$$

$$+ [\exp[tA(r)] - \exp[(t - r)A(r)]]\, [f(r) - f(t)] +$$

$$+ \int_0^r \int_{r-s}^{t-s} A(r)^2 \exp[\sigma A(r)][f(s) - f(r)]\, d\sigma\, ds +$$

$$+ [\exp[tA(t)] - 1]\, [f(t) - f(r)] +$$

$$+ [\exp[tA(t)] - \exp[tA(r)]] f(r) + \int_r^t A(r) \exp[\sigma A(r)][f(r) - f(0)]\, d\sigma +$$

$$+ \int_r^t [A(r) \exp[\sigma A(r)] - A(0) \exp[\sigma A(0)]] f(0)\, d\sigma +$$

$$+ [A(t) \exp[tA(t)] - A(r) \exp[tA(r)]] x +$$

$$+ \int_r^t [A(r)^2 \exp[\sigma A(r)] - A(0)^2 \exp[\sigma A(0)]] x\, d\sigma +$$

$$+ [\exp[tA(0) - \exp[rA(0)]] [A(0)x + f(0)] = \sum_{i=1}^{11} I_i .$$

All terms but I_{11} can be easily estimated by Lemma 1.10 (i)-(ii). Thus we get:

$$\|L(f, x)(t) - L(f, x)(r)\|_E \leqslant C(t-r)^\beta + \|I_{11}\|_E ,$$

and (i) follows by Lemma 1.8 (iii). More precisely we have

$$\|L(f, x)\|_{C^\beta(E)} \leqslant C\{\|f\|_{C^\beta(E)} + \|x\|_{D_{A(0)}} + \|A(0)x + f(0)\|_{D_{A(0)}(\beta,\infty)}\} . \tag{2.3}$$

(ii) If $0 \leqslant t \leqslant T$ we can write for each $\xi > 0$:

$$\begin{aligned} &\xi^{1-\beta} A(t) \exp[\xi A(t)][L(f, x)(t) + f(t)] = \\ &= \xi^{1-\beta} \int_0^t A(t)^2 \exp[(\xi + t - s) A(t)][f(s) - f(t)]\, ds + \\ &+ \xi^{1-\beta} A(t) \exp[(\xi + t) A(t)][f(t) - f(0)] + \\ &+ \xi^{1-\beta} [A(t) \exp[(\xi + t) A(t)] - A(0) \exp[(\xi + t) A(0)]] f(0) + \\ &+ \xi^{1-\beta} [A(t)^2 \exp[(\xi + t) A(t)] - A(0)^2 \exp[(\xi + t) A(0)]] x + \\ &+ \xi^{1-\beta} A(0) \exp[(\xi + t) A(0)][f(0) + A(0)x] = \sum_{j=1}^{5} J_j ; \end{aligned} \tag{2.4}$$

by Lemma 1.10 (i)-(ii) we get

$$[L(f, x)(t) + f(t)]_{D_{A(t)}(\beta,\infty)} \leqslant c + \sup_{\xi > 0} \|J_5\|_E$$

and (ii) follows by Lemma 1.8 (iv) and $(1.6)_2$. More precisely we obtain

$$\|L(f, x) + f\|_{B(D_A(\beta,\infty))} \leqslant c\{\|f\|_{C^\beta(E)} + \|x\|_{D_A(0)} + \\ + \|A(0)x + f(0)\|_{D_A(\beta,\infty)}\} . \tag{2.5}$$

(iii) The fact that $L(f, x) \in C_+(E)$ will follow by (iv); thus we have just to prove continuity at $t = 0$. We have instead of (2.1):

$$L(f, x)(t) - A(0)x = \int_0^{t/2} A(t) \exp[(t-s)A(t)][f(s) - f(0)]\, ds + \\ + \int_{t/2}^{t} A(t) \exp[(t-s)A(t)][f(s) - f(t)]\, ds + \\ + [\exp[t/2)A(t)] - 1][f(t) - f(0)] + [\exp[tA(t)] - \exp[tA(0)]]f(0) + \\ + [A(t)\exp[tA(t)] - A(0)\exp[tA(0)]]x + \\ + [\exp[tA(0)] - 1][A(0)x + f(0)] = \sum_{i=1}^{6} I_i .$$

By Lemmata 1.10 (i)-(ii) and 1.8 (ii) it is easily seen that

$$\|L(f, x)(t) - A(0)x\|_E \leqslant \|I_2\|_E + o(1) \qquad \text{as } t \to 0^+;$$

on the other hand by Lemma 1.10 (i)

$$\|I_2\|_E \leqslant c \int_{t/2}^{t} \frac{1}{t-s} \|f(t) - f(s)\|_E^{\frac{1}{2}+\frac{1}{2}}\, ds \leqslant \\ \leqslant \frac{c}{t^{\beta/2}} \int_{t/2}^{t} \frac{1}{(t-s)^{1-\beta/2}} [f]_{0,\beta}^{\frac{1}{2}} \cdot \sup_{r\in[t/2,t]} \|f(r) - f(t)\|_E^{\frac{1}{2}} ,$$

and this clearly implies that $L(f, x)(t) \to A(0)x$ as $t \to 0^+$.

Moreover by (2.10) below and by Convention 1.2 it will follow that

$$\|L(f, x)\|_{C(E)} \leqslant c\{\|f\|_{Z_{0,\beta}(E)} + \|x\|_{D_A(0)}\} . \tag{2.6}$$

(iv) Similarly to (2.1) we write

$$(2.7)\qquad L(f,x)(t)=\int_0^{t/2}A(t)\exp[(t-s)A(t)]f(s)\,ds+$$
$$+\int_{t/2}^{t}A(t)\exp[(t-s)A(t)][f(s)-f(t)]\,ds+$$
$$+[\exp[(t/2)A(t)]-1]f(t)+A(t)\exp[tA(t)x];$$

hence (1.9) and Lemma 1.10 (ii) yield

$$(2.8)\qquad \|L(f,x)(t)\|_E\leqslant\frac{c}{t^{\mu}}\{\|f\|_{Z_{\mu,\beta}(E)}+\|x\|_{D_{A(0)}(1-\mu,\infty)}\}\,.$$

Next, if $0<t/2\leqslant r\leqslant s\leqslant t\leqslant T$, by (2.7) similarly to (2.2) we get:

$$(2.9)\qquad L(f,x)(s)-L(f,x)(r)=\int_{r/2}^{s/2}A(s)\exp[(s-\sigma)A(s)]f(\sigma)\,d\sigma+$$
$$+\int_0^{r/2}[A(s)\exp[(s-\sigma)A(s)]-A(r)\exp[(s-\sigma)A(r)]]f(\sigma)\,d\sigma+$$
$$+\int_0^{r/2}\int_{r-\sigma}^{s-\sigma}A(r)^2\exp[\xi A(r)]f(\sigma)\,d\xi\,d\sigma+\Big[\int_r^s-\int_{r/2}^{s/2}\Big]A(s)\exp[(s-\sigma)A(s)]\cdot$$
$$\cdot[f(\sigma)-f(s)]\,d\sigma+\int_{r/2}^{r}[A(s)\exp[(s-\sigma)A(s)]-A(r)\exp[(s-\sigma)A(r)]]\cdot$$
$$\cdot[f(\sigma)-f(s)]\,d\sigma+[\exp[(s-r/2)A(r)]-\exp[(s-r)A(r)]][f(r)-f(s)]+$$
$$+\int_{r/2}^{r}\int_{r-\sigma}^{s-\sigma}A(r)^2\exp[\xi A(r)][f(\sigma)-f(r)]\,d\xi\,d\sigma+[\exp[(s/2)A(s)]-1]\cdot$$
$$\cdot[f(s)-f(r)]+[\exp[(s/2)A(s)]-\exp[(s/2)A(r)]]f(r)+$$
$$+\int_{r/2}^{s/2}A(r)\exp[\xi A(r)]f(r)\,d\xi+[A(s)\exp[sA(s)]-A(r)\exp[sA(r)]]x+$$
$$+\int_r^s A(r)^2\exp[\xi A(r)]x\,d\xi;$$

consequently by Lemma 1.10 (i)-(ii) and (1.11) we deduce, recalling (2.8),

$$\|L(f, x)\|_{Z_{\mu,\beta}(E)} \leqslant c\{\|f\|_{Z_{\mu,\beta}(E)} + \|x\|_{D_{A(0)}(1-\mu,\infty)}\}\,, \tag{2.10}$$

and (iv) is proved.

(v) Similarly to (2.4) we write for $0 < t/2 \leqslant s \leqslant t \leqslant T$ and for each $\xi > 0$:

$$\begin{aligned} \xi^{1-\beta} A(s) \exp[\xi A(s)][L(f, x)(s) + f(s)] = \\ = \xi^{1-\beta} \int_0^{s/2} A(s)^2 \exp[(\xi + s - \sigma) A(s)] f(\sigma)\, d\sigma + \\ + \xi^{1-\beta} \int_{s/2}^{s} A(s)^2 \exp[(\xi + s - \sigma) A(s)][f(\sigma) - f(s)]\, d\sigma + \\ + \xi^{1-\beta} A(s) \exp[(\xi + s/2) A(s)] f(s) + \xi^{1-\beta} A(s)^2 \exp[(\xi + s) A(s)] x; \end{aligned} \tag{2.11}$$

now Lemma 1.10 (ii) and (2.8) easily lead to

$$\|L(f, x) + f\|_{Z_\mu(D_A(\beta,\infty))} \leqslant c\{\|f\|_{Z_{\mu,\beta}(E)} + \|x\|_{D_{A(0)}(1-\mu,\infty)}\}\,, \tag{2.12}$$

which proves (v).

(vi) If $0 \leqslant t \leqslant T$, similarly to (2.7) we have

$$\begin{aligned} L(f, x)(t) = -\int_0^{t/2} \int_{t-s}^{t} A(t)^2 \exp[\sigma A(t)] f(s)\, d\sigma\, ds + \\ + A(t) \exp[tA(t)] \int_0^{t/2} f(s)\, ds + \int_{t/2}^{t} A(t) \exp[(t - s) A(t)][f(s) - f(t)]\, ds + \\ + [\exp[(t/2) A(t)] - 1] f(t) + A(t) \exp[tA(t)] x; \end{aligned} \tag{2.13}$$

hence by Lemma 1.10 (ii) we obtain

$$\|L(f, x)(t)\|_E \leqslant \frac{c}{t^\mu} \{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + [f]_{*,E}\}\,. \tag{2.14}$$

Next, if $0 < t/2 \leqslant r \leqslant s \leqslant t \leqslant T$ we have by (2.13)

$$L(f, x)(s) - L(f, x)(r) = -\int_{r/2}^{s/2} \int_{s-\sigma}^{s} A(s)^2 \exp[\xi A(s)] f(\sigma)\, d\xi\, d\sigma -$$

$$-\int_{0}^{r/2} \int_{s-\sigma}^{s} [A(s)^2 \exp[\xi A(s)] - A(r)^2 \exp[\xi A(r)]]\, f(\sigma)\, d\xi\, d\sigma -$$

$$-\int_{0}^{r/2} \left[\int_{s-\sigma}^{s} - \int_{r-\sigma}^{r} \right] A(r)^2 \exp[\xi A(r)] f(\sigma)\, d\xi\, d\sigma +$$

$$+ [A(s) \exp[sA(s)] - A(r) \exp[sA(r)]] \int_{0}^{s/2} f(\sigma)\, d\sigma +$$

$$+ \int_{r}^{s} A(r)^2 \exp[\xi A(r)]\, d\xi \int_{0}^{s/2} f(\sigma)\, d\sigma + A(r) \exp[rA(r)] \int_{r/2}^{s/2} f(\sigma)\, d\sigma + \sum_{i=4}^{13} I_i$$

where $I_4, I_5, \ldots, I_{13}$ are the same terms occurring in (2.9); thus denoting by $J_1, \ldots, J_7$ the remaining terms above we have

$$L(f, x)(s) - L(f, x)(r) = \sum_{j=1}^{7} J_j + \sum_{i=4}^{13} I_i .$$

Now some attention must be paid in extimating $\|J_3 + J_4\|_E$: indeed, we have by Lemma 1.10 (ii)

$$\|J_3 + J_4\|_E \leqslant \int_{0}^{r/2} \left\| \left[\int_{s-\sigma}^{s} - \int_{r-\sigma}^{r} \right] \ldots d\xi \right\|_E^{1-\beta} \left\| \left[\int_{r}^{s} - \int_{r-\sigma}^{s-\sigma} \right. \ldots d\xi \right\|_E^{\beta} d\sigma \leqslant$$

$$\leqslant \int_{0}^{r/2} \left\{ \left[\int_{s-\sigma}^{s} + \int_{r-\sigma}^{r} \right] \frac{d\xi}{\xi^2} \right\}^{1-\beta} \left\{ \left[\int_{r}^{s} + \int_{r-\sigma}^{s-\sigma} \right] \frac{d\xi}{\xi^2} \right\}^{\beta} \frac{d\sigma}{\sigma^{\mu}} \|f\|_{B_\mu(E)} \leqslant$$

$$\leqslant c \int_{0}^{r/2} \left(\frac{\sigma}{(r/2 - \sigma)\, t} \right)^{1-\beta} \left(\frac{s-r}{t^2} \right)^{\beta} \frac{d\sigma}{\sigma^{\mu}} \|f\|_{B_\mu(E)} \leqslant c\, \frac{(s-r)^{\beta}}{t^{\mu+\beta}} \|f\|_{B_\mu(E)};$$

the other terms are easily estimated by Lemma 1.10 (i)-(ii). Summing up, and recalling (2.14), we get

$$\|L(f, x)\|_{Z_{\mu,\beta}(E)} \leqslant c\{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + [f]_{*,E}\} . \tag{2.15}$$

By (1.13) it remains to show that $L(f, x) \in I_\mu(E)$, i.e. (by (2.14)) that there exists

$$E - \lim_{a \to 0^+} \int_a^T L(f, x)(t)\, dt .$$

Fix $0 < a \leqslant b \leqslant T$. We have for each $t \in [a, b]$:

$$\begin{aligned}
\int_a^b L(f, x)(t)\, dt = & \int_a^b \int_0^{t/2} [A(t) \exp[(t-s)A(t)] - A(s) \exp[(t-s)A(s)]] \cdot \\
& \cdot f(s)\, ds\, dt + \int_a^b \int_0^{t/2} A(s) \exp[(t-s)A(s)] f(s)\, ds\, dt + \\
& + \int_a^b \int_{t/2}^t A(t) \exp[(t-s)A(t)][f(s) - f(t)]\, ds\, dt + \\
& + \int_a^b [\exp[(t/2)A(t)] - 1] f(t)\, dt + \\
& + \int_a^b [A(t) \exp[tA(t)] - A(0) \exp[tA(0)]] x\, dt + \\
& + [\exp[bA(0)] - \exp[aA(0)]] x = \sum_{i=1}^6 I_i .
\end{aligned}$$

Now both I_2 and I_3 are absolutely convergent integrals, so that using Fubini's Theorem it is not difficult to see that:

$$\begin{aligned}
I_2 = & \int_0^{a/2} [\exp[(b-s)A(s)] - \exp[(a-s)A(s)]] f(s)\, ds + \\
& + \int_{a/2}^{b/2} [\exp[(b-s)A(s)] - \exp[sA(s)]] f(s)\, ds ,
\end{aligned}$$

$$I_3 = \lim_{\varepsilon\to 0^+} \int_a^b \int_{t/2}^{(t-\varepsilon)\vee t/2} A(t) \exp[(t-s)A(t)][f(s)-f(t)]\,ds\,dt =$$

$$= \int_a^b \int_{t/2}^{t} [A(t)\exp[(t-s)A(t)] - A(s)\exp[(t-s)A(s)]]\, f(s)\,ds\,dt +$$

$$+ \lim_{\varepsilon\to 0^+} \Big\{ \int_a^b \int_{t/2}^{(t-\varepsilon)\vee t/2} A(s)\exp[(t-s)A(s)] f(s)\,ds\,dt -$$

$$- \int_a^b \int_{t/2}^{(t-\varepsilon)\wedge t/2)} A(t)\exp[(t-s)A(t)] f(t)\,ds\,dt \Big\} =$$

$$= \int_a^b \int_{t/2}^{t} [A(t)\exp[(t-s)A(t)] - A(s)\exp[(t-s)A(s)]]\, f(s)\,ds\,dt +$$

$$+ \int_{a/2}^{b/2} \exp[sA(s)] f(s)\,ds + \int_{b/2}^{b} \exp[(b-s)A(s)] f(s)\,ds -$$

$$- \int_{a/2}^{a} \exp[(a-s)A(s)] f(s)\,ds - \int_a^b \exp[(s/2)A(s)] f(s)\,ds;$$

in the last equality we have used again Fubini's Theorem. Summing up, we easily get

$$(2.16) \qquad \int_a^b L(f,x)(t)\,dt = \int_a^b \int_0^t [A(t)\exp[(t-s)A(t)] -$$

$$- A(s)\exp[(t-s)A(s)]]\, f(s)\,ds\,dt +$$

$$+ \int_0^b [\exp[(b-s)A(s)] - 1] f(s)\,ds - \int_0^a [\exp[(a-s)A(s)] - 1] f(s)\,ds +$$

$$+ \int_a^b [A(t)\exp[tA(t)] - A(0)\exp[tA(0)]]\, x\,dt +$$

$$+ [\exp[bA(0)] - \exp[aA(0)]]\, x\,,$$

and consequently by Lemma 1.10 (i)-(ii) we deduce

$$\|\int_a^b L(f,x)(t)\,dt\|_E \leqslant c(b-a)^\delta\{\|f\|_{L^1(E)} + \|x\|_E\} +$$

$$+ c\int_0^b \|f(s)\|_E\,ds + \|[\exp[(b-a)A(0)]-1]x\|_E;$$

thus recalling Lemma 1.8 (ii) we conclude that

$$\|\int_a^b L(f,x)(s)\,ds\|_E = o(1) \qquad \text{as } a, b \to 0^+;$$

moreover we also get

$$[L(f,x)]_{*,E} \leqslant c\{\|x\|_E + \|f\|_{L^1(E)}\} \tag{2.17}$$

which, together with (2.15), implies

$$\|L(f,x)\|_{Z^*_{\mu,\beta}(E)} \leqslant c\{\|x\|_E + \|f\|_{L^1(E)} + \|f\|_{Z_{\mu,\beta}(E)}\}\,. \tag{2.18}$$

(vii) By (2.13) if $0 < t/2 \leqslant s \leqslant t \leqslant T$ we have for each $\xi > 0$ similarly to (2.11):

$$\xi^{1-\beta}A(s)\exp[\xi A(s)][L(f,x)(s) + f(s)] =$$

$$= -\xi^{1-\beta}\int_0^{s/2}\int_{s-\sigma}^{s} A(s)^3\exp[(\xi+\eta)A(s)]f(s)\,d\eta\,ds +$$

$$+ \xi^{1-\beta}A(s)^2\exp[(\xi+s)A(s)]\int_0^{s/2} f(\sigma)\,d\sigma +$$

$$+ \xi^{1-\beta}\int_{s/2}^{s} A(s)^2\exp[(\xi+s-\sigma)A(s)][f(\sigma)-f(s)]\,d\sigma +$$

$$+ \xi^{1-\beta}A(s)\exp[(\xi+s/2)A(s)]f(s) + \xi^{1-\beta}A(s)^2\exp[(\xi+s)A(s)]x;$$

hence Lemma 1.10 (ii) yields

$$[L(f, x)(s) + f(s)]_{D_{A(s)}(\beta,\infty)} \leqslant \frac{c}{t^{\mu+\beta}} \{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + [f]_{*,E}\} .$$

This, together with (2.14), shows that

$$\|L(f, x) + f\|_{Z_\mu(D_A(\beta,\infty))} \leqslant c\{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + [f]_{*,E}\}; \tag{2.19}$$

recalling (2.17), we deduce that

$$\|L(f, x) + f\|_{Z^*_\mu(D_A(\beta,\infty))} \leqslant c\{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + \|f\|_{L^1(E)}\} . \tag{2.20}$$

The proof of Proposition 2.1 is complete. ■

PROPOSITION 2.2. *Fix* $\beta \in]0, \delta]$. *If* $x \in D_{A(0)}$, $f \in B(D_A(\beta, \infty))$, *then*:

(i) $L(f, x) \in C^\beta(E)$ *if and only if* $A(0)x \in D_{A(0)}(\beta, \infty)$;

(ii) $L(f, x) \in B(D_A(\beta, \infty))$ *if and only if* $A(0)x \in D_{A(0)}(\beta, \infty)$.

If $x \in D_{A(0)}$, $f \in C(E) \cap Z_0(D_A(\beta, \infty))$, *then*:

(iii) $L(f, x) \in C(E)$ *if and only if* $A(0)x + f(0) \in \overline{D_{A(0)}}$.

Fix also $\mu \in [0, 1[$. *If* $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_\mu(D_A(\beta, \infty))$, *then*

(iv) $L(f, x) \in Z_{\mu,\beta}(E)$;

(v) $L(f, x) \in Z_\mu(D_A(\beta, \infty))$.

Finally fix $\mu \in [1, 1+\beta[$. *If* $x \in \overline{D_{A(0)}}$, $f \in Z_\mu(D_A(\beta, \infty)) \cap L^1(E)$, *then*

(vi) $L(f, x) \in Z^*_{\mu,\beta}(E)$;

(vii) $L(f, x) \in Z^*_\mu(D_A(\beta, \infty))$.

PROOF. It is quite analogous to the proof of Proposition 2.1: the required splittings are just slightly different from the corresponding ones in that proof, and again only Lemmata 1.10 and 1.8 have to be applied. Thus we omit the details, writing down only the precise inequalities that can be obtained for each statement:

$$\|L(f, x)\|_{C^\beta(E)} + \|L(f, x)\|_{B(D_A(\beta,\infty))} \leqslant \tag{2.21}$$

$$\leqslant c\{\|f\|_{B(D_A(\beta,\infty))} + \|x\|_{D_{A(0)}} + \|A(0)x\|_{D_{A(0)}(\beta,\infty)}\} \qquad \text{(cases (i)-(ii))};$$

$$\|L(f, x)\|_{C(E)} \leqslant c\{\|f\|_{Z_0(D_A(\beta,\infty))} + \|x\|_{D_{A(0)}}\} \qquad \text{(case (iii))}; \tag{2.22}$$

$$\|L(f, x)\|_{Z_{\mu,\beta}(E)} + \|L(f, x)\|_{Z_\mu(D_A(\beta,\infty))} \leqslant \leqslant c\{\|f\|_{Z_\mu(D_A(\beta,\infty))} + \|x\|_{D_{A(0)}(1-\mu,\infty)}\} \qquad \text{(cases (iv)-(v))}; \tag{2.23}$$

$$\|L(f, x)\|_{Z_{\mu,\beta}(E)} + \|L(f, x)\|_{Z_\mu(D_A(\beta,\infty))} \leqslant \leqslant c\{\|f\|_{Z_\mu(D_A(\beta,\infty))} + [f]_{*,E} + \|x\|_E\}\,, \tag{2.24}$$

$$\|L(f, x)\|_{Z^*_{\mu,\beta}(E)} + \|L(f, x)\|_{Z^*_\mu(D_A(\beta,\infty))} \leqslant \leqslant c\{\|f\|_{Z_\mu(D_A(\beta,\infty))} + \|f\|_{L^1(E)} + \|x\|_E\} \qquad \text{(cases (vi)-(vii))}\,. \tag{2.25}$$

■

(b) The operator Q.

We start with the following lemma, concerning the kernel $Q(t, s)$ defined in (1.19).

LEMMA 2.3. *We have*

(i) $$\|Q(t, s)\|_{\mathfrak{L}(E)} \leqslant c \sum_{i=1}^{k} \frac{1}{(t-s)^{1+\beta_i-\alpha_i}} \quad \forall 0 \leqslant s < t \leqslant T;$$

(ii) $$\|Q(t, s) - Q(\sigma, s)\|_{\mathfrak{L}(E)} \leqslant c \sum_{i=1}^{k} \left[\frac{(t-\sigma)^{\alpha_i}}{(t-s)^{1+\beta_i}} + \int_{\sigma-s}^{t-s} \frac{(\sigma-s)^{\alpha_i}}{\xi^{2+\beta_i}}\, d\xi\right] \quad \forall 0 \leqslant s < \sigma \leqslant t \leqslant T;$$

(iii) $$\|Q(t, s) - Q(t, r)\|_{\mathfrak{L}(E)} \leqslant c \sum_{i=1}^{k} \left[\frac{(s-r)^{\alpha_i}}{(t-r)^{1+\beta_i}} + \int_{t-s}^{t-r} \frac{(t-s)^{\alpha_i}}{\xi^{2+\beta_i}}\, d\xi\right] \quad \forall 0 \leqslant r \leqslant s < t \leqslant T;$$

(iv) $$\left\|\int_a^\sigma [Q(t, s) - Q(\sigma, s)]y\, ds\right\|_E \leqslant c(t-\sigma)^\delta \|y\|_E \quad \forall 0 \leqslant a \leqslant \sigma \leqslant t \leqslant T\,,\ \forall y \in E;$$

(v) $$\|Q(t, s) - Q(\sigma, s) - Q(t, r) + Q(\sigma, r)\|_{\mathfrak{L}(E)} \leqslant c\, \frac{(t-\sigma)^\delta (s-r)^\delta}{(\sigma-s)^{1+\delta}} \quad \forall 0 \leqslant r \leqslant s < \sigma \leqslant t \leqslant T\,.$$

PROOF. (i) Trivial consequence of Lemma 1.11 (i).

(ii) By (1.19) we have

$$(2.26)\qquad Q(t,s)-Q(\sigma,s)=A(t)^2\exp[(t-s)A(t)][A(t)^{-1}-A(\sigma)^{-1}]+$$
$$+[A(t)^2\exp[(t-s)A(t)]-A(\sigma)^2\exp[(t-s)A(\sigma)]][A(\sigma)^{-1}-A(s)^{-1}]+$$
$$+\int_{\sigma-s}^{t-s}A(\sigma)^3\exp[\xi A(\sigma)][A(\sigma)^{-1}-A(s)^{-1}]\,d\xi\,,$$

and the result follows easily by Lemma 1.11 (i)-(ii).

(iii) Similar to (ii) (with the « dual » splitting and using again Lemma 1.11 (i)-(ii)).

(iv) It follows by integrating (2.26) over $]a,\sigma[$ with the use of Lemma 1.11 (i)-(ii).

(v) By (2.26) we can write:

$$Q(t,s)-Q(\sigma,s)-Q(t,r)+Q(\sigma,r)=$$
$$=-\int_{t-s}^{t-r}A(t)^3\exp[\xi A(t)][A(t)^{-1}-A(\sigma)^{-1}]\,d\xi-$$
$$-\int_{t-s}^{t-r}[A(t)^3\exp[\xi A(t)]-A(\sigma)^3\exp[\xi A(\sigma)]]\,[A(\sigma)^{-1}-A(s)^{-1}]\,d\xi-$$
$$-[A(t)^2\exp[(t-r)A(t)]-A(\sigma)^2\exp[(t-r)A(\sigma)]]\,[A(s)^{-1}-A(r)^{-1}]+$$
$$+\Big[\int_{\sigma-s}^{t-s}-\int_{\sigma-r}^{t-r}\Big]A(\sigma)^3\exp[\xi A(\sigma)][A(\sigma)^{-1}-A(s)^{-1}]\,d\xi-$$
$$-\int_{\sigma-r}^{t-r}A(\sigma)^3\exp[\xi A(\sigma)][A(s)^{-1}-A(r)^{-1}]\,d\xi=\sum_{i=1}^{6}I_i\,.$$

The sum I_4+I_5 can be estimated by Lemma 1.11 (i) as follows:

$$\|I_4+I_5\|_{\mathfrak{L}(E)}=\Big\|\int_s^r\frac{d}{d\eta}\Big[\int_{\sigma-\eta}^{t-\eta}A(\sigma)^3\exp[\xi A(\sigma)][A(\sigma)^{-1}-A(s)^{-1}]\,d\xi\Big]\,d\eta\Big\|_{\mathfrak{L}(E)}=$$
$$=\Big\|\int_s^r\int_{\sigma-\eta}^{t-\eta}A(\sigma)^4\exp[\xi A(\sigma)][A(\sigma)^{-1}-A(s)^{-1}]\,d\xi\,d\eta\Big\|_{\mathfrak{L}(E)}\leqslant$$

$$\leqslant c\int_r^s\int_{\sigma-\eta}^{t-\eta}(\sigma-s)^\delta\frac{d\xi}{\xi^3}\,d\eta\leqslant c(\sigma-s)^\delta\left[\frac{1}{\sigma-s}-\frac{1}{\sigma-r}-\frac{1}{t-s}+\frac{1}{t-r}\right]=$$

$$=c(\sigma-s)^\delta\frac{(s-r)(t-\sigma)(t+\sigma-s-r)}{(\sigma-s)(\sigma-r)(t-s)(t-r)}\leqslant c\,\frac{(t-\sigma)^\delta(s-r)^\delta}{(\sigma-s)^{1+\delta}};$$

the other terms are easily estimated by Lemma 1.11 (i)-(ii) and the result follows. ■

PROPOSITION 2.4. *We have*:

(i) $Q\in\mathfrak{L}(B(E),\,C^\varepsilon(E))\ \forall\varepsilon\in\,]0,\,\delta[\,;$

(ii) $Q\in\mathfrak{L}(C^\varepsilon(E),\,C^\delta(E))\ \forall\varepsilon\in\,]0,\,1[\,;$

(iii) $Q\in\mathfrak{L}(B_\mu(E),\,Z_{\mu-\delta,\varepsilon}(E))\ \forall\mu\in[0,\,1[,\ \forall\varepsilon\in\,]0,\,\delta[\,;$

(iv) $Q\in\mathfrak{L}(Z_{\mu,\varepsilon}(E),\,Z_{\mu-\delta,\delta}(E))\ \forall\mu\in[0,\,1[,\ \forall\varepsilon\in\,]0,\,1[\,;$

(v) $Q\in\mathfrak{L}\big(B_\mu(E),\ Z_{\mu-\delta}(D_A(\delta,\,\infty))\big)\ \forall\mu\in[0,\,1[.$

(vi) $Q\in\mathfrak{L}(I_\mu(E),\,Z_{\mu-\delta,\varepsilon}(E))\ \forall\mu\in[1,\,1+\delta[,\ \forall\varepsilon\in\,]0,\,\delta[\,;$

(vii) $Q\in\mathfrak{L}(Z^*_{\mu,\varepsilon}(E),\,Z_{\mu-\delta,\delta}(E))\ \forall\mu\in[1,\,1+\delta[,\ \forall\varepsilon\in\,]0,\,1[\,;$

(viii) $Q\in\mathfrak{L}\big(I_\mu(E),\ Z_{\mu-\delta}(D_A(\delta,\,\infty))\big)\ \forall\mu\in[1,\,1+\delta[.$

PROOF. (i) Let $g\in B(E)$, If $0\leqslant\sigma\leqslant t\leqslant T$ we have:

$$Qg(t)-Qg(\sigma)=\tag{2.27}$$

$$=\int_\sigma^t Q(t,s)g(s)\,ds+\int_0^\sigma[Q(t,s)-Q(\sigma,s)]g(s)\,ds=I_1+I_2\,,$$

and consequently by Lemma 2.3 (i)-(ii) we easily get

$$\|Qg(t)-Qg(\sigma)\|_E\leqslant c(t-\sigma)^\delta\left[1+\log\left(1+\frac{\sigma}{t-\sigma}\right)\right]\|g\|_{B(E)}\,,$$

which clearly implies (since $Qg(0)=0$)

$$\|Qg\|_{C^\varepsilon(E)}\leqslant c(\varepsilon)\|g\|_{B(E)}\qquad\forall\varepsilon\in\,]0,\,\delta[\,.\tag{2.28}$$

(ii) Let $g \in C^\varepsilon(E)$. If in (2.28) we replace I_2 by

$$\int_0^\sigma [Q(t, s) - Q(\sigma, s)][g(s) - g(\sigma)]\, ds + \int_0^\sigma [Q(t, s) - Q(\sigma, s)] g(\sigma)\, ds\,,$$

then Lemma 2.3 (i)-(ii)-(iv) yields

$$\|Qg\|_{C^\varepsilon(E)} \leqslant c \|g\|_{C^\varepsilon(E)}\,. \tag{2.29}$$

(iii) Let $g \in B_\mu(E)$. Assume first $\mu \in [0, \delta[$: then by (2.27) and Lemma 2.3 (i)-(ii) we have

$$\|Qg\|_{C^{[\mu-\delta]}(E)} \leqslant c \|g\|_{B_\mu(E)}; \tag{2.30}$$

on the other hand if $\mu \in [\delta, 1[$ by Lemma 2.3 (i) we readily obtain

$$\|Qg\|_{B_{\mu-\delta}(E)} \leqslant c \|g\|_{B_\mu(E)}\,. \tag{2.31}$$

Next, if $0 < t/2 \leqslant r \leqslant \sigma \leqslant t \leqslant T$ we have as in (2.27)

$$\begin{aligned} Qg(\sigma) - Qg(r) &= \int_r^\sigma Q(\sigma, s) g(s)\, ds + \\ &+ \left[\int_0^{r/2} + \int_{r/2}^r\right] [Q(\sigma, s) - Q(r, s)] g(s)\, ds = J_1 + J_2 + J_3\,, \end{aligned} \tag{2.32}$$

hence by Lemma 2.3 (i)-(ii) it is easy to deduce

$$\|Qg(\sigma) - Qg(r)\|_E \leqslant \frac{c}{t^\mu} (\sigma - r)^\delta \left[1 + \log\left(1 + \frac{t}{\sigma - r}\right)\right] \|g\|_{B_\mu(E)}\,,$$

which, together with (2.30) or (2.31), clearly implies

$$\|Qg\|_{Z_{\mu-\delta,\varepsilon}(E)} \leqslant c(\varepsilon) \|g\|_{B_\mu(E)} \qquad \forall \varepsilon \in]0, \delta[\,. \tag{2.33}$$

(iv) Let $g \in Z_{\mu,\varepsilon}(E)$. Replace in (2.32) I_3 by

$$\int_{r/2}^r [Q(\sigma, s) - Q(r, s)][g(s) - g(r)]\, ds + \int_{r/2}^r [Q(\sigma, s) - Q(r, s)] g(r)\, ds;$$

then by Lemma 2.3 (i)-(ii)-(iv) and recalling (2.30) or (2.31) we conclude that

$$\|Qg\|_{Z_{\mu-\delta,\,\delta}(E)} \leqslant c\|g\|_{Z_{\mu,\delta}(E)}\,. \tag{2.34}$$

(v) Let $g \in B_\mu(E)$. Assume first $\mu \in [0, \delta[$: then Lemma 1.11 (i) easily leads to

$$\|Qg\|_{B(D_A(|\mu-\delta|,\infty))} \leqslant c\|g\|_{B_\mu(E)}; \tag{2.35}$$

on the other hand if $\mu \in [\delta, 1[$ as in (iii) we have (2.31). Next, if $0 < t/2 \leqslant \sigma \leqslant t \leqslant T$ we have for each $\xi > 0$:

$$\xi^{1-\delta}A(\sigma)\exp[\xi A(\sigma)]Qg(\sigma) = \tag{2.36}$$

$$= \xi^{1-\delta}\Big[\int_0^{\sigma/2} + \int_{\sigma/2}^{\sigma}\Big] A(\sigma)^3 \exp[(\xi + \sigma - s)A(\sigma)][A(\sigma)^{-1} - A(s)^{-1}]g(s)\,ds\,,$$

so that by Lemma 1.11 (i)

$$[Qg(\sigma)]_{D_A(\sigma)(\delta,\infty)} \leqslant$$

$$\leqslant c \sup_{\xi>0}\left\{\frac{\xi^{1-\delta}t^{1-\mu}}{(\xi + t/4)^{2-\delta}} + \frac{\xi^{1-\delta}}{(t/4)^\mu}\left[\frac{1}{\xi^{1-\delta}} - \frac{1}{(\xi + \sigma/2)^{1-\delta}}\right]\right\}\cdot \|g\|_{B_\mu(E)} \leqslant \frac{c}{t^\mu}\|g\|_{B_\mu(E)};$$

this, together with (2.35) or (2.31), shows that

$$\|Qg\|_{Z_{\mu-\delta}(D_A(\delta,\infty))} \leqslant c\|g\|_{B_\mu(E)}\,. \tag{2.37}$$

(vi) Let $g \in I_\mu(E)$. We have

$$Qg(t) = \int_0^t [Q(t, s) - Q(t, 0)]g(s)\,ds + Q(t, 0)\int_0^t g(s)\,ds\,,$$

and by Lemma 2.3 (i)-(iii) we get

$$\|Qg\|_{B_{\mu-\delta}(E)} \leqslant c\|g\|_{I_\mu(E)}\,. \tag{2.38}$$

Next, if $0 < t/2 \leqslant r \leqslant \sigma \leqslant t \leqslant T$ we replace in (2.32) I_2 by

$$(2.39) \qquad \int_0^{r/2} [Q(\sigma, s) - Q(r, s) - Q(\sigma, 0) + Q(r, 0)] g(s)\, ds + \\ + [Q(\sigma, 0) - Q(r, 0)] \int_0^{r/2} g(s)\, ds ,$$

so that by Lemma 2.3 (i)-(ii)-(v) we find

$$\|Qg(\sigma) - Qg(t)\|_E \leqslant \frac{c}{t^\mu} (\sigma - r)^\delta \left[1 + \log\left(1 + \frac{t}{\sigma - r}\right)\right] \|g\|_{B_\mu(E)};$$

this, recalling (2.38), implies

$$(2.40) \qquad \|Qg\|_{B_{\mu-\delta,\varepsilon}(E)} \leqslant c(\varepsilon) \|g\|_{I_\mu(E)} \qquad \forall \varepsilon \in]0, \delta[.$$

(vii) Let $g \in Z^*_{\mu,\varepsilon}(E)$. As in the proof of (vi) we have (2.38). Next, if $0 < t/2 \leqslant r \leqslant \sigma \leqslant t \leqslant T$ we replace in (2.32) I_2 by (2.39) and I_3 by

$$\int_{r/2}^{r} [Q(\sigma, s) - Q(r, s)] [g(s) - g(r)]\, ds + \int_{r/2}^{r} [Q(\sigma, s) - Q(r, s)] g(r)\, ds .$$

Now I_1 and the terms in (2.39) can be estimated as in the proof of (vi), whereas the terms replacing I_3 can be estimated by Lemma 2.3 (ii)-(iv), obtaining

$$\|Qg(\sigma) - Qg(r)\|_E \leqslant \frac{c}{t^\mu} (\sigma - r)^\delta \|g\|_{Z_{\mu,\varepsilon}(E)} .$$

Thus by (2.38) we conclude that

$$(2.41) \qquad \|Qg\|_{Z_{\mu-\delta,\delta}(E)} \leqslant c \|g\|_{Z^*_{\mu,\varepsilon}(E)} .$$

(viii) Let $g \in I_\mu(E)$. As in the proof of (vi) we have (2.38).

Moreover for $0 < t/2 \leqslant \sigma \leqslant t \leqslant T$ and for each $\xi > 0$ we have, starting from (2.36):

$$\xi^{1-\delta} A(\sigma) \exp[\xi A(\sigma)] Qg(\sigma) =$$

$$= -\xi^{1-\delta} \int_0^{\sigma/2} \int_{\sigma-s}^{\sigma} A(\sigma)^4 \exp[(\xi+\eta) A(\sigma)] [A(\sigma)^{-1} - A(s)^{-1}] g(s)\, d\eta\, ds -$$

$$- \xi^{1-\delta} \int_0^{\sigma/2} [A(\sigma)^3 \exp[(\xi+\sigma) A(\sigma)] - A(s)^3 \exp[(\xi+\sigma) A(s)]] \cdot$$

$$\cdot [A(s)^{-1} - A(0)^{-1}] g(s)\, ds -$$

$$- \xi^{1-\delta} \int_0^{\sigma/2} A(s)^3 \exp[(\xi+\sigma) A(s)] [A(s)^{-1} - A(0)^{-1}] g(s)\, ds +$$

$$+ \xi^{1-\delta} A(\sigma)^3 \exp[(\xi+\sigma) A(\sigma)] [A(\sigma)^{-1} - A(0)^{-1}] \int_0^{\sigma/2} g(s)\, ds +$$

$$+ \xi^{1-\delta} \int_{\sigma/2}^{\sigma} A(\sigma)^3 \exp[(\xi+\sigma-s) A(\sigma)] [A(\sigma)^{-1} - A(s)^{-1}] g(s)\, ds\,.$$

Now by Lemma 1.11 (i)-(ii) we easily get

$$[Qg(\sigma)]_{D_A(\sigma)(\delta,\infty)} \leqslant \frac{c}{t^\mu} \|g\|_{I_\mu(E)}\,,$$

hence recalling (2.38) we find

$$\|Qg\|_{Z_{\mu-\delta}(D_A(\delta,\infty))} \leqslant c \|g\|_{I_\mu(E)}\,. \tag{2.42}$$

The proof of Proposition 2.4 is complete. ■

(*c*) *The operator* $(1-Q)^{-1}$.

We need the following elementary lemma:

LEMMA 2.5. *Let* $\alpha, \beta, \gamma, \varrho \in [0, 1[$ *with* $\gamma < \alpha$ *and* $\gamma + 1 - \alpha - \beta - \varrho > 0$. *Then*:

(i) $\sup_{s>} [s^\varrho \exp[-sr]] = \varrho^\varrho r^{-\varrho} \exp[-\varrho] \quad \forall r > 0;$

(ii) $\lim_{\omega \to +\infty} \sup_{t \in]0,T]} \omega^\varrho t^\gamma \int_0^t \frac{\exp[-\omega(t-s)]}{(t-s)^\beta s^\alpha} ds = 0 .$

PROOF. Tedious but easy. ■

PROPOSITION 2.6. *We have*:

(i) $(1-Q)^{-1} \in \mathfrak{L}(B_\mu(E)) \ \forall \mu \in [0, 1[;$

(ii) $(1-Q)^{-1} \in \mathfrak{L}(I_\mu(E)) \ \forall \mu \in [1, 1+\delta[;$

(iii) $(1-Q)^{-1} \in \mathfrak{L}(X)$ *where* X *is any of the following spaces*:

(*a*) $C^\beta(E)$, (*b*) $C(E)$, (*c*) $C_\mu(E)$, (*d*) $Z_{\mu,\beta}(E)$, (*e*) $Z_\mu(D_A(\beta, \infty))$,

where $\beta \in]0, \delta]$, $\mu \in [0, 1[$, *and*

(*f*) $C_\mu(E) \cap I_\mu(E)$, (*g*) $Z^*_{\mu,\beta}(E)$, (*h*) $Z^*_\mu(D_A(\beta, \infty))$

where $\beta \in]0, \delta]$, $\mu \in [1, 1+\delta[$.

PROOF. (i) For each $\omega > 0$ introduce in $B_\mu(E)$ the following norm:

$$\|g\|_{\omega, B_\mu(E)} = \sup_{t \in]0,T]} t^\mu \exp[-\omega t] \|g(t)\|_E , \tag{2.43}$$

which is obviously equivalent to the usual one. We will show that for sufficiently large ω

$$\|Qg\|_{\omega, B_\mu(E)} \leqslant \tfrac{1}{2} \|g\|_{\omega, B_\mu(E)}; \tag{2.44}$$

this will prove that $(1-Q)^{-1} = \sum_{n=0}^{\infty} Q^n$ exists in $\mathfrak{L}(B_\mu(E))$. Now (2.44) follows easily by Lemma 2.3 (i) and Lemma 2.5 (ii) (with $\varrho = 0$, $\alpha = \gamma = \mu$, $\beta = 1 - \delta$).

(ii) Set for each $\omega > 0$

$$\|g\|_{\omega, I_\mu(E)} = \|g\|_{\omega, B_\mu(E)} + [g]_{\omega,*,E},$$

where $\|\cdot\|_{\omega, B_\mu(E)}$ is defined in (2.43) and

$$[g]_{\omega,*,E} = \sup\left\{\left\|\int_a^b \exp[-\omega s] g(s)\, ds\right\|_E : 0 < a \leqslant b \leqslant T\right\}.$$

Again we have to show that for large ω

$$\|Qg\|_{\omega, I_\mu(E)} \leqslant \tfrac{1}{2}\|g\|_{\omega, I_\mu(E)}. \tag{2.45}$$

Now if $0 < t \leqslant T$ we have

$$\begin{aligned} \exp[-\omega t] Qg(t) &= \\ &= \int_0^t \exp[-\omega(t-s)][Q(t,s) - Q(t,0)] \exp[-\omega s] g(s)\, ds + \\ &+ \int_0^t [\exp[-\omega(t-s)] - \exp[-\omega t]] Q(t,0) \exp[-\omega s] g(s)\, ds + \\ &+ \exp[-\omega t] Q(t,0) \int_0^t \exp[-\omega s] g(s)\, ds; \end{aligned} \tag{2.46}$$

hence by Lemma 2.3 (i)-(iii) we check

$$\begin{aligned} & t^\mu \exp[-\omega t] \|Qg(t)\|_E \leqslant \\ & \leqslant c\left\{t^{\mu-1}\int_0^t \frac{\exp[-\omega(t-s)]}{s^{\mu-\delta}}\, ds + t^{\mu-\delta}\omega^\delta \int_0^t \frac{\exp[-\omega(t-s)]}{s^{\mu-\delta}}\, ds\right\} \|g\|_{\omega, B_\mu(E)} + \\ & + c t^{\mu-\delta} \exp[-\omega t] [g]_{\omega,*,E}, \end{aligned}$$

so that by Lemma 2.5 (i)-(ii) we deduce for large ω

$$\|Qg\|_{\omega, B_\mu(E)} \leqslant \tfrac{1}{4}\|g\|_{\omega, I_\mu(E)}. \tag{2.47}$$

Next, if $0 < a \leqslant b \leqslant T$ we have by (2.46) and Lemma 2.3 (i)-(iii):

$$\left\| \int_a^b \exp[-\omega t] Qg(t)\, dt \right\|_E \leqslant c \left\{ \int_0^T \int_0^t \exp[-\omega(t-s)] \frac{ds\, dt}{s^{\mu-\delta} t} + \right.$$

$$\left. + \int_0^T \int_0^t \exp[-\omega(t-s)] \omega^\delta \frac{ds\, dt}{s^{\mu-\delta} t^{1-\delta}} \right\} \|g\|_{\omega, B_\mu(E)} +$$

$$+ c \int_0^T \frac{\exp[-\omega t]\, dt}{t^{1-\delta}} [g]_{\omega,*,E},$$

which easily implies for large ω

$$[Qg]_{\omega,*,E} \leqslant \tfrac{1}{4} \|g\|_{\omega, I_\mu(E)}; \tag{2.48}$$

by (2.47) and (2.48) we obtain (2.45).

(iii) This is a simple corollary of (i), (ii) and the statements of Proposition 2.4. Indeed, let $h = (1-Q)^{-1} g$; then, by definition h solves the equation

$$h = g - Qh. \tag{2.49}$$

Thus to prove (*a*) and (*b*) we have the following chains of implications (the last of which is due to (2.49)):

$$\left.\begin{array}{l} g \in C^\beta(E),\ \beta \in]0, \delta[\\ g \in C(E) \end{array}\right\} \Rightarrow g \in B(E) \Rightarrow h \in B(E) \Rightarrow$$

$$\Rightarrow Qh \in C^\varepsilon(E) \qquad \forall \varepsilon \in]0, \delta[\Rightarrow \left\{ \begin{array}{l} h \in C^\beta(E) \\ h \in C(E); \end{array} \right.$$

$$g \in C^\delta(E) \Rightarrow g \in C^{\delta/2}(E) \Rightarrow h \in C^{\delta/2}(E) \Rightarrow Qg \in C^\delta(E) \Rightarrow h \in C^\delta(E).$$

The proof of the remaining statements of (iii) is analogous and can be omitted. This completes the proof of Proposition 2.6. ■

3. The representation formula. Uniqueness.

The technical results of the preceding section allow us to give sense to the heuristic argument used at the end of Section 1, and to introduce the representation formula for strict and classical solutions of (0.1).

As a consequence we will obtain some uniqueness results for such solutions. To begin with, set (just formally)

$$w = (1-Q)^{-1}L(f, x)\,, \tag{3.1}$$

where $x \in E$ and $f\colon]0, T] \to E$ are prescribed data and Q, $L(f, x)$ are defined by (1.21)-(1.19) and (1.20). The following result summarizes the cases in which the function w is well defined.

PROPOSITION 3.1. *Let δ be defined by* (1.8). *If $\beta \in]0, \delta]$, we have:*

(i) *if $x \in D_{A(0)}$, $f \in C^\beta(E)$, then $w \in C^\beta(E)$ if and only if $A(0)x + f(0) \in D_{A(0)}(\beta, \infty)$;*

(ii) *if $x \in D_{A(0)}$, $f \in C(E) \cap Z_{0,\beta}(E)$, then $w \in C(E)$ if and only if $A(0)x + f(0) \in \overline{D_{A(0)}}$;*

(iii) *if $x \in D_{A(0)}$, $f \in B(D_A(\beta, \infty))$, then $w \in C^\beta(E) \cap B(D_A(\beta, \infty))$ if and only if $A(0)x \in D_{A(0)}(\beta, \infty)$;*

(iv) *if $x \in D_{A(0)}$, $f \in C(E) \cap Z_0(D_A(\beta, \infty))$, then $w \in C(E)$ if and only if $A(0)x + f(0) \in \overline{D_{A(0)}}$;*

Further, if $\beta \in]0, \delta]$ and $\mu \in [0, 1[$, we have:

(v) *if $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_{\mu,\beta}(E)$, then $w \in Z_{\mu,\beta}(E)$;*

(vi) *if $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_\mu(D_A(\beta, \infty))$, then $w \in Z_{\mu,\beta}(E) \cap Z_\mu(D_A(\beta, \infty))$.*

Finally, if $\beta \in]0, \delta]$ and $\mu \in [1, 1+\beta[$, we have:

(vii) *if $x \in \overline{D_{A(0)}}$, $f \in Z_{\mu,\beta}(E) \cap L^1(E)$, then $w \in Z^*_{\mu,\beta}(E)$;*

(viii) *if $x \in \overline{D_{A(0)}}$, $f \in Z_\mu(D_A(\beta, \infty)) \cap L^1(E)$, then $w \in Z^*_{\mu,\beta}(E) \cap Z^*_\mu(D_A(\beta, \infty))$.*

PROOF. It is a straightforward consequence of (3.1) and Propositions 2.1, 2.2 and 2.4. ■

We need the following property of classical solutions:

LEMMA 3.2. *Let $x \in E$, $f \in C_+(E)$ and let u be a classical solution of* (0.1). *If f satisfies in addition*

$$\exists E - \lim_{a \to 0^+} \int_a^T f(s)\, ds = y$$

then Au has the same property.

PROOF. We have for each $a \in]0, T]$

$$u(T) - u(a) = \int_a^T u'(s)\,ds = \int_a^T A(s)u(s)\,ds + \int_a^T f(s)\,ds;$$

as $u \in C(E)$ and $u(0) = x$ we get

$$E - \lim_{a \to 0} \int_a^T A(s)u(s)\,ds = u(T) - x - y\,. \qquad \blacksquare$$

Let us make precise the heuristic argument given at the end of Section 1.

THEOREM 3.3. *Let* $\beta \in]0, \delta]$, $\mu \in [1, 1+\beta[$, $\lambda \in [1, 1+\delta[$; *fix* $x \in \overline{D_{A(0)}}$, $f \in Z_{\mu,\beta}(E) \cup Z_\mu(D_A(\beta, \infty))$ *and moreover suppose* $f \in L^1(E)$. *If* u *is a classical solution of* (0.1) *such that* $u \in I_\lambda(D_A)$, *then we have* $Au = w$ *with* w *defined by* (3.1), *i.e. the representation formula* (1.22) *holds.*

PROOF. Fix $t \in]0, T]$ and set $v(s) = \exp[(t-s)A(t)]u(s)$, $s \in [0, t]$. Write down $v'(s)$, pick $\varepsilon \in]0, t[$ and integrate $v'(s)$ in $[\varepsilon, t]$: it is easy to see that

$$\begin{aligned} u(t) - \exp[(t-\varepsilon)A(t)] &= \\ &= \int_\varepsilon^t A(t)\exp[(t-s)A(t)][A(t)^{-1} - A(s)^{-1}]A(s)u(s)\,ds + \\ &\qquad + \int_\varepsilon^t \exp[(t-s)A(t)]f(s)\,ds\,, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} u(t) = A(t)^{-1}\int_\varepsilon^t [Q(t,s) - Q(t,0)]A(s)u(s)\,ds + \\ + A(t)^{-1}Q(t,0)\int_\varepsilon^t A(s)u(s)\,ds + \int_\varepsilon^t \exp[(t-s)A(t)]f(s)\,ds\,. \end{aligned}$$

Since $Au \in I_\lambda(E)$ and $f \in L^1(E)$, by Lemmata 3.2 and 2.3 (i)-(iii) as $\varepsilon \to 0^+$ we deduce that

$$u(t) = A(t)^{-1}[Q(Au)](t) + \int_0^t \exp[(t-s)A(t)]f(s)\,ds;$$

finally, operating with $A(t)$, we get $Au = Q(Au) + L(f, x)$, and the result follows. ■

COROLLARY 3.4. *Let $\beta \in]0, \delta]$, $\mu \in [1, 1+\beta[$; fix $x \in D_{A(0)}$, $f \in C(E)$. If u is a strict solution of* (0.1), *then we have $Au = w$ with w defined by* (3.1). ■

THEOREM 3.5. *For each $x \in E$ and $f \in C_+(E)$, the classical solution of* (0.1) *in the class $\bigcup_{\mu \in [0,1+\delta[} I_\mu(D_A)$ is unique.*

PROOF. Trivial consequence of Theorem 3.3. ■

COROLLARY 3.6. *For each $x \in D_{A(0)}$ and $f \in C(E)$ the strict solution of* (0.1) *is unique.* ■

Concerning existence of strict and classical solutions, a necessary condition is given by:

PROPOSITION 3.7. (i) *If a strict solution u of* (0.1) *exists, then $f \in C(E)$, $x \in D_{A(0)}$ and $A(0)x + f(0) \in \overline{D_{A(0)}}$.*

(ii) *If a classical solution u of* (0.1) *exists in the class $\bigcup_{\mu \in [0,1+\delta[} B_\mu(D_A)$, then $x \in \overline{D_{A(0)}}$.*

PROOF. – (i) Obviously $f \in C(E)$ and $x \in D_{A(0)}$: Next, clearly we have

$$A(0)x + f(0) = u'(0) = E - \lim_{t \to 0^+} \frac{u(t) - x}{t}; \tag{3.2}$$

on the other hand we can write

$$\frac{u(t)-x}{t} = \frac{1}{t}\left[R\left(\frac{1}{t}, A(t)\right) - R\left(\frac{1}{t}, A(0)\right)\right]\frac{u(t)-x}{t} + \tag{3.3}$$

$$+ \frac{1}{t} R\left(\frac{1}{t}, A(0)\right)\left[\frac{u(t)-x}{t} - [A(0)x + f(0)]\right] +$$

$$+\frac{1}{t}R\left(\frac{1}{t}, A(0)\right)[A(0)x+f(0)]-\frac{1}{t}R\left(\frac{1}{t}, A(t)\right)[A(t)u(t)-A(0)x]-$$
$$-\frac{1}{t}A(t)R\left(\frac{1}{t}, A(t)\right)[A(t)^{-1}-A(0)^{-1}]A(0)x=\sum_{i=1}^{5}I_i\,.$$

Moreover by Lemma 1.9 (ii) and Hypothesis II we get

$$\|I_1\|_E+\|I_2\|_E+\|I_4\|_E+\|I_5\|_E=o(1) \quad \text{as } t\to 0^+;$$

hence by (3.2) and (3.3)

$$A(0)x+f(0)=E-\lim_{t\to 0^+}I_3=E-\lim_{t\to 0^+}\frac{1}{t}R\left(\frac{1}{t}, A(0)\right)[A(0)x+f(0)]\,,$$

which evidently yields the result.

(ii) Let $\mu\in[0, 1+\delta[$ be such that $Au\in B_\mu(E)$. Write

$$u(t)=\left[R\left(\frac{1}{t}, A(t)\right)-R\left(\frac{1}{t}, A(0)\right)\right]\left[\frac{1}{t}-A(t)\right]u(t)+$$
$$+R\left(\frac{1}{t}, A(0)\right)\left[\frac{1}{t}-A(t)\right]u(t)=J_1+J_2;$$

obviously $u(t)-J_1=J_2\in D_{A(0)}$: Now by Lemma 1.9 (ii) we get

$$\|J_1\|_E\leqslant ct^{1+\delta}\left\{\frac{1}{t}\|u\|_{C(E)}+\frac{1}{t^\mu}\|Au\|_{B_\mu(E)}\right\}=o(1) \quad \text{as } t\to 0^+\,,$$

so that $x=E-\lim_{t\to 0^+}[u(t)-J_1]\in\overline{D_{A(0)}}$. ■

We will see in Section 6 that the compatibility conditions of the preceding proposition are also sufficient for existence of strict, or classical, solutions of (0.1), provided the data x, f are slightly more regular: in fact the function $u=A^{-1}w$, with w defined by (3.1), will turn out to solve Problem (0.1). We will obtain u as the limit as $n\to\infty$, in suitable norms, of functions u_n solving certain problems which in some sense approach Problem (0.1) as $n\to\infty$. Such problems have the same form as (0.1), with $A(t)$ replaced by the bounded operator $nA(t)R(n, A(t))$ (its Yosida approximation). This will be done in the next sections.

4. The approximating problems.

We consider here the problems

$$(4.1)_n \qquad \begin{cases} u_n'(t) - A_n(t)u_n(t) = f(t)\,, & t \in [0, T] \\ u_n(0) = x_n\,, \end{cases}$$

where $n \in \mathbf{N}^+$, $A_n(t) = nA(t)R(n, A(t))$ is the Yosida approximation of $A(t)$, $x_n = nR(n, A(0))x$, x is an element of E and $f\colon]0, T] \to E$ is a fixed function. This section starts with a survey of the main properties of the operators $A_n(t)$; next, we prove existence and uniqueness of the solution of $(4.1)_n$ and a representation formula for it (see $(4.2)_n$ below), provided the data are sufficiently regular.

LEMMA 4.1. *Let* $\theta \in]\pi/2, \theta_0[$. *Then*

$$\frac{1}{|\lambda + n|} \leqslant \frac{3}{\sin\theta[1 + |\lambda|]} \wedge \frac{[1 + \operatorname{tg}^2\theta]^{\frac{1}{2}}}{|\operatorname{tg}\theta| n} \qquad \forall \lambda \in S_\theta\,, \quad \forall n \in \mathbf{N}^+\,.$$

PROOF. Tedious but elementary. ■

LEMMA 4.2. *Let* $\theta \in]\pi/2, \theta_0[$. *Then for* $\lambda \in S_\theta$, $n \in \mathbf{N}^+$ *and* $0 \leqslant s \leqslant t \leqslant T$ *we have*:

(i) $\varrho(A_n(t)) \supseteq S_\theta$ and

$$R(\lambda, A_n(t)) = \frac{1}{\lambda + n}[n - A(t)]R\left(\frac{\lambda n}{\lambda + n}, A(t)\right);$$

(ii) $\|R(\lambda, A_n(t))\|_{\mathcal{L}(E)} \leqslant \dfrac{c(\theta)}{1 + |\lambda|}$;

(iii) $R(\lambda, A_n(t)) - R(\lambda, A(t)) =$

$$= \frac{1}{\lambda + n} A(t)R\left(\frac{\lambda n}{\lambda + n}, A(t)\right) A(t)R(\lambda, A(t));$$

(iv) $A_n(t)R(\lambda, A_n(t)) = \dfrac{n}{\lambda + n} A(t)R\left(\dfrac{\lambda n}{\lambda + n}, A(t)\right)$;

(v) $R(\lambda, A_n(t)) - R(\lambda, A_n(s)) = -\dfrac{n^2}{(\lambda + n)^2} A(t)R\left(\dfrac{\lambda n}{\lambda + n}, A(t)\right)\cdot$

$$\cdot[A(t)^{-1} - A(s)^{-1}]A(s)R\left(\frac{\lambda n}{\lambda + n}, A(s)\right).$$

PROOF. Straightforward. ■

LEMMA 4.3. *Let* $\theta \in]\pi/2, \theta_0[$, *fix* $\beta \in [0, 1]$, $\mu \in [0, \infty[$, $m \in \mathbb{N}$. *Then for* $\xi > 0$, $n \in \mathbb{N}^+$ *and* $t \in [0, T]$ *we have*:

(i) $$\|A_n(t)^m \exp[\xi A_n(t)]\|_{\mathcal{L}(D_{A(t)}(\beta,\infty),E)} \leqslant \frac{c(\theta, \beta, m)}{\xi^{(m-\beta)\vee 0}};$$

(ii) $$\|A_n(t)^m \exp[\xi A_n(t)] - A(t)^m \exp[\xi A(t)]\|_{\mathcal{L}(D_{A(t)}(\beta,\infty),E)} \leqslant \leqslant \frac{c(\theta, \beta, m)}{n^\eta \xi^{m-\beta+\eta}} \quad \forall \eta \in [0, 1];$$

(iii) $$\lim_{n\to\infty} \sup_{\xi>0} \xi^m \|[A_n(t)^m \exp[\xi A_n(t)] - A(t)^m \exp[\xi A(t)]]\, y\|_E = 0 \quad \forall y \in \overline{D_{A(t)}};$$

(iv) $$\lim_{n\to\infty} \sup_{0<p\leqslant q\leqslant r\leqslant T} \Big\|\int_p^q (r-s)^m [A_n(s)^m \exp[(r-s)A_n(s)] - - A(s) \exp[(r-s)A(s)]]\, f(s)\, ds\Big\|_E = 0 \quad \forall f \in L^1(E) \cap B_\mu(E);$$

(v) $$\|A_n(t)^m \exp[\xi A_n(t)] - A_n(s)^m \exp[\xi A_n(s)]\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)} \leqslant \leqslant c(\theta, \beta, m) \sum_{i=1}^k \frac{(t-s)^{\alpha_i}}{\xi^{m+\beta_i-\beta}}.$$

PROOF. (i) Easy consequence of Lemmata 4.2 (iv) and 4.1.

(ii) Easy consequence of Lemma 4.2 (iii), $(1.6)_3$ and Lemma 4.1.

(iii) Fix $\varepsilon > 0$ and select $z \in D_{A(t)}$ such that $\|y - z\|_E < \varepsilon$. Then setting $B = A_n(t)^m \exp[\xi A_n(t)] - A(t)^m \exp[\xi A(t)]$ and using (i)-(ii) and Lemma 1.10 (ii) we get

$$\|By\|_E \leqslant \frac{c}{\xi^m} \|y - z\|_E + \frac{c}{n\xi^m} \|z\|_{D_{A(t)}},$$

so that

$$\limsup_{n\to\infty} \sup_{\xi>0} \xi^m \|By\|_E \leqslant c\varepsilon \quad \forall \varepsilon > 0.$$

(iv) Fix $\varepsilon \in]0, T[$ and choose $\sigma_\varepsilon > 0$ such that

$$\sup_{a\in[0, T-\sigma_\varepsilon]} \int_a^{a+\sigma_\varepsilon} \|f(s)\|_E\, ds \leqslant \varepsilon.$$

Then if $q-p\leqslant 3\sigma_\varepsilon$ we have by (ii)

$$\Big\|\int_p^q (r-s)^m[A_n(s)^m \exp[(r-s)A_n(s)] - \\ - A(s)^m \exp[(r-s)A(s)]]f(s)\,ds\Big\|_E \leqslant c\varepsilon\,,$$

whereas if $q-p\geqslant 3\sigma_\varepsilon$ we get, again by (ii)

$$\Big\|\int_p^{p+\sigma_\varepsilon} + \int_{p+\sigma_\varepsilon}^{q-\sigma_\varepsilon} + \int_{q-\sigma_\varepsilon}^{q} (r-s)^m[A_n(s)\exp[(r-s)A_n(s)] - \\ - A(s)\exp[(r-s)A(s)]]\,f(s)\,ds\Big\|_E \leqslant c\varepsilon + \frac{c}{n^\eta \sigma_\varepsilon^{\mu+\eta}}\,,$$

and (iv) follows readily.

(v) Easy consequence of Lemma 4.2 (v), $(1.6)_3$ and Lemma 4.1. ■

LEMMA 4.4. *Let $\theta\in\,]\pi/2, \theta_0[$ and fix $m\in\mathbf{N}^+$. Then for $\xi>0$, $n\in\mathbf{N}^+$, $0\leqslant r\leqslant s\leqslant t\leqslant T$ we have:*

(i) $\|A_n(t)^m \exp[\xi A_n(t)][A(t)^{-1}-A(s)^{-1}]\|_{\mathcal{L}(E)} \leqslant c(\theta, m)\sum_{i=1}^k \frac{(t-s)^{\alpha_i}}{\xi^{m-1+\beta_i}}$;

(ii) $\|[A_n(t)^m \exp[\xi A_n(t)] - A(t)^m \exp[\xi A(t)]]\,[A(t)^{-1}-A(s)^{-1}]\|_{\mathcal{L}(E)} \leqslant$
$$\leqslant \frac{c(\theta, m)}{n^\eta}\sum_{i=1}^k \frac{(t-s)^{\alpha_i}}{\xi^{m-1+\beta_i+\eta}}\ \forall\eta\in[0,1];$$

(iii) $\|[A_n(t)^m \exp[\xi A_n(t)] - A_n(s)^m \exp[\xi A_n(s)]]\,[A(s)^{-1}-A(r)^{-1}]\|_{\mathcal{L}(E)} \leqslant$
$$\leqslant c(\theta, m)\sum_{i,j=1}^k \frac{(t-s)^{\alpha_i}(s-r)^{\alpha_j}}{\xi^{m-1+\beta_i+\beta_j}}\,.$$

PROOF. (i) It follows by Lemmata 4.2 (iv) and 4.1.

(ii) It follows by Lemmata 4.2 (iii) and 4.1.

(iii) It follows by Lemmata 4.2 (v) and 4.1. ■

We are now ready to prove existence of the solution of Problem $(4.1)_n$ and a representation formula for it.

PROPOSITION 4.5. *Set $x_n = nR(n, A(0))x$ with $x\in E$ and $f\in C(E)$ (resp. $f\in C_\mu(E)$ with $\mu\in[0,1[$, $f\in L^1(E)$). Then for each $n\in\mathbf{N}^+$ Pro-*

blem $(4.1)_n$ *has a unique solution* $u_n \in C(E)$ *such that* $u_n' \in C(E)$ (*resp.* $u_n' \in C_\mu(E)$, $u_n' \in L^1(E)$); *moreover* u_n *is given by*

$$(4.2)_n \qquad u_n(t) = A_n(t)^{-1}[(1-Q_n)^{-1}L_n(f, x_n)](t)\,, \qquad t \in [0, T]\,,$$

where the operator Q_n *and the function* $L_n(f, x_n)$ *are defined by*:

$$(4.3)_n \qquad Q_n g(t) = \int_0^t Q_n(t, s) g(s)\, ds\,,$$

$$(4.4)_n \qquad Q_n(t, s) = A_n(t)^2 \exp[(t-s)A_n(t)][A(t)^{-1} - A(s)^{-1}]\,,$$

$$(4.5)_n \qquad L_n(f, x_n)(t) = \\ = \int_0^t A_n(t) \exp[(t-s)A_n(t)] f(s)\, ds + A_n(t) \exp[tA_n(t)] x_n\,.$$

PROOF. For fixed $n \in \mathbf{N}^+$ Lemma 1.9 (i) yields

$$\|A_n(t) - A_n(s)\|_{\mathfrak{L}(E)} \leqslant c \sum_{i=1}^{k} n^{1+\beta_i}(t-s)^{\alpha_i} \qquad \forall 0 \leqslant s \leqslant t \leqslant T\,,$$

and consequently by the method of successive approximations we get existence and uniqueness of the solution of $(4.1)_n$. Formula $(4.2)_n$ follows by the same argument of the proof of Theorem 3.3 (in an even simpler way). ■

5. Technicalities, II.

In this section we study the regularity and convergence properties of the functions $L_n(f, x_n)$ and the operators Q_n defined by $(4.5)_n$ and $(4.3)_n$-$(4.4)_n$. We recall that $x_n = nR(n, A(0))x$.

(*a*) *The functions* $L_n(f, r_n)$.

PROPOSITION 5.1. $L_n(f, x_n) \in C(E)$ whenever $x_n \in E$, $f \in L^1(E)$.

PROOF. It is quite easy since, for fixed n,

$$(t, s) \mapsto A_n(t) \exp[(t-s)A_n(t)] \quad \text{and } t \mapsto A_n(t) \exp[tA_n(t)]$$

are continuous functions with values in $\mathfrak{L}(E)$. ■

PROPOSITION 5.2. *Let* δ *be defined by* (1.8) *and fix* $\beta \in]0, \delta]$, $\mu \in [0, 1[$. *If either* (i) $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_{\mu,\beta}(E)$, *or* (ii) $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_\mu(D_A(\beta, \infty))$, *then* $L_n(f, x_n) \to L(f, x)$ *in* $B_{\mu+\eta}(E)$ *as* $n \to \infty$, *for each* $\eta \in]0, (1-\mu)\wedge\beta[$.

Fix now $\beta \in]0, \delta]$, $\mu \in [1, 1+\beta[$. *If either* (iii) $x \in \overline{D_{A(0)}}$, $f \in Z_{\mu,\beta}(E) \cap L^1(E)$, *or* (iv) $x \in \overline{D_{A(0)}}$, $f \in Z_\mu(D_A(\beta, \infty)) \cap L^1(E)$, *then* $L_n(f, x_n) \to L(f, x)$ *in* $I_{\mu+\eta}(E)$ *as* $n \to \infty$, *for each* $\eta \in]0, 1+\beta-\mu[$.

PROOF. We write

$$(5.1) \qquad L_n(f, x_n)(t) - L(f, x)(t) = \\ = \int_0^t [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]] f(s)\, ds + \\ + [A(t) \exp[tA_n(t)]x_n - A(t) \exp[tA(t)]x] = U + V:$$

we will split U and V in different ways.

(i) Fix $\eta \in]0, (1-\mu)\wedge\beta[$. By (5.1), for U we have

$$(5.2) \qquad U = \int_0^{t/2} [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]] f(s)\, ds + \\ + \int_{t/2}^t [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]][f(s) - f(t)]\, ds + \\ + [\exp[(t/2)A_n(t)] - \exp[(t/2)A(t)]] f(t)\,,$$

and by Lemma 4.3 (ii)

$$\|U\|_E \leqslant \frac{c}{n^\eta} \frac{1}{t^{\mu+\eta}} \|f\|_{Z_{\mu,\beta}(E)}\,.$$

Concerning V, we fix $\varepsilon \in]0, T[$ and distinguish two cases:

(*a*) $t \in]0, \varepsilon[$,

(*b*) $t \in [\varepsilon, T]$.

In case (*b*), we split

$$(5.3) \qquad V = A_n(t) \exp[tA_n(t)][x_n - x] + \\ + [A_n(t) \exp[tA_n(t)] - A(t) \exp[tA(t)]]x\,,$$

and as $x_n - x = A(0)R(n, A(0))x$, by Lemma 5.2 (i)-(ii) and $(1.6)_3$ we deduce

$$\|V\|_E \leqslant \frac{c}{\varepsilon \cdot n^{1-\mu}} \|x\|_{D_{A(0)}(1-\mu,\infty)} + \frac{c}{\varepsilon^{1+\eta} n^{\eta}} \|x\|_E;$$

in case (a) we write

$$\begin{aligned}(5.4)\qquad V = {} & [A_n(t) \exp[tA_n(t)] - A_n(0) \exp[tA_n(0)]] x_n + \\ & + [A_n(0) \exp[tA_n(0)] - A(0) \exp[tA(0)]] x_n + \\ & + A(0) \exp[tA(0)][x_n - x] + [A(0) \exp[tA(0)] - A(t) \exp[tA(t)]] x,\end{aligned}$$

so that by Lemmata 4.3 (ii)-(v) and 1.10 (i)-(ii) we check

$$(5.5)\qquad \|V\|_E \leqslant c\left[\frac{1}{n^{\eta}} + \varepsilon^{\eta}\right] \frac{1}{t^{\mu+\eta}} \|x\|_{D_{A(0)}(1-\mu,\infty)}.$$

Consequently we get

$$(5.6)\qquad \limsup_{n\to\infty} \sup_{t\in]0,T]} t^{\mu+\eta} \|L_n(f, x_n)(t) - L(f, x)(t)\|_E \leqslant c\varepsilon^{\eta} \qquad \forall \varepsilon > 0,$$

and (i) is proved.

(ii) Fix again $\eta \in]0, [1-\mu) \wedge \beta[$. We write now U as follows:

$$\begin{aligned}(5.7)\qquad U = {} & \int_0^{t/2} [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]] f(s)\, ds + \\ & + \int_{t/2}^{t} [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)] - \\ & - A_n(s) \exp[(t-s)A_n(s)] + A(s) \exp[(t-s)A(s)]] f(s)\, ds + \\ & + \int_{/2}^{t} [A_n(s) \exp[(t-s)A_n(s)] - A(s) \exp[(t-s)A(s)]] f(s)\, ds = \sum_{j=1}^{3} I_i.\end{aligned}$$

It is easy to show that

$$\|I_1\|_E + \|I_3\|_E \leqslant \frac{c}{n^\eta t^{\mu+\eta}} \|f\|_{Z_\mu(D_A(\beta,\infty))};$$

the estimate of I_2 is more delicate: if $\alpha \in]0, (\delta+\beta)/(\eta+\delta+\beta)[$ by Lemmata 4.3 (ii)-(v) and 1.10 (i) we have

$$(5.8) \qquad \|I_2\|_E = \|\int_{t/2}^{t} [A_1 - A_2 - A_3 + A_4] f(s)\, ds\|_E \leqslant$$

$$\leqslant \int_{t/2}^{t} [\|A_1 - A_2\|_{\mathcal{L}(E)} + \|A_3 - A_4\|_{\mathcal{L}(E)}]^\alpha \|f(s)\|_E^\alpha \cdot$$

$$\cdot [\|A_1 - A_3\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)} + \|A_2 - A_4\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)}]^{1-\alpha} \|f(s)\|^{1-\alpha}_{D_{A(s)}(\beta,\infty)} ds \leqslant$$

$$\leqslant \frac{c}{n^{\eta\alpha}} \frac{1}{t^{\mu+\eta\alpha-\delta(1-\alpha)}} \|f\|_{Z_\mu(D_A(\beta,\infty))} .$$

On the other hand, exactly as in the proof of (i), we obtain (5.5): summing up, we again get (5.6), which yields the result.

(iii) Fix $\eta \in]0, 1+\beta-\mu[$. For U we use (5.2), easily obtaining by Lemma 4.3 (ii)

$$\|U\|_E \leqslant \frac{c}{n^\eta t^{\mu+\eta}} \{\|f\|_{L^1(E)} + \|f\|_{Z_{\mu,\beta}(E)}\} .$$

For V we again write (5.3) and (5.4) in cases $t \in [\varepsilon, T]$ and $t \in]0, \varepsilon[$ respectively: hence proceeding as in the proof of (iii) we get (5.5) (with $\mu = 1$) and finally (5.6), i.e.

$$(5.9) \qquad L_n(f, x_n) \to L(f, x) \quad \text{in } B_{\mu+\eta}(E) .$$

Next, we have to show that

$$(5.10) \qquad \lim_{n\to\infty} \sup_{0<a\leqslant b\leqslant T} \|\int_a^b [L_n(f, x_n)(t) - L(f, x)(t)]\, dt\|_E = 0 .$$

We start from (2.16), which clearly holds similarly for $L_n(f, x_n)(t)$, obtaining

$$(5.11)\qquad \int_a^b [L_n(f, x_n)(t) - L(f, x)(t)]\,dt =$$

$$= \int_a^b \int_0^t [A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)] -$$

$$- A_n(s) \exp[(t-s)A_n(s)] + A(s) \exp[(t-s)A(s)]]\, f(s)\, ds\, dt +$$

$$+ \int_0^a [\exp[(b-s)A_n(s)] - \exp[(b-s)A(s)] - \exp[(a-s)A_n(s)] +$$

$$+ \exp[(a-s)A(s)]]\, f(s)\, ds +$$

$$+ \int_a^b [\exp[(b-s)A_n(s)] - \exp[(b-s)A(s)]]\, f(s)\, ds + \int_a^b V\, dt = \sum_{j=1}^{4} J_j\,,$$

where V has been introduced in (5.1).

Now, proceeding as in (5.8), if $\alpha \in]0, \delta/(1+\delta)[$ we easily have

$$(5.12)\qquad \|J_1\|_E \leqslant \frac{c}{\eta^{\eta\alpha}} \|f\|_{B_\mu(E)};$$

moreover by Lemma 4.3 (iv)

$$(5.13)\qquad \|J_2\|_E + \|J_3\|_E \leqslant 3 \cdot \sup_{0 < p \leqslant q \leqslant r \leqslant T} \cdot$$

$$\cdot \Big\| \int_p^q [\exp(r-s)A_n(s)] - \exp[(r-s)A(s)]]\, f(s)\, ds \Big\|_E = o(1) \quad \text{as } n \to \infty.$$

Finally, concerning J_4, fix $\varepsilon \in]0, T[$ and suppose first $b \in]0, \varepsilon]$; then by using (5.4) and Lemmata 4.3 (ii)-(v) and 1.10 (i)-(ii) we easily have:

$$\|J_4\|_E \leqslant c(b-a)^\delta \|x\|_E + c\|x_n - x\|_E +$$

$$+ c \sup_{t \in]0, \varepsilon]} \|[\exp[tA_n(0)] - \exp[tA(0)]]\, x\|_E\,,$$

so that Lemma 4.3 (iii) yields

$$\limsup_{n\to\infty} \sup_{0<a\leqslant b\leqslant\varepsilon} \|J_4\|_E \leqslant c\varepsilon^\delta \qquad \forall \varepsilon \in]0, T[\,. \tag{5.14}$$

Suppose now $b \in]\varepsilon, T]$; then if $0 < a \leqslant \varepsilon$ we split

$$J_4 = \left[\int_a^\varepsilon + \int_\varepsilon^b\right] V\, dt\,,$$

and the first integral can be estimated by (5.14). Thus we can assume that $\varepsilon < a \leqslant b \leqslant T$, and writing

$$V = A_n(t) \exp[tA_n(t)](x_n - x) + [A_n(t) \exp[tA_n(t)] - A(t) \exp[tA(t)]]x,$$

by Lemma 4.3 (i)-(ii) we obtain

$$\|J_4\|_E \leqslant \frac{c}{\varepsilon}\left\{\|x_n - x\|_E + \frac{\|x\|_E}{n^\eta}\right\},$$

which, together with (5.12) implies

$$\lim_{n\to\infty} \sup_{0<a\leqslant b\leqslant T} \|J_4\|_E = 0\,. \tag{5.15}$$

By (5.12), (5.13), (5.15) and (5.11) we have (5.10), and recalling (5.9) the result follows.

(iv) In order to estimate U (see (5.1)) we use (5.7): by Lemmata 4.3 (ii) (with $\eta \in]0, \beta[$), 4.3 (v) and 1.10 (i), proceeding as in the proof of (ii) we deduce that

$$\|U\|_E \leqslant c\left\{\frac{1}{n^\eta}\frac{1}{t^{1+\eta}}\|f\|_{L^1(E)} + \frac{c}{n^{\eta\alpha}}\frac{1}{t^{\mu+\eta\alpha-\delta(1-\alpha)}}\|f\|_{Z_\mu(D_A(\beta,\infty))} + \frac{c}{n^\eta}\frac{1}{t^{\mu+\eta}}\{f\}_{\mu,\beta}\right\},$$

so that

$$\lim_{n\to\infty} \sup_{t\in]0,T]} t^{\mu+\eta}\|U\|_E = 0\,. \tag{5.16}$$

For V we still have (5.5) (with $\mu = 1$) which, together with (5.16), yields (5.9).

Finally we have to prove (5.10); but the argument used in (iii) still works, since we just used there the fact that $f \in B_\mu(E) \cap L^1(E)$: this proves (iv). Proposition 5.2 is completely proved. ■

(*b*) *The operators* Q_n.

LEMMA 5.3. *Let* $Q_n(t, s)$ *be defined by* $(4.4)_n$, *and fix* $\theta \in]\pi/2, \theta_0[$. *Then we have*:

(i) $$\|Q_n(t, s)\|_{\mathfrak{L}(E)} \leqslant c(\theta) \sum_{i=1}^{k} \frac{1}{(t-s)^{1+\beta_i-\alpha_i}} \quad \forall n \in \mathbb{N}^+, \quad \forall 0 \leqslant s < t \leqslant T;$$

(ii) $$\|Q_n(t, s) - Q_n(\sigma, s)\|_{\mathfrak{L}(E)} \leqslant$$
$$\leqslant c(\theta) \sum_{i=1}^{k} \left[\frac{(t-\sigma)^{\alpha_i}}{(t-s)^{1+\beta_i}} + \int_{\sigma-s}^{t-s} \frac{(\sigma-s)^{\alpha_i}}{\xi^{2+\beta_i}} d\xi \right] \forall n \in \mathbb{N}^+, \quad \forall 0 \leqslant s < \sigma \leqslant t \leqslant T;$$

(iii) $$\|Q_n(t, s) - Q_n(t, r)\|_{\mathfrak{L}(E)} \leqslant c(\theta) \sum_{i=1}^{k} \left[\frac{(s-r)^{\alpha_i}}{(t-r)^{1+\beta_i}} + \right.$$
$$\left. + \int_{t-s}^{t-r} \frac{(t-s)^{\alpha_i}}{\xi^{2+\beta_i}} d\xi \right] \quad \forall n \in \mathbb{N}^+, \quad \forall 0 \leqslant r \leqslant s < t \leqslant T;$$

(iv) $$\|Q_n(t, s) - Q(t, s)\|_{\mathfrak{L}(E)} \leqslant \frac{c(\theta)}{n^\eta} \sum_{i=1}^{k} \frac{1}{(t-s)^{1+\beta_i-\alpha_i+\eta}}$$
$$\forall \eta \in [0, 1], \quad \forall n \in \mathbb{N}^+, \quad \forall 0 \leqslant s < t \leqslant T;$$

(v) $$\|Q_n(t, s) - Q(t, s) - Q_n(t, r) + Q(t, r)\|_{\mathfrak{L}(E)} \leqslant c(\theta) \cdot$$
$$\cdot \frac{(s-r)^{\delta\alpha}}{n^{\eta(1-\alpha)}(t-s)^{1-(\delta-\eta)(1-\alpha)}} \quad \forall \eta, \alpha \in [0, 1], \quad \forall n \in \mathbb{N}^+, \quad \forall 0 \leqslant r \leqslant s < t \leqslant T.$$

PROOF. (i) It follows by Lemma 4.4 (i).

(ii) Exactly as in the proof of Lemma 2.3 (ii) (cfr. (2.26)), using Lemma 4.4 (i)-(iii).

(iii) Exactly as in the proof of Lemma 2.3 (iii), using Lemma 4.4 (i)-(iii).

(iv) It follows by Lemma 4.4 (ii).

(v) By (iii)-(iv) and Lemma 2.3 (iii) we have easily:

$$\|Q_n(t, s) - Q(t, s) - Q_n(t, r) + Q(t, r)\|_{\mathfrak{L}(E)} \leqslant$$
$$\leqslant [\|A_1 - A_2\|_{\mathfrak{L}(E)} + \|A_3 - A_4\|_{\mathfrak{L}(E)}]^{1-\alpha} [\|A_1 - A_3\|_{\mathfrak{L}(E)} + \|A_2 - A_4\|_{\mathfrak{L}(E)}]^\alpha \leqslant$$
$$\leqslant c(\theta) \left[\frac{1}{n^\eta (t-s)^{1-\delta+\eta}} \right]^{1-\alpha} \left[\frac{(s-r)^\delta}{t-s} \right]^\alpha. \quad ■$$

PROPOSITION 5.4. *For each* $\mu \in [0, 1[$ *and* $\lambda \in [1, 1+\delta[$ *we have* $Q_n \in \mathfrak{L}(B_\mu(E)) \cap \mathfrak{L}(I_\lambda(E))$ $\forall n \in \mathbb{N}^+$ *and*

$$Q_n \to Q \text{ as } n \to \infty \text{ im } \mathfrak{L}(B_\mu(E)) \text{ and in } \mathfrak{L}(I_\lambda(E)) .$$

PROOF. The first part is obvious by Proposition 2.4, since the kernels $Q_n(t, s)$ have the same properties as $Q(t, s)$. Next, if $g \in B_\mu(E)$ we have by Lemma 5.3 (iv) (with $\eta = \delta/2$):

$$\|Q_n g(t) - Qg(t)\|_E \leqslant \frac{c}{n^{\delta/2} t^{\mu-\delta/2}} \|g\|_{B_\mu(E)};$$

this shows that $Q_n \to Q$ in $\mathfrak{L}(B_\mu(E))$ as $n \to \infty$.

On the other hand if $g \in I_\mu(E)$ we can write

$$(5.17) \qquad Q_n g(t) - Qg(t) = \\ = \int_0^t [Q_n(t, s) - Q(t, s) - Q_n(t, 0) + Q(t, 0)] g(s)\, ds + \\ + [Q_n(t, 0) - Q(t, 0)] \int_0^t g(s)\, ds ,$$

so that by Lemma 5.3 (iv)-(v) (with $\eta = \delta/2$ and $\alpha \in](\lambda-1)/\delta, 1[$) it is not difficult to check

$$\|Q_n g(t) - Qg(t)\|_E \leqslant \frac{c}{n^{(1-\alpha)\delta/2}} \frac{1}{t^{\lambda-(1-\alpha)\delta/2}} \|g\|_{B_\lambda(E)} + \frac{c}{n^{\delta/2} t^{1-\delta/2}} [g]_{*,E};$$

this shows that $Q_n \to Q$ in $\mathfrak{L}(B_\lambda(E))$ as $n \to \infty$. Finally we have to verify that

$$(5.18) \qquad [Q_n g - Qg]_{*,E} = o(1) \qquad \text{as } n \to \infty;$$

now by (5.17) and Lemma 5.3 (iv)-(v) (with $\eta = \delta/2$ and $\alpha \in$

$\in](\lambda-1-\delta/2)/(\delta/2), 1[)$ we have for $0<a\leqslant b\leqslant T$:

$$\|\int_a^b [Q_n g(t)-Qg(t)]\,dt\|_E \leqslant$$
$$\leqslant c\left\{\frac{1}{n^{(1-\alpha)\delta/2}}\int_a^b \frac{dt}{t^{\lambda-(1+\alpha)\delta/2}}\|g\|_{B_\lambda(E)}+\frac{1}{n^{\delta/2}}\int_a^b \frac{dt}{t^{1-\delta/2}}[g]_{*,E}\right\}=\frac{c}{n^{(1-\alpha)\delta/2}}\|g\|_{I_\lambda(E)},$$

and (5.18) is proved. ■

(*c*) *The operators* $(1-Q_n)^{-1}$.

PROPOSITION 5.5. *We have*:

(i) $\|(1-Q_n)^{-1}\|_{\mathfrak{L}(B_\mu(E))}\leqslant c(\mu)\ \forall\mu\in[0,1[,\ \forall n\in\mathbf{N}^+$;

(ii) $\|(1-Q_n)^{-1}\|_{\mathfrak{L}(I_\mu(E))}\leqslant c(\mu)\ \forall\mu\in[1,1+\delta[,\ \forall n\in\mathbf{N}^+$;

(iii) $(1-Q_n)^{-1}\to(1-Q)^{-1}$ *as* $n\to\infty$ *in* $\mathfrak{L}(B_\mu(E))\ \forall\mu\in[0,1[$ *and in* $\mathfrak{L}(I_\mu(E))\ \forall\mu\in[1,1+\delta[$.

PROOF. (i)-(ii) Exactly as in the proof of Proposition 2.6 (i)-(ii), using Lemma 5.3 instead of Lemma 2.3.

(iii) We have

$$(1-Q_n)^{-1}g-(1-Q)^{-1}g=(1-Q_n)^{-1}[Q_n-Q](1-Q)^{-1}g,$$

and the result follows by (i)-(ii) and Propositions 5.4 and 2.6 (i)-(ii). ■

6. Strict and classical solutions.

We are now ready to show that the function $u(t)=A(t)^{-1}w(t)$, with w defined by (3.1), is in fact the strict, or classical, solution of Problem (0.1) under suitable assumptions on the data x, f; we will also prove the maximal regularity properties of such solutions. Since the right number of (0.1) can be chosen to be regular in time as well as in space, for each kind of solution we have two distinct results. We start with strict solutions.

THEOREM 6.1. *Fix* $\beta\in]0,\delta]$, *let* $x\in D_{A(0)}$, $f\in C(E)\cap Z_{0,\beta}(E)$ *and suppose that* $A(0)x+f(0)\in\overline{D_{A(0)}}$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is the unique strict solution of* (0.1);

(ii) $u' \in Z_{0,\beta}(E) \cap Z_0(D_A(\beta, \infty))$ *and* $Au \in Z_{0,\beta}(E)$; *in addition*

$$\|u'\|_{Z_{0,\beta}(E)} + \|u'\|_{Z_0(D_A(\beta,\infty))} + \|Au\|_{Z_{0,\beta}(E)} \leqslant c\{\|x\|_{D_{A(0)}} + \|f\|_{Z_{0,\beta}(E)}\}\,. \tag{6.1}$$

Let now $x \in D_{A(0)}$, $f \in C^\beta(E)$. *Then*:

(iii) $u' \in C^\beta(E) \cap B(D_A(\beta, \infty))$ *and* $Au \in C^\beta(E)$ *if and only if* $A(0)x + f(0) \in D_{A(0)}(\beta, \infty)$: *in this case we have also*

$$\|u'\|_{C^\beta(E)} + \|u'\|_{B(D_A(\beta,\infty))} + \|Au\|_{C^\beta(E)} \leqslant \\ \leqslant c\{\|x\|_{D_{A(0)}} + \|f\|_{C^\beta(E)} + \|A(0)x + f(0)\|_{D_{A(0)}(\beta,\infty)}\}\,. \tag{6.2}$$

PROOF. (i) Set $x_n = nR(n, A(0))x$, $n \in \mathbb{N}^+$. By Prop. 4.5, Problem $(4.1)_n$ has a unique solution $u_n \in C^1(E)$, which is given by $(4.2)_n$. By Proposition 5.2 (i) and 5.5 (iii), and taking into account that $A_n(t)^{-1} = A(t)^{-1} - 1/n$, we deduce that $u_n \to A^{-1}w$ in $B_\eta(E)$ as $n \to \infty$, for each $\eta \in]0, \beta[$. On the other hand, as $u_n' = A_n u_n + f$, by $(4.2)_n$ and Propositions 5.2 (i) and 5.5 (iii) we also get $u_n' \to w + f$ in $B_\eta(E)$ as $n \to \infty$, for each $\eta \in]0, \beta[$. This implies that $u = A^{-1}w \in C^1_+(E)$ and

$$u' = Au + f \quad \text{in }]0, T]\,. \tag{6.3}$$

But Prop 3.1 (ii) yields $Au + f \in C(E)$ and $[Au + f]_{t=0} = A(0)x + f(0)$, hence by (6.3) it is easily verified that

$$\exists u'(0) = A(0)x + f(0) = \lim_{t\to 0^+} u'(t);$$

As, clearly, $u(0) = x$, this shows that u is a strict solution of (0.1). Uniqueness follows by Corollary 3.6.

(ii) By (6.3) and Prop. 3.1 (v) we get $Au, u' \in Z_{0,\beta}(E)$. On the other hand, Proposition 2.1 (v) and 2.6 (iii) yield $(1-Q)^{-1}(L(f, x) + f) \in Z_0(D_A(\beta, \infty))$; hence

$$u' = (1-Q)^{-1}(L(f, x)) + f = \\ = (1-Q)^{-1}(L(f, x) + f) - Q(1-Q)^{-1}f\,, \tag{6.4}$$

so that $u' \in Z_0(D_A(\beta, \infty))$ by Propositions 2.6 (iii) and (2.4) (iv), and (ii) is proved. Estimate (6.1) follows by (6.3), (6.4), (2.10), (2.12), Propositions 2.6 (iii)-2.4 (iv) and (2.37).

(iii) By Prop. 3.1 (i) and (6.3), we have $u', Au \in C^\beta(E)$ if and only if $A(0)x + f(0) \in D_{A(0)}(\beta, \infty)$. In addition, bu Propositions 2.1 (ii) and 2.6 (iii), $(1-Q)^{-1}(L(f, x) + f) \in B(D_A(\beta, \infty))$ if and only if $A(0)x + f(0) \in D_{A(0)}(\beta, \infty)$; by (6.4) and Propositions 2.6 (iii) and 2.4 (v), this is also equivalent to $u' \in B(D_A(\beta, \infty))$. Finally, estimate (6.2) follows by (6.3), (6.4), (2.3), (2.5), Propositions 2.6 (iii)-2.4 (v) and (2.37). ■

THEOREM 6.2. *Fix* $\beta \in]0, \delta]$, *let* $x \in D_{A(0)}$, $f \in C(E) \cap Z_0(D_A(\beta, \infty))$ *and suppose that* $A(0)x + f(0) \in \overline{D_{A(0)}}$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is the unique strict solution of* (0.1);

(ii) $u' \in Z_0(D_A(\beta, \infty))$ and $Au \in Z_0(D_A(\beta, \infty)) \cap Z_{0,\beta}(E)$; *in addition*

$$\|u'\|_{Z_0(D_A(\beta,\infty))} + \|Au\|_{Z_0(D_A(\beta,\infty))} + \|Au\|_{Z_{0,\beta}(E)} \leqslant c\{\|x\|_{D_{A(0)}} + \|f\|_{Z_0(D_A(\beta,\infty))}\}\,. \tag{6.5}$$

Let now $x \in D_{A(0)}$, $f \in B(D_A(\beta, \infty))$. *Then*:

(iii) $u' \in B(D_A(\beta, \infty))$ *and* $Au \in B(D_A(\beta, \infty)) \cap C^\beta(E)$ *if and only if* $A(0)x \in D_{A(0)}(\beta, \infty)$; *in this case we have also*

$$\|u'\|_{B(D_A(\beta,\infty))} + \|Au\|_{B(D_A(\beta,\infty))} + \|Au\|_{C^\beta(E)} \leqslant c\{\|x\|_{D_{A(0)}} + \|f\|_{B(D_A(\beta,\infty))} + \|A(0)x\|_{D_{A(0)}(\beta,\infty)}\}\,. \tag{6.6}$$

PROOF. Quite similar to the proof of Theorem 6.1. ■

Let us consider now classical solutions. We start with the case of solutions which are weakly singular at $t = 0$ (i.e. such that $Au \in B_\mu(E)$ for some $\mu \in [0, 1[$).

THEOREM 6.3. *Fix* $\beta \in]0, \delta]$, $\mu \in [0, 1[$ *and let* $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_{\mu,\beta}(E)$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is a classical solution of* (0.1) *and is unique in the class* $\bigcup_{0 \leqslant \mu < 1+\delta} I_\mu(D_A)$;

(ii) $u' \in Z_{\mu,\beta}(E) \cap Z_\mu(D_A(\beta, \infty))$ *and* $Au \in Z_{\mu,\beta}(E)$; *in addition*

$$\|u'\|_{Z_{\mu,\beta}(E)} + \|u'\|_{Z_\mu(D_A(\beta,\infty))} + \\ + \|Au\|_{Z_{\mu,\beta}(E)} \leqslant c\{\|x\|_{D_{A(0)}(1-\mu,\infty)} + \|f\|_{Z_{\mu,\beta}(E)}\}\,. \tag{6.7}$$

PROOF. (i) Set $x_n = nR(n, A(0))x$, and let $u_n \in C^1(E)$ be the solution of Problem $(4.1)_n$, given by Prop. 4.5. By Propositions 5.2 (i) and 5.5. (iii) we have, as $n \to \infty$, $u_n \to A^{-1}w$ and $u_n' = A_n u_n + f \to \to w + f$ in $B_{\mu+\eta}(E)$ for each $\eta \in]0, (1-\mu)\wedge\beta[$. Hence we get $u \in C^1_+(E)$ and (6.3) holds. Moreover we get

$$\|u(t) - u(s)\|_E \leqslant \int_s^t \|u'(r)\|_E\, dr \leqslant \\ \leqslant \int_s^t [\|A(r)u(r)\|_E + \|f(r)\|_E]\, dr = 0((t-s)^{1-\mu-\eta}) \quad \text{as } t - s \to 0^+\,, \tag{6.8}$$

so that $u \in C(E)$ and, clearly, $u(0) = x$. Uniqueness follows by Theorem 3.5.

(ii) Quite similar to the proof of Theorem 6.1 (ii). ■

THEOREM 6.4. *Fix* $\beta \in]0, \delta]$, $\mu \in [0, 1[$, *and let* $x \in D_{A(0)}(1-\mu, \infty)$, $f \in Z_\mu(D_A(\beta, \infty))$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is a classical solution of* (0.1) *and is unique in the class* $\bigcup_{0 \leqslant \mu < 1+\delta} I_\mu(D_A)$;

(ii) $u' \in Z_\mu(D_A(\beta, \infty))$ *and* $Au \in Z_\mu(D_A(\beta, \infty)) \cap Z_{\mu,\beta}(E)$; *in addition*

$$\|u'\|_{Z_\mu(D_A(\beta,\infty))} + \|Au\|_{Z_\mu(D_A(\beta,\infty))} + \\ + \|Au\|_{Z_{\mu,\beta}(E)} \leqslant c\{\|x\|_{D_{A(0)}(1-\mu,\infty)} + \|f\|_{Z_\mu(D_A(\beta,\infty))}\}\,. \tag{6.9}$$

PROOF. Quite similar to that of Theorems 6.3 (i) and 6.2 (ii). ■

Finally we treat the case of strongly singular classical solutions (i.e. such that $Au \in I_\mu(E)$ for some $\mu \in [1, 1+\delta[$).

THEOREM 6.5. Fix $\beta \in]0, \delta]$, $\mu \in [1, 1+\beta[$ and let $x \in \overline{D_{A(0)}}$, $f \in Z_{\mu,\beta}(E) \cap L^1(E)$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is a classical solution of* (0.1) *and is unique in the class* $\bigcup_{0\leqslant\mu<1+\delta} I_\mu(D_A)$;

(ii) $u' \in Z^*_{\mu,\beta}(E) \cap Z^*_\mu(D_A(\beta, \infty))$ and $Au \in Z^*_{\mu,\beta}(E)$; *in addition*

$$\|u'\|_{Z^*_{\mu,\beta}(E)} + \|u'\|_{Z^*_\mu(D_A(\beta,\infty))} + \|Au\|_{Z^*_{\mu,\beta}(E)} \leqslant \leqslant c\{\|x\|_E + \|f\|_{Z_{\mu,\beta}(E)} + \|f\|_{L^1(E)}\}\,. \tag{6.10}$$

PROOF. (i) The solution $u_n \in C^1(E)$ of $(4.1)_n$ satisfies, as $n \to \infty$, $u_n \to A^{-1}w$ and $u_n' \to w + f$ in $I_{\mu+\eta}(E)$ for each $\eta \in]0, 1+\beta-\mu[$ (Propositions 5.2 (iii) and 5.5 (iii)). Thus we have once more (6.3). Next, as $Au \in I_{\mu+\eta}(E)$, we have

$$\|u(t) - u(s)\|_E = \|\int_s^t u'(r)\,dr\|_E \leqslant \|\int_s^t A(r)u(r)\,dr\|_E + + \int_s^t \|f(r)\|_E\,dr = o(1) \quad \text{as } t - s \to 0^+\,, \tag{6.11}$$

so that $u \in C(E)$ and, clearly $u(0) = x$. Uniqueness follows again by Theorem 3.5.

(ii) Similar to the proof of Theorem 6.1 (ii). ■

THEOREM 6.6. *Fix* $\beta \in]0, \delta]$, $\mu \in [1, 1+\beta[$ *and let* $x \in \overline{D_{A(0)}}$, $f \in Z_\mu(D_A(\beta, \infty)) \cap L^1(E)$. *Then*:

(i) *the function* $u = A^{-1}w$, *with* w *defined by* (3.1), *is a classical solution of* (0.1) *and is unique in the class* $\bigcup_{0\leqslant\mu<1+\delta} I_\mu(D_A)$;

(ii) $u' \in Z^*_\mu(D_A(\beta, \infty))$ *and* $Au \in Z^*_\mu(D_A(\beta, \infty)) \cap Z^*_{\mu,\beta}(E)$; *in addition*

$$\|u'\|_{Z^*_\mu(D_A(\beta,\infty))} + \|Au\|_{Z^*_\mu(D_A(\beta,\infty))} + \|Au\|_{Z^*_{\mu,\beta}(E)} \leqslant \leqslant c\{\|x\|_E + \|f\|_{Z_\mu(D_A(\beta,\infty))} + \|f\|_{L^1(E)}\}\,. \tag{6.12}$$

PROOF. Quite similar to that of Theorems 6.5 (i) and 6.2 (ii). ■

REMARK 6.7. All results of this section can be improved in the following way. First of all, if A is the generator of a bounded analytic

semigroup, we consider the « continuous interpolation spaces » $D_A(\beta)$ introduced by Da Prato-Grisvard [9] (see also Butzer-Berens [7]), and characterized, in analogy with $(1.6)_1$, $(1.6)_2$, $(1.6)_3$, by

$$(6.13)_1 \qquad D_A(\beta) = \{x \in D_A(\beta, \infty) : \lim_{s\to 0^+} s^{-\beta}\|[\exp[sA]-1]x\|_E = 0\} ,$$

$$(6.13)_2 \qquad D_A(\beta) = \{x \in D_A(\beta, \infty) : \lim_{s\to 0^+} s^{1-\beta}\|A \exp[sA]x\|_E = 0\}$$

$$(6.13)_3 \qquad D_A(\beta) = \{x \in D_A(\beta, \infty) : \lim_{|\lambda|\to\infty,\, \lambda\in\varrho(A)} |\lambda|^\beta \|AR(\lambda, A)x\|_E = 0\} .$$

It is known that $D_A(\beta)$ coincides with the closure of D_A in the norm of $D_A(\beta, \infty)$ (see e.g. Sinestrari [16, prop. 1.8]). Introduce moreover the « little-Hölder » spaces, which are defined by

$$h^\alpha(Y) = \left\{ f \in C^\alpha(Y) : \lim_{r\to 0^+} \sup \left\{ \frac{\|f(t)-f(s)\|_Y}{|t-s|^\alpha} : t, s \in [0, T], 0 < |t-s| \leqslant r \right\} = 0 \right\}.$$

Replace now the inequality of Hypothesis II by the stronger one

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(E)} \leqslant$$
$$\leqslant \omega(t-s)\cdot \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1} \qquad \forall \lambda \in S_{\theta_0} - \{0\} , \quad \forall 0 \leqslant s \leqslant t \leqslant T,$$

where $\omega : [0, T] \to [0, \infty[$ is a non-decreasing, continuous function satisfying $\omega(0) = 0$.

Then we can improve all results of this section by performing, in each statement, the following modifications: instead of

$$C^\beta(E) , \quad B(D_A(\beta, \infty)) , \quad D_{A(0)}(\eta, \infty) , \quad Z_{\mu,\beta}(E) , \quad Z_\mu(D_A(\beta, \infty)) ,$$
$$Z^*_{\mu,\beta}(E) , \quad Z^*_\mu(D_A(\beta, \infty))$$

read:

$$h^\beta(E) , \quad C(D_A(\beta)) , \quad D_{A(0)}(\eta) , \quad \bar{Z}_{\mu,\beta}(E) , \quad \bar{Z}_\mu(D_A(\beta)) ,$$
$$\bar{Z}^*_{\mu,\beta}E) , \quad \bar{Z}^*_\mu(D_A(\beta)) ,$$

where the spaces $\bar{Z}$, $\bar{Z}^*$ are obtained by the corresponding spaces Z, Z^* by replacing in Definition 1.4, similarly, C^β by h^β and B, B_+ by C, C_+.

The proof of these results needs a series of technicalities which are closely related to those of Section 2, 4 and 5; we do not go into further details. The corresponding results in the case of constant domains (i.e. under Hypotheses I and 7.1 below) are explicitly stated and proved in Acquistapace-Terreni [2], [3].

7. Comparisons and examples.

As remarked in the Introduction, Problem (0.1) in the parabolic case (i.e. under Hypothesis I) has been studied by several authors, with different assumptions in place of our Hypothesis II. Let us shortly recall the main kinds of assumptions used in the literature.

The simpler situation is the constant-domain case (Tanabe [18], Sobolevskii [17], Acquistapace-Terreni [2], [3]):

HYPOTHESIS 7.1. (i) $D_{A(t)} = D_{A(0)}$ $\forall t \in [0, T]$,

(ii) there exist $B > 0$, $\alpha \in]0, 1[$ such that

$$\|A(t)A(s)^{-1} - 1\|_{\mathcal{L}(E)} \leqslant B|t-s|^{\alpha} \qquad \forall s, t \in [0, T] .$$

If the domains are not constant, many situations may occur. In some cases, the domains vary with t, but there are some intermediate spaces between $D_{A(t)}$ and E which do not change; more precisely, it can be assumed (Kato [11], [12]) that the domain of some fractional power of $-A(t)$ is constant:

HYPOTHESIS 7.2. (i) There exists $\varrho \in]0, 1[$ such that

$$\varrho^{-1} \in \mathbf{N}^{+} , \qquad D_{[-A(t)]^{\varrho}} = D_{[-A(0)]^{\varrho}} \qquad \forall t \in [0, T];$$

(ii) there exist $B > 0$ and $\alpha \in]1-\varrho, 1[$ such that

$$\|[-A(t)]^{\varrho}[-A(s)]^{\varrho} - 1\|_{\mathcal{L}(E)} \leqslant B|t-s|^{\alpha} \qquad \forall s, t \in [0, T] .$$

On the other hand, it can be supposed (Acquistapace-Terreni [5]) that some interpolation space $D_{A(t)}(\varrho, \infty)$ is independent of t:

HYPOTHESIS 7.3. (i) There exists $\varrho \in]0, 1[$ such that (with uniformly equivalent norms)

$$D_{A(t)}(\varrho, \infty) = D_{A(0)}(\varrho, \infty) \qquad \forall t \in [0, T];$$

(ii) there exist $B > 0$ and $\alpha \in]1-\varrho, 1[$ such that

$$\|A(t)^{-1} - A(s)^{-1}\|_{\mathfrak{L}(E, D_{A(0)}(\varrho,\infty))} \leqslant B|t-s|^{\alpha} \qquad \forall s, t \in [0, T] .$$

In the case of totally variable domains, several kinds of assumptions can be made. First of all we have (Kato-Tanabe [13], Acquistapace-Terreni [1]):

HYPOTHESIS 7.4. (i) $t \mapsto R(\lambda, A(t)) \in C^1(\mathfrak{L}(E))$ $\forall \lambda \in S_{\theta_0}$;

(ii) there exist $K > 0$ and $\varrho \in]0, 1[$ such that

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathfrak{L}(E)} \leqslant \frac{K}{1 + |\lambda|^{\varrho}} \qquad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0, T];$$

(iii) there exist $B > 0$ and $\eta \in]0, 1[$ such that

$$\left\| \frac{d}{dt} A(t)^{-1} - \frac{d}{ds} A(s)^{-1} \right\|_{\mathfrak{L}(E)} \leqslant B|t-s|^{\eta} \qquad \forall s, t \in [0, T] .$$

Next, one can suppose (Tanabe [19]):

HYPOTHESIS 7.5. (i) $t \mapsto A(t)^{-1} \in C^1(\mathfrak{L}(E))$;

(ii) there exists $\varrho \in]0, 1[$ such that

$$\frac{d}{dt} A(t)^{-1} \in \mathfrak{L}(E, D_{[-A(t)]^{\varrho}}) \qquad \forall t \in [0, T];$$

(iii) there exists $B > 0$ such that

$$\left\| [-A(t)]^{\varrho} \frac{d}{dt} A(t)^{-1} \right\|_{\mathfrak{L}(E)} \leqslant B \qquad \forall t \in [0, T] .$$

Further, another possible assumption is (Yagi [21]):

HYPOTHESIS 7.6. (i) $t \mapsto A(t)^{-1} \in C^1(\mathfrak{L}(E))$;

(ii) there exist $B > 0$ and $\varrho \in]0, 1[$ such that

$$\left\| A(t) R(\lambda, A(t)) \frac{d}{dt} A(t)^{-1} \right\|_{\mathfrak{L}(E)} \leqslant \frac{B}{1 + |\lambda|^{\varrho}} \qquad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0, T] .$$

Finally it can be assumed the following (Yagi [22]):

HYPOTHESIS 7.7. (i) $t \mapsto R(\lambda, A(t)) \in C^1(\mathcal{L}(E))$ $\forall \lambda \in S_{\theta_0}$;

(ii) there exist $K > 0$ and $\varrho \in]0, 1[$ such that

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathcal{L}(E)} \leqslant \frac{K}{1 + |\lambda|^\varrho} \quad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0, T];$$

(iii) there exist $B > 0$, $k \in \mathbb{N}^+$ and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{R}$ such that

$$-1 \leqslant \beta_j < \alpha_j \leqslant 1 \quad \text{for } j = 1, \ldots, k$$

and

$$\left\| A(t) R(\lambda, A(t)) \frac{d}{dt} A(t)^{-1} - A(s) R(\lambda, A(s)) \frac{d}{ds} A(s)^{-1} \right\|_{\mathcal{L}(E)} \leqslant$$

$$\leqslant B \sum_{j=1}^{k} |t - s|^{\alpha_j} |\lambda|^{\beta_j} \quad \forall \lambda \in S_{\theta_0} - \{0\}, \quad s, t \in [0, T].$$

REMARK 7.8. It is easily seen, by $(1.6)_3$ and the inclusions

$$D_{[-A(t)]^\varrho} \hookrightarrow D_{A(t)}(\varrho, \infty) \hookrightarrow D_{[-A(t)]^\sigma} \quad \forall \sigma \in]0, \varrho[$$

(see Triebel [20, formulae 1.15.2 (3) and 1.13.2 (3*a*)]), that Hyp. 7.5 implies Hyp. 7.6 and, conversely, Hyp. 7.6 implies Hyp. 7.5 with ϱ replaced by any smaller σ. Similarly it is easy to show that Hyp. 7.4, as well as Hyp. 7.5, is stronger than Hyp. 7.7. Finally we note that Hyp. 7.6 implies Hyp. 7.4 (i)-(ii) (but not (iii)), in view of the identity

$$\frac{\partial}{\partial t} R(\lambda, A(t)) = A(t) R(\lambda, A(t)) \frac{d}{dt} A(t)^{-1} A(t) R(\lambda, A(t)). \tag{7.1}$$

In order to analyze the connections between Hypotheses 7.1, ..., 7.7 and our Hypothesis II, we divide such assumptions into two classes, (A) and (B): class (A) consists of Hypotheses 7.1, 7.2, 7.3, 7.5 and 7.6, whereas class (B) contains Hypotheses 7.4 and 7.7. We have the following result:

THEOREM 7.9. *Assume Hypothesis* I. *Then Hypothesis* II *is weaker than any assumption from class* (A), *and is independent of any assumption from class* (B).

PROOF. Clearly, Hyp. 7.1 is stronger than Hypothesis II in view of

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\|_{\mathfrak{L}(E)} =$$
$$= \|R(\lambda, A(t))[1 - A(t)A(s)^{-1}]\|_{\mathfrak{L}(E)} \leqslant \frac{MB}{1 + |\lambda|} |t - s|^{\alpha} .$$

Assume now Hyp. 7.2, set $m = \varrho^{-1}$ and write

$$(7.2) \qquad A(t)^{-1} - A(s)^{-1} =$$
$$= -\sum_{j=1}^{m} [-A(t)]^{-j/m} \{[-A(t)]^{1/m} [-A(s)]^{-1/m} - 1\} [-A(s)]^{-1+j/m};$$

as $D_{[-A(t)]^{\varrho}} \hookrightarrow D_{A(t)}(\varrho, \infty)$, by $(1.6)_3$ and (7.2) we easily obtain for each $x \in E$

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]x\|_E \leqslant \frac{1}{|\lambda|^{\varrho}} \|[A(t)^{-1} - A(s)^{-1}]x\|_{D_{A(t)}(\varrho,\infty)} \leqslant$$
$$\leqslant \frac{c}{|\lambda|^{\varrho}} \|[A(t)^{-1} - A(s)^{-1}]x\|_{D_{[-A(t)]^{\varrho}}} \leqslant \frac{cB}{|\lambda|^{\varrho}} (t - s)^{\alpha} \|x\|_E .$$

Similarly it is clear that Hyp. 7.3 implies

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\|_{\mathfrak{L}(E)} \leqslant \frac{B}{|\lambda|^{\varrho}} (t - s)^{\alpha} .$$

Finally, by Remark 7.8, Hyp. 7.5 is stronger than Hyp. 7.6; on the other hand if Hyp. 7.6 holds, then we can write:

$$A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}] =$$
$$= \int_s^t [A(t)R(\lambda, A(t)) - A(r)R(\lambda, A(r))] \frac{d}{dr} A(r)^{-1} dr +$$
$$+ \int_s^t A(r)R(\lambda, A(r)) \frac{d}{dr} A(r)^{-1} dr =$$
$$= \int_s^t \int_r^t \lambda \frac{\partial}{\partial \sigma} R(\lambda, A(\sigma)) \frac{d}{dr} A(r)^{-1} d\sigma\, dr + \int_s^t A(r)R(\lambda, A(r)) \frac{d}{dr} A(r)^{-1} dr ,$$

and hence by Hyp. 7.4 (ii) (which is true because of (7.1)) we get

$$\|A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]\|_{\mathfrak{L}(E)} \leqslant \frac{K^2}{2}|\lambda|^{1-\varrho}(t-s)^2 + \frac{B}{|\lambda|^{\varrho}}(t-s).$$

The first part of the proof is complete.

In order to prove the second statement, we need two lemmata.

LEMMA 7.10. *Under Hypotheses* I, II, *if in addition* $t \mapsto A(t)^{-1} \in C^1(\mathfrak{L}(E))$ *then the range of* $(d/dt)A(t)^{-1}$ *satisfies*

$$R\left(\frac{d}{dt}A(t)^{-1}\right) \subseteq \overline{D_{A(t)}} \quad \forall t \in [0, T].$$

PROOF. Let $x \in E$. For $0 \leqslant t < t+h \leqslant T$ we have:

$$\begin{aligned}
&\frac{A(t+h)^{-1} - A(t)^{-1}}{h}x = \\
&\quad = \frac{1}{h}R\left[\left(\frac{1}{h}, A(t+h)\right) - R\left(\frac{1}{h}, A(t)\right)\right]\frac{A(t+h)^{-1} - A(t)^{-1}}{h}x + \\
&\quad + \frac{1}{h}R\left(\frac{1}{h}, A(t)\right)\left[\frac{A(t+h)^{-1} - A(t)^{-1}}{h} - \frac{d}{dt}A(t)^{-1}\right]x + \\
&\quad + \frac{1}{h}R\left(\frac{1}{h}, A(t)\right)\frac{d}{dt}A(t)^{-1}x - \\
&\qquad - \frac{1}{h}A(t+h)R\left(\frac{1}{h}, A(t+h)\right)[A(t+h)^{-1} - A(t)^{-1}] = \sum_{i=1}^{4} I_i,
\end{aligned}$$

and it is easy to see, by Lemma 1.9 (ii) and Hypothesis II, that

$$\|I_1\|_E + \|I_2\|_E + \|I_4\|_E = o(1) \quad \text{as } h \to 0^+,$$

whereas, clearly, $I_3 \in D_{A(t)}$. Thus we get

$$\frac{d}{dt}A(t)^{-1}x = \lim_{h \to 0^+} I_3 \in \overline{D_{A(t)}}.$$

A similar and even simpler argument leads to the same result when $t = T$. ■

LEMMA 7.11. *Let* $z \in \mathbb{C}$, $f \in C([0,1], \mathbb{C})$, $\beta \in C([0,T],]0, \infty[)$. *For each* $t \in [0,T]$, $\theta_0 \in]\pi/2, \pi[$ *and* $\lambda \in S_{\theta_0}$ *there exists a unique solution* $u \in C^2([0,1], \mathbb{C})$ *of the problem*

$$(7.3) \qquad \begin{cases} \lambda u(x) - u''(x) = f(x)\,, & x \in [0,1] \\ u(0) = 0 \\ u(1) + \beta(t) u'(1) = z; \end{cases}$$

moreover we have (*denoting by* $\|\cdot\|_\infty$ *the usual sup-norm*):

$$[1 + |\lambda|]\|u\|_\infty + [1 + |\lambda|^{\frac{1}{2}}]\|u'\|_\infty + \|u''\|_\infty \leqslant$$
$$\leqslant C\{\|f\|_\infty + [1 + |\lambda|^\nu]|z|\} \qquad \forall \lambda \in S_{\theta_0}\,,$$

where C *depends on* θ_0 *and* $\|\beta\|_\infty$ *but is independent of* t, *and*

$$\nu = \begin{cases} 1 & \text{if } \min\limits_{t \in [0,T]} \beta(t) = 0 \\ \frac{1}{2} & \text{if } \min\limits_{t \in [0,T]} \beta(t) > 0\,. \end{cases}$$

PROOF. See Acquistapace-Terreni [4, Prop. 3.1]. ■

We are now ready to prove the second part of Theorem 7.9. Let us show that Hypothesis II does not imply Hyp. 7.7. Take in (7.3)

$$(7.4) \qquad \beta(t) = 1 + t^{\frac{2}{3}};$$

then if we set

$$(7.5) \qquad E = C([0,1], \mathbb{C})\,,$$

$$(7.6) \qquad \begin{cases} D_{A(t)} = \{u \in C^2([0,1], \mathbb{C}) \colon u(0) = 0,\ u(1) + \beta(t) u'(1) = 0\} \\ A(t)u = u'' \end{cases}$$

then Problem 7.3 is a particular case of (0.1) and, by Lemma 7.11, Hypothesis I is fulfilled. In order to verify that Hypothesis II holds too, fix $0 \leqslant s \leqslant t \leqslant T$, pick $f \in E$ and set

$$v = A(s)^{-1} f\,, \qquad u = R(\lambda, A(t))[\lambda - A(s)]v\,.$$

Then it is easily seen that v and u solve respectively

$$\begin{cases} v'' = f & \text{in } [0,1] \\ v(0) = 0 \\ v(1) + \beta(s)v'(1) = 0\,, \end{cases} \qquad \begin{cases} \lambda u - u'' = \lambda v - f & \text{in } [0,1] \\ u(0) = 0 \\ u(1) + \beta(t)u'(1) = 0\,, \end{cases}$$

consequently the function

$$w = A(t)R(\lambda, A(t))[A(t)^{-1} - A(s)^{-1}]f = v - u$$

solves

$$\begin{cases} \lambda w - w'' = 0 & \text{in } [0,1] \\ w(0) = 0 \\ w(1) + \beta(t)w'(1) = [\beta(s) - \beta(t)]v'(1)\,. \end{cases}$$

Hence by Lemma 7.11 and (7.4) we deduce that

$$[1 + |\lambda|]\,\|w\|_\infty \leqslant c[1 + |\lambda|^{\frac{1}{2}}]|\beta(s) - \beta(t)||v'(1)| \leqslant c[1 + |\lambda|^{\frac{1}{2}}](t-s)^{\frac{3}{2}}\|f\|_\infty\,,$$

so that Hypothesis II is fulfilled.

On the other hand, as β is not differentiable it is easy to see that $t \to A(t)^{-1} \notin C^1(\mathfrak{L}(E))$, since

$$(7.7) \qquad [A(t)^{-1}f](x) = \int_0^x (x-y)f(y)\,dy - \\ -\left[\int_0^1 f(y)\,dy - \frac{1}{1+\beta(t)}\int_1^0 yf(y)\,dy\right]\cdot x\,, \qquad x \in [0,1]\,.$$

In view of Remark 7.8, this shows that no assumption from class (B) can hold.

Conversely, let us show that Hyp 7.4 (and hence Hyp. 7.7 too) does not imply Hypothesis II. Choose $\beta(t) = t$, and let E, $\{A(t)\}$ be given by (7.5), (7.6). By (7.7) it is clear that

$$(7.8) \qquad \left[\frac{d}{dt}A(t)^{-1}f\right](x) = -\frac{1}{(1+t)^2}\int_0^1 yf(y)\,dy\cdot x\,, \qquad x \in [0,1]\,,$$

so that Hypothesis 7.4 (iii) holds with any $\eta \in]0, 1[$; by Acquistapace-Terreni [4, Prop. 3.2] we get Hypothesis 7.4 (i)-(ii) (with $\varrho = \frac{1}{2}$). Now suppose by contradiction that Hypothesis II also holds: then by Lemma 7.10 we get

$$R\left(\left[\frac{d}{dt} A(t)^{-1}\right]_{t=0}\right) \subseteq \overline{D_{A(0)}} = \{u \in C([0, 1], \mathbb{C}) : u(0) = u(1) = 0\};$$

but if we take $f \equiv 1$, then (7.8) yields

$$\left[\left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} f\right](x) = -\frac{x}{2}, \quad x \in [0, 1],$$

and this function does not belong to $\overline{D_{A(0)}}$.

Theorem 7.9 is completely proved. ■

REMARK 7.12. Although the assumptions from class (B) are independent of ours, it is to be noted that they always require continuous differentiability for the map $t \mapsto R(\lambda, A(t))$, so that from this point of view our hypotheses are indeed less restrictive. In addition, as we have already remarked, Hypotheses I and II make it possible to use a unified method for solving (0.1) in any situation (i.e. constant domains, variable domains, intermediate cases), with minor smoothness assumptions on the resolvent operator.

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