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Baer-Elation Planes.

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SUMMARY - Translation planes of even order q^2 and kernel $GF(q)$ that admit an elation group (Baer group) of order q and a non-trivial Baer group (Elation group) are studied. A classification of these « Baer-Elation » planes is determined. Aside from the classifications, the main result is that translation planes of order q^2 and kernel $GF(q)$ which admits a Baer group of order q and elations with at least two axes (in the translation complement) are the translation planes of Hall.

1. Introduction.

A translation plane π of order p^{2r} will be said to be a *Baer-Elation* plane if an only if there exists a nontrivial Baer p-group and a nontrivial elation group in the translation complement.

By Foulser [2], any Baer-Elation plane must be of even order. Also, there are quite a number of examples of Baer-Elation planes. For example, the Hall and Desarguesian plane are Baer-Elation planes. Biliotti-Menichetti [1] have studied translation planes which are derived from semifield planes and which admit elations with more than one axis. The number of elation axes -1 gives the kernel of the

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plane. Hence, (see also Johnson-Rahilly [9]) the only such plane of dimension 2 (kernel $GF(q)$, order q^2) admits $1+q$ elation axes. The plane turns out to be Hall by a result of Johnson and Rahilly. Translation planes of order q^2 which are derived from semifield planes admit a Baer collineation group of order q. However, the existence of a Baer group does not (or may not) imply that the plane is derivable or even, *if so*, *derivable* from a semifield plane

(1.1) DEFINITION. Let π be a translation plane of even order q^2. Let $\mathcal{B}$ be a 2-group which fixes a Baer subplane pointwise of order 2^b and $\mathcal{E}$ an elation group of order 2^e where $\mathcal{B}$, $\mathcal{E}$ are subgroups of the translation complement. We assume $\mathcal{E}$ normalizes $\mathcal{B}$. We call a plane π with groups $\mathcal{B}$ and $\mathcal{E}$ above, a *Baer-Elation plane of type* $(2^b, 2^e)$. Note that $2^b, 2^e \leqslant q$ (see Foulser [2] for $2^b \leqslant q$). Also, note that if $\mathcal{E}$ normalizes $\mathcal{B}$ then $\mathcal{E}$ centralizes $\mathcal{B}$.

When one of the groups $\mathcal{B}$ or $\mathcal{E}$ is large, the other group tends to be small. For example, we note the following:

(1.2) THEOREM (Jha-Johnson [4], [5]).

(1) Let π be a Baer-Elation plane of even order q^2 and type $(2^b, q)$. Then $b = 1$.

(2) Let π be a Baer-Elation plane of even order q^2 and type $(\geqslant 2\sqrt{q}, 2^e)$. Then $e = 1$. (Also, note that it is not necessary to assume that $\mathcal{E}$ normalizes $\mathcal{B}$ in this case. See Jha-Johnson [7].)

In this article, we consider Baer-Elation planes of order q^2 and type $(q, 2)$ or $(2, q)$. And, although many of our arguments may be extended for planes or arbitrary dimension, we consider only those planes of dimension 2 but we make no assumption as to the possible derivation of these planes.

In sections 2 and 3, we consider the classification of $(2, q)$ or $(q, 2)$ planes of order q^2 and dimension 2 (note, we assume the groups are in the *linear* translation complement in the $(2, q)$-situation).

In section 4, we consider planes of type $(q, 2)$ with several elation axes (or type $(2, q)$ with several Baer axes). Here we obtain the rather surprising result that the plane must be Hall (or type $(2, q)$ is Desarguesian). (Contrast this result with the work of Biliotti-Menichetti [1] and Johnson-Rahilly [9].)

This paper probably raises more questions then it answers and several problems and questions are listed in section 5.

2. The structure of Baer-elation planes of order q^2 and type $(q, 2)$.

(2.1) Let p be a translation plane of even order q^2 and kernel F isomorphic to $GF(q)$. Assume that π is a $(q, 2)$-plane with collineation groups $\mathcal{B}$ and $\mathcal{E}$ in the translation complement such that $\mathcal{B}$ is a Baer 2-group (fixes a Baer subplane π_0 pointwise) and $\mathcal{E}$ is an elation group with axis $\mathcal{L}$.

Assume the conditions of (2.1) in the following.

(2.2) NOTES.

(1) $\mathcal{B}$ centralizes $\mathcal{E}$.

(2) $\mathcal{B}$ is in the *linear* translation complement.

PROOF 1. By Jha-Johnson [4] if $|\mathcal{B}| \geqslant 2\sqrt{q}$ then any elation group has order $\leqslant 2$. Hence, if $\mathcal{B}$ does not centralize $\mathcal{E}$ then $\mathcal{B}$ must move the axis of $\mathcal{E}$ so that $\mathcal{E}$ cannot, in turn, normalize $\mathcal{B}$. In this case, there are at least two Baer groups of order $|\mathcal{B}|$ and by Jha-Johnson [7], the plane must be Hall.

PROOF 2. If $|\mathcal{B}| > 2$ then by Foulser [2], the fixed point subplane of $\mathcal{B}$ must be a F-subspace. Since, in any case, $\mathcal{B} \subseteq \Gamma L(4, F)$ this forces $\mathcal{B}$ to be in $GL(4, F)$, $\mathcal{B}$ is in the linear translation complement.

Let $\mathcal{N}$ denote the net of π of degree $q + 1$ which is defined by π_0. That is, the components of π_0 are components of $\mathcal{N}$.

(2.3) LEMMA. Coordinates may be chosen so that

$$\pi = \{(x_1, x_2, y_1, y_2): x_i, y_i \in F, i = 1, 2\}, \quad \pi_0 = \{(0, x_2, 0, y_2): x_2, y_2 \in F\}$$

and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} : a \in F \right\}.$$

Furthermore, the components of $\mathcal{N}$ (see (2.2) and following) have the form

$$\{(0, 0, y_1 y_2): y_i \in F, i = 1, 2\} \equiv (x = 0),$$

and f is a function on F such that

$$\left\{\left(x_1, x_2, (x_1, x_2)\begin{bmatrix} a & f(a) \\ 0 & a \end{bmatrix}\right): a \text{ fixed} \in F,\ x_1, x_2 \in F \text{ and } f \text{ is a function on } F \text{ such that } f(0) = f(1) = 0\right\} \equiv \left(y + x\begin{bmatrix} a & f(a) \\ 0 & a \end{bmatrix}\right).$$

When convenient, we write $(x_1, x_2, y_1, y_2) = (x, y)$.

Proof. Choose components belonging to π_0 as $(x = \mathcal{O})$, $(y = \mathcal{O})$, and $(y = x)$. $\mathcal{B}$ is in the linear translation complement, fixes

$$\pi_0 = \{(0, x_2, 0, y_2): x_2, y_2 \in F\}$$

pointwise and is elementary abelian. All of this implies that

$$\mathcal{B} = \left\{\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}, a \in F\right\}.$$

The components of π_0 on π_0 clearly have the form

$$x = \mathcal{O}, \qquad y = x\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

for various elements a, b, c in F. However, b and c must be determined by a so that $b = f(a)$ and $c = g(a)$ for functions $f, g: F \to F$. By using the form of $\mathcal{B}$ we have that

$$\begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{fixes } y = x\begin{bmatrix} a & f(a) \\ 0 & g(a) \end{bmatrix}$$

if and only if

$$\left(x\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, x\begin{bmatrix} a & f(a) \\ 0 & g(a) \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) \in \left(y = x\begin{bmatrix} a & f(a) \\ 0 & g(a) \end{bmatrix}\right).$$

This implies $ab = bg(a)$ so that $a = g(a)$.

(2.4) LEMMA. Let $\mathcal{E} = \langle\sigma\rangle$ denote the elation group of order 2. Then coordinates may be chosen so that

$$\sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

PROOF. σ centralizes $\mathcal{B}$. Hence, choose the axis of σ to be $x = \mathcal{O}$ then σ maps $y = \mathcal{O}$ onto $y = x\begin{bmatrix} \bar{a} & f(\bar{a}) \\ 0 & \bar{a} \end{bmatrix}$. A basis change so that this latter component is $y = x$ would not alter the form of $\mathcal{B}$ so as σ has the form $\begin{bmatrix} I & C \\ \mathcal{O} & I \end{bmatrix}$ clearly $C = I$ $\left(\text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

(2.5) LEMMA. $\mathcal{B}\mathcal{E}$ contains precisely $q-1$ Baer involutions τ_a for $a \in F - \{0\}$ not in $\mathcal{B}$. Each Baer involution fixes pointwise a Baer subplane π_a which shares precisely the component $x = \mathcal{O}$ with π_b for $a \neq b$.

PROOF. $|\mathcal{B}\mathcal{E} - \mathcal{B} - \mathcal{E}| = q - 1$ and $\mathcal{B}\mathcal{E}$ is elementary abelian. The $q-1$ Baer involutions are

$$\tau_a = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} 1 & a & 1 & a \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{for } a \neq 0 .$$

(2.6) LEMMA. If

$$\tau_a = \begin{bmatrix} 1 & a & 1 & a \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{for } a \neq 0$$

then the components of π_a have the following form:

$$x = \mathcal{O}, \qquad y = x\begin{bmatrix} u & G(u, a^{-1}) \\ a^{-1} & u + 1 \end{bmatrix}$$

where G is a function from $F \times (F - \{0\})$ to F.

PROOF. A component

$$y = x \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

is fixed by τ_a if and only if

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

Equating entries we have:

(1,1): $m_1 + am_4 = a + m_1 a + m_2$,

(1,2): $m_2 + am_4 = a + m_1 a + m_2$.

Hence, $m_3 a = 1$ and $m_4 = m_1 + 1$. Since there are q components $\neq (x = \mathcal{O})$ which are fixed by τ_a, m_1 takes on all possible entries in F. Thus, m_2 depends uniquely on a^{-1} and m_1 as any two matrices in F. Thus, m_2 depends uniquely on a^{-1} and m_1 as any two matrices which define components have nonsingular differences. Let $G(0, a^{-1}) = g(a^{-1})$.

Hence, we have:

(2.7) Lemma. There exist exactly $(q - 1)$ components of the form

$$y = x \begin{bmatrix} 0 & g(a^{-})^1 \\ a^{-1} & 1 \end{bmatrix} \qquad \text{for } a^{-1} \in F - \{0\}.$$

The components of π_a for $a \neq 0$ have the form:

$$x = \mathcal{O}, \qquad y = x \begin{bmatrix} ba^{-1} & b^2 a^{-1} + b + g(a^{-1}) \\ a^{-1} & ba^{-1} + 1 \end{bmatrix} \qquad \text{for all } b \in F.$$

PROOF. $\mathcal{B}\mathcal{E}$ is elementary abelian so $\mathcal{B}$ must leave each subplane π_a invariant and act transitively on the components not equal to $(x = \mathcal{O})$. Hence, the $\mathcal{B}$-images of

$$y = x \begin{bmatrix} 0 & g(a^{-1}) \\ a^{-1} & 1 \end{bmatrix}$$

have the following form:

$$y = x\begin{bmatrix} 0 & g(a^{-}) \\ a^{-1} & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$y = x\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & g(a^{-1}) \\ a^{-1} & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} =$$

$$= \left(y = x\begin{bmatrix} ba^{-1} & g(a^{-1}) + b + b^2a^{-1} \\ a^{-1} & ba^{-1} + 1 \end{bmatrix}\right).$$

Thus, we may give the basic structure for $(q, 2)$-planes of dimension 2 as follows:

(2.8) **Theorem.** Let π be a translation plane of even order q^2 and kernel F isomorphic to $GF(q)$. If π admits a Baer group of order q and a nontrivial elation group then there exists a coordinatization so that the following subspace define a spread of π: There exist functions $f, g\colon F \to F$ such that

(1) $f(a) = f(a + 1)$, for all $a \in F$; $f(0) = g(0) = 0$,

(2) $d + d^2 + f(d)a^{-1} \neq g(a^{-1})a^{-1}$, for all $d, a \neq 0$ in F,

(3) $g(d) + g(c) \neq (1 + t(dc/c + d))$ for all $0 \neq d \neq c \neq 0$, t of F

and the components for π have the form:

$$x = \mathcal{O}, \qquad y = x\begin{bmatrix} u & f(u) \\ 0 & u \end{bmatrix},$$

$$y = x\begin{bmatrix} ba^{-1} & b^2a^{-1} + b + g(a^{-1}) \\ a^{-1} & ba^{-1} + 1 \end{bmatrix} \qquad \text{for all } u, b, a \neq 0$$

of F and $(x, y) = (x_1, x_2, y_1, y_2)$ for $x_i, y_i \in F$, $i = 1, 2$. Conversely, if there are functions f, g on a field F isomorphic to $GF(q)$ satisfiyng (1), (2), (3) then a translation plane of order q and type $(q, 2)$ may be obtained.

Proof. By Jha-Johnson ([7], [5]), $\mathcal{B}$ centralizes $\mathcal{E}$ and $|\mathcal{E}| = 2$. $f(s + 1) = f(s)$ since $\mathcal{E}$ exists. The conditions given in 2), 3) are the requirements that the matrices and the differences of any two are nonsingular.

That is,

$$\begin{bmatrix} d & f(d) \\ 0 & d \end{bmatrix} + \begin{bmatrix} ba^{-1} & b^2a^{-1} + b + g(a^{-1}) \\ a^{-1} & ba^{-1} + 1 \end{bmatrix}$$

must have nonzero determinant so that

$$(d + ba^{-1})(ba^{-1} + 1 + d) + \big(f(d) + b^2a^{-1} + b + g(a^{-1})\big)a^{-1} \neq 0$$

for all $d, a \neq 0$ in F.

This expression reduces to $\big(f(d) + g(a^{-1})\big)a^{-1} \neq d(d+1)$ which is (2). Also,

$$\begin{bmatrix} bd & g(d) + b^2d + b \\ d & bd + 1 \end{bmatrix} + \begin{bmatrix} rk & g(k) + r^2k + r \\ k & rk + 1 \end{bmatrix} \qquad \text{for } dk \neq 0$$

has nonzero determinant if and only if

$$(bd + rk)^2 + (d + k)\big(g(d) + b^2d + b + g(k) + r^2k + r\big) \neq 0 .$$

Let $d = k$. Then, for distinct matrices we must have $b \neq r$ so we want $k^2(b + r)^2 \neq 0$ which is automatically satisfied. Hence, nonsingularity is guaranteed if $d = k$. So assuming $d \neq k$ and letting $b + r = t$ we have $g(d) + g(k) \neq t\big(1 + t(dk/d + k)\big)$. This is condition (3).

(2.9) COROLLARY. Under the conditions of (2.8), if f is identically zero then π is derivable and a derived plane is a $(2\ q)$-plane which has components:

$$x = 0\,, \qquad y = x\begin{bmatrix} b & g(a^{-1}) \\ a & a + b \end{bmatrix} \qquad \text{for all } a, b \in F$$

where $g(0) = 0$ and

$$g(d) + g(k) \neq t\big(1 + t(dk/d + k)\big) \qquad \text{for } 0 \neq d \neq k \neq 0\,,\ t \in F\,.$$

PROOF Derive the net $\mathcal{N}$. Now apply Jha-Johnson [8].

(2.10) THEOREM. Let π be a translation plane of even order q^2 and kernel F isomorphic to $GF(q)$. Assume π admits a Baer group of order q and a nontrivial elation group. Let the net defined by the Baer axis be denoted by $\mathcal{N}$.

(1) Then the components of $\pi - \mathcal{N}$ uniquely determine $\mathcal{N}$.

(2) Given a function $g\colon F \to F$ satisfying condition (3) of (2.8), $\big(g(d) + g(c) \neq t(1 + t(dc/c + d))$ for $0 \neq c \neq d \neq 0$, $t \in F$ and $g(0) = 0\big)$

then there exists at most one function f satisfying (1) and (2) of (2.8) $\big(\bar{f}(a) = f(a) + 1$ and $d + d^2 + f(d)a^{-1} \neq g(a^{-1})a^{-1}$ for all d, $a \neq 0$ in $F\big)$.

Proof. Suppose $\pi = \mathcal{N} \cup \mathcal{M}$ and $\pi_1 = \mathcal{N}_1 \cup \mathcal{M}$ where

$$\mathcal{N}\colon x = \mathcal{O}\ ,\ y = x\begin{bmatrix} a & f(a) \\ 0 & a \end{bmatrix} \text{ and } \bar{\eta}\colon x = \mathcal{O}\ ,\ y = x\begin{bmatrix} a & \bar{f}(a) \\ 0 & a \end{bmatrix}$$

for all $a \in F$. Then $\mathcal{N}$ and $\mathcal{N}_1$ are mutual replacements so that $\mathcal{N}$ is either equal to $\mathcal{N}_1$ or $\mathcal{N}_1$ is the derived net of $\mathcal{N}$. But then π_1 is also a $(2, q)$-plane which cannot be the case by Jha-Johnson [5].

Hence, if a plane exists, the function g uniquely determines the function f.

(2.11) Corollary. Under the conditions of (2.10), if $g(a) = t_0/a$ with $Tr\,(t_0) \neq 0$ for $a \neq 0$ then f is identically zero and the corresponding plane is a Hall plane.

Proof. Condition (3) for g becomes $t_0/d = t_0/c \neq t\big(1 + t(dc/(d + g))\big)$ for $d \neq c$. Let $dc/(d + c) = Z$. Then $t_0/Z \neq t(1 + tZ)$. Letting $tZ = x$, $t_0 \neq x^2 + x$. That is, $x^2 + x + t_0$ is irreducible over F, there is a corresponding Hall plane with components

$$y = x\begin{bmatrix} ba^{-1} & b^2a^{-1} + b + t_0 a \\ a^{-1} & ba^{-1} + 1 \end{bmatrix} \qquad \text{for } a \neq 0 \text{ in } F$$

and

$$y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \qquad \text{for } u \text{ in } F\,.$$

Hence, by (2.10) f is uniquely determined as the zero function.

3. The structure of Baer-elation planes of order q^2 and type $(2, q)$.

(3.1) Let π be a translation plane of even order q^2 and kernel F isomorphic to $GF(q)$. Assume that π is a $(2, q)$-plane with collineation groups $\mathcal{B}$ and $\mathcal{E}$ in the linear translation complement where $\mathcal{B}$ is a Baer 2-group of order 2, $\mathcal{E}$ is an elation group of order q which centralizes $\mathcal{B}$. Let $\mathcal{B}$ fix π_0 pointwise and let $\mathcal{E}$ have axis $\mathcal{L}$.

Let $\mathcal{N}$ denote the net of π of degree $q+1$ which is defined by π_0.

Assume the conditions of (3.1) in the following.

(3.2) LEMMA. Coordinates may be chosen so that

$$\pi = \{(x_1, x_2, y_1, y_2) : x_i, y_i \in F, i = 1, 2\},$$

$$\pi_0 = \{(0, x_2, 0, y_2) : x_2, y_2 \in F\}, \qquad \mathcal{B} = \langle \tau \rangle .$$

$$\tau = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 & u & m(u) \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : u \in F \right\}$$

and m is an additive function on F such that $m(0) = m(1) = 0$.

PROOF. Choose $x = \mathcal{O}$, $y = \mathcal{O}$, $y = x$ in π_0 and change bases, if necessary so that τ has the required form. As in (2.3), the components of $\mathcal{N}$ have the form

$$x = \mathcal{O}, \qquad y = x \begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix}$$

for some function m on F such that $m(0) = m(1) = 0$. Hence, $\mathcal{E}$, being transitive on $\mathcal{N} - (x = \mathcal{O})$, has the form

$$\left\{ \begin{bmatrix} I & \begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix} \\ \mathcal{O} & I \end{bmatrix} \quad u \in F \right\}.$$

However, as $\mathcal{E}$ is a group, it follows that m is an additive function.

(3.3) LEMMA. There exist functions g, f on F where f is $1-1$ such that

$$y = x\begin{bmatrix} g(v) & f(v) \\ v & 0 \end{bmatrix}$$

is a component for all $v \in F$.

PROOF. Consider an arbitrary component

$$y = x\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \text{ then } \{[m_3, m_4]\} = F \times F .$$

Hence, consider the components

$$y = x\begin{bmatrix} - & - \\ v & 0 \end{bmatrix}.$$

Since $[v, 0]$ completely determines the (1,1) and (1,2)-entries, we must have functions of v, g, f such that the (1,1)-entry is $g(v)$ and the (1,2)-entry is $f(u)$. And,

$$\begin{bmatrix} g(u) & f(u) \\ u & 0 \end{bmatrix} + \begin{bmatrix} g(v) & f(v) \\ v & 0 \end{bmatrix}$$

is nonsingular so that f is $1-1$.

(3.4) LEMMA. The components of π have the form

$$x = O , \qquad y = x\begin{bmatrix} u + g & f(v) + m(u) \\ v & u \end{bmatrix}.$$

PROOF. Apply the group $\mathcal{E}$ to

$$y = x\begin{bmatrix} g(v) & f(v) \\ v & 0 \end{bmatrix}.$$

(3.5) LEMMA. $\mathcal{E}$ contains exactly q Baer involutions $= \tau\mathcal{E} =$

$$= \left\{ \tau\sigma_a = \begin{bmatrix} 1 & 1 & a & m(a) + a \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} : a \in F \right\}.$$

PROOF. $\mathcal{BE}$ is elementary abelian with Baer involutions $\mathcal{BE} - \mathcal{E}$.

(3.6) LEMMA. $g(a) = m(a) + a$.

PROOF. $\tau\sigma_a$ fixes the component

$$y = x\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

if and only if $a = m_3$ and $m_1 = m_4 + m(a) + a$. (See the argument to (2.6)). If $m_4 = u$ then, by (3.4), $u + g(a) = m_1 = u + M(a) + a$. Hence, $g(a) = m(a) + a$.

Thus we have:

(3.7) THEOREM. Let π be a translation plane of even order q^2 and kernel F isomorphic to $GF(q)$. Let π admit an elation group $\mathcal{E}$ of order q and a nontrivial Baer 2-group such that $\mathcal{E}$ normalizes $\mathcal{B}$ and $\mathcal{E}$, $\mathcal{B}$ are in the linear translation complement. Then there exist functions f, m on F such that

1) f is $1-1$.

2) m is additive and $m(0) = m(1) = 0$.

3) $$\begin{bmatrix} u + v + m(u) & f(v) + m(u) \\ v & u \end{bmatrix} + \begin{bmatrix} a + b + m(a) & f(b) + m(a) \\ b & a \end{bmatrix}$$

is nonsingular when $u \neq a$ or $v \neq b$ and the components of π may be represented in the form

$$x = 0\,, \qquad y = x\begin{bmatrix} u + v + m(v) & f(v) + m(u) \\ v & u \end{bmatrix} \qquad \text{for all } u, v \in F\,.$$

Conversely, functions satisfying the above conditions give rise to a translation plane of order q^2 and type $(2, q)$.

PROOF. By Jha-Johnson [4], $|\mathcal{B}| = 2$. By the various lemmas (3.2)-(3.6), we have the proof of (3.7).

(3.8) **Theorem.** Let π be a translation plane of even order q^2 and dimension 2 which is a $(2, q)$-plane. If the net defined by the Baer subplane is derivable then coordinates may be chosen so that the components for π have the form

$$x = \mathcal{O}\,, \qquad y = x\begin{bmatrix} u+v & f(v) \\ v & u \end{bmatrix}$$

for all $u, v \in F$ where f is a $1-1$ function on F and $x^2 + x + f(v)/v$ is irreducible for all $v \neq 0$ of F.

Conversely, a $1-1$ function f such that $x^2 + x + f(v)/v$ is irreducible for all $v \neq 0$ gives rise to a $(2, q)$-plane of order q^2 which is derivable.

Proof.

$$y = x\begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix}, \qquad x = \mathcal{O}\,,$$

derivable implies $m \equiv 0$ since m is additive and derivability implies m is also multiplicative. But, $m(1) = 0$ so that $m \equiv 0$. Now apply (3.7).

Note that for derivable planes with $m \equiv \mathcal{O}$, we have the connection between the $(q, 2)$ and $(2, q)$-spreads (as noted in Jha-Johnson [8])

$$\underbrace{\begin{gathered} x = \mathcal{O} \\ y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \\ y = x\begin{bmatrix} ba^{-1} & b^2a^{-1} + b + g(a^{-1}) \\ a^{-1} & ba^{-1} + 1 \end{bmatrix} \end{gathered}}_{(q,2)\text{-plane}} \quad \longleftrightarrow \quad \underbrace{\begin{gathered} x = \mathcal{O} \\ y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \\ y = x\begin{bmatrix} a+b & g(a^{-1}) \\ a & b \end{bmatrix} \end{gathered}}_{(2,q)\text{-plane}}$$

where $f(a) = g(a^{-1})$.

Also, we may actually derive any elation orbit union $(x = \mathcal{O})$ (in a $(2, q)$-derivable plane) as noted in Jha-Johnson [6].

If we change bases so that

$$y = x\begin{bmatrix} a_0 & f(a_0) \\ a_0 & 0 \end{bmatrix} \qquad \text{for } a_0 \neq 0$$

takes the place $y = \mathcal{O}$ and the orbit of $\mathcal{E}$ union $x = \mathcal{O}$ becomes a derivable net, we obtain the corresponding function

$$f_{a_0}(a) = f(a_0 + a) + f(a_0) .$$

That is, there are also the coordinates

$$x = \mathcal{O} , \qquad y = x\begin{bmatrix} a+b & f_{a_0}(a) \\ a & b \end{bmatrix}, \quad y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$

for the $(2, q)$-plane where $f_{a_0}(a) = f(a_0 + a) + f(a_0)$ for a_0 fixed in F and for all a in F. Let $g_{a_0}(a^{-1}) = f_a(a)$. The corresponding $(q, 2)$-plane has coordinates,

$$x = \mathcal{O} , \qquad y = x\begin{bmatrix} ba^{-1} & b^2a^{-1} + b + g_{a_0}(a^{-1}) \\ a^{-1} & ba^{-1} + 1 \end{bmatrix}.$$

As $g_{a_0}(a^{-1}) = g(a_0 + a)^{-1} + g(a_0^{-1})$, this second $(q, 2)$-plane may not be isomorphic to the original. Again, see Jha-Johnson [6] for a few more details regarding this construction.

4. Type $(q, 2)$-planes of dimension 2 with several elation axes.

(4.1) Assume π is a translation plane of order q^2 with kernel F isomorphic to $GF(q)$. Assume π admits a Baer group $\mathcal{B}$ of order q admits elations in the translation complement with at least two axes.

For the following, assume the conditions of (4.1).

(4.2) Lemma. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be distinct nontrivial elation groups with axes $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \neq \mathcal{L}_2$. Then $\mathcal{B}$ centralizes $\mathcal{E}_1$, $\mathcal{E}_2$ and $|\mathcal{E}_1| = |\mathcal{E}_2| = 2$. Also, $\mathcal{B}$ is in the linear translation complement.

Proof. Jha-Johnson [4]. Note π_0 is a kernel subplane by Foulser and so the kernel of π_0 is the kernel of π (or π is Desarguesian).

Assune π, π_0 and $\mathcal{E}_1$ and $\mathcal{B}$ have the form given in section 2 (2.3), (2.4). Assume, without loss of generality that $\mathcal{E}_2$ has axis $y = \mathcal{O}$ and $\mathcal{E}_2 = \langle \varrho \rangle$ where

$$\varrho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & a_2 & 1 & 0 \\ a_3 & a_4 & 0 & 1 \end{bmatrix} \qquad \text{for } a_i \in F , \ i = 1, 2, 3, 4 .$$

The net $\mathcal{N}$ has the form

$$y = x\begin{bmatrix} a & f(a) \\ 0 & a \end{bmatrix}, \qquad x = 0$$

where f is $1-1$ as in section 2. Also, ϱ must leave $\mathcal{N}$ invariant as ϱ centralizes $\mathfrak{B}$. Since ϱ must map $x = \mathcal{O}$ onto

$$y = x\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1},$$

it must be that $a_3 = 0$. That is,

$$x = \mathcal{O} \xrightarrow{\varrho} y = x\begin{bmatrix} a_1^{-1} & a_1^{-1}a_4^{-1}a_2 \\ 0 & a_4^{-1} \end{bmatrix}$$

so that $a_1 = a_4$. Also,

$$y = x\begin{bmatrix} a & f(a) \\ 0 & a \end{bmatrix}$$

is mapped onto

$$y = x\frac{1}{d}\begin{bmatrix} 1 + aa_1 & aa_2 + f(a)a_1 \\ 0 & 1 + aa_1 \end{bmatrix}\begin{bmatrix} a & f(0) \\ 0 & a \end{bmatrix}$$

where $d = (1 + aa_1)^2$ provided $1 + aa_1 \neq 0$. Hence,

$$f\left((1 + aa_1)\frac{a}{d}\right) = \frac{1}{d}\left\{(aa_2 + f(a)a_1)a + (1 + aa_1)f(a)\right\},$$

since this matrix must be

$$\begin{bmatrix} \dfrac{(1 + aa_1)a}{d} & f\left(\dfrac{(1 + aa_1)a}{d}\right) \\ 0 & \dfrac{(1 + aa_1)a}{d} \end{bmatrix}.$$

That is, $f((1 + aa_1)^{-1}a) = (1 + aa_1)^{-1}(a^2a_2 + f(a))$ for $1 + aa_1 \neq 0$.

(4.3) LEMMA. If

$$\varrho = \begin{bmatrix} I & \mathcal{O} \\ a_1a_2 & I \\ 0\ a_1 & I \end{bmatrix} \qquad \text{then } a_2 = 0\,.$$

PROOF. Let

$$x_b = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \varrho .$$

For $b \neq 0$, this element is a Baer involution which fixes $y = O$ but fixes no other component of $\mathcal{N}$. Hence, there exist q components of the form

$$y = x \begin{bmatrix} da^{-1} & d^2a^{-1} + d + g(a^{-1}) \\ a^{-1} & da^{-1} + 1 \end{bmatrix}$$

which are fixed by x_b by (2.8). Such a component is fixed by x_b if and only if

$$\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} da^{-1} & F(d, a) \\ a^{-1} & da^{-1} + 1 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 + ba_1 \\ 0 & a_1 \end{bmatrix} \right) \begin{bmatrix} da^{-1} & F(d, a) \\ a^{-1} & da^{-1} + 1 \end{bmatrix} = $$
$$= \begin{bmatrix} da^{-1} & F(d, a) \\ a^{-1} & da^{-1} + 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

where $F(d, a) = d^2a^{-1} + d + g(a^{-1})$. Now let $b = a_1^{-1}a_2$ if $a_2 \neq 0$. Working out the (2,1)-entries, we have:

$$\begin{bmatrix} 1 + da^{-1}a_1 & b + da^{-1}(a_2 + ba_1) + F(d, a)a_1 \\ a^{-1}a_1 & 1 + a^{-1}(a_2 + ba_1) + (da^{-1} + 1)a_1 \end{bmatrix} \begin{bmatrix} da^{-1} & F(d, a) \\ a^{-1} & da^{-1} + 1 \end{bmatrix} =$$
$$= \begin{bmatrix} da^{-1} & da^{-1}b + F(d, a) \\ a^{-1} & a^{-1}b + da^{-1} + 1 \end{bmatrix}$$

so that $a^{-1}a_1 da^{-1} + \big(1 + a^{-1}(a_2 + ba_1) + (da^{-1} + 1)a^1\big)a^{-1} = a^{-1}$. If $b = a^{22}a_2$ for $a_2 \neq 0$ then we have $(1 + a_1)a^{-1} = a^{-1}$ or rather $a_1 = 0$. This cannot be so $a_2 = 0$.

(4.4) LEMMA. $g(a^{-1}) = a/a_1$.

PROOF. Since $a_2 = 0$, we have, equating the (i, j)-entries of the matrix equations considered in (4.3):

$$(1,2) \qquad (1 + da^{-1}a_1)F(d_1 a) + \big(b + da^{-1}ba_1 + F(d, a)a_1\big)(da^{-1} + 1) =$$
$$= da^{-1}b + F(d, a)$$

so that

$$(b + da^{-1}ba_1)(da^{-1} + 1) + F(d, a)a_1 = da^{-1}b$$

where

$$F(d, a) = d^2a^{-1} + d + g(a^{-1}) .$$

Hence,

$$bda^{-1} + b + d^2a^{-2} + ba_1 + da^{-1}ba_1 + d^2a^{-1}a_1 + da_1 + g(a^{-1})a_1 = da^{-1}b$$

so that

$$(1,2)' \qquad b + d^2a^{-2}ba_1 + da^{-1}ba_1 + d^2a^{-1}a_1 + da_1 = g(a^{-1})a_1 .$$

$$(1,1) \qquad (1 + da^{-1}a_1)da^{-1} + \big(b + da^{-1}(ba_1) + F(d, a)a_1\big)a^{-1} = da^{-1} .$$

Hence, we have:

$$d^2a^{-1}a_1 + b + da^{-1}ba_1 + d^2a^{-1}a_1 + da_1 + g(a^{-1})a_1 = 0 .$$

so that

$$(1,1)' \qquad b + da^{-1}ba_1 + da_1 = g(a^{-1})a_1 .$$

Thus, equating (1,2)′ and (1,1)′ we have $d^2a^{-2}ba_1 + d^2a^{-1}a_1 = 0$. That is, $a = b$. Now replacing $a = b$ in (1,1)′, we obtain,

$$g(a^{-1})a_1 = a .$$

If we now consider (2.11), it follows that f is identically zero.

(4.5) THEOREM. Let π be a translation plane of order q^2 with kernel F isomorphic to $GF(q)$. Assume π admits a Baer group of order q. If π admits affine elations with at least two distinct axes (in the translation complement) then π is a Hall plane and conversely a Hall plane admits such collineation groups.

PROOF. Apply the previous lemmas and (2.11).

We also obtain the corresponding result for $(2, q)$ planes although in this case we must assume a normalizing property.

(4.6) THEOREM. Let π be a translation plane of even order q^2 with kernel F isomorphic to $GF(q)$. Assume π admits an elation group $\mathcal{E}$ of order q. Also, assume π admits distinct Baer 2-groups $\mathcal{B}_1$, $\mathcal{B}_2$ in the linear translation complement where $\mathcal{E}$ normalizes $\mathcal{B}_1$ and $\mathcal{B}_2$ but $\mathcal{B}_2 \not\subseteq \mathcal{E}\mathcal{B}_1$. Then π is Desarguesian.

PROOF. By section 3, $|\mathcal{B}_1| = |\mathcal{B}_2| = 2$ and since $\mathcal{E}$ centralizes $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_2 \not\subseteq \mathcal{E}\mathcal{B}_1$ there must be two Baer subplanes which are pointwise fixed by involutions in $\mathcal{B}_2\mathcal{B}_1\mathcal{E}$ which belong to the same net of degree $1+q$. That is, we may assume if π_i is the associated Baer subplane pointwise fixed by $\mathcal{B}_i$, $i = 1, 2$ then π_i belongs to $\mathcal{N}$. Hence, by Foulser [3], $\mathcal{N}$ must be derivable as π_i, $i = 1, 2$ must be Desarguesian subplanes. (Note, the components are completely defined by the Baer involutions in $\mathcal{B}_1\mathcal{E}$ ($\mathcal{B}_2\mathcal{E}$). Since two corresponding subplanes therefore must overlap and $\mathcal{E}$ fixes each and acts transitively on the components of each, it must be that subplanes belong to the *same* net of degree $1+q$.)

The result now follows from (4.5).

5. Questions and open problems.

(5.1) In sections 2 and 3, Baer-elation planes of dimension 2, order q^2 and type $(2, q)$ or $(q, 2)$ were developed. Determine a classification of type $(2, q)$ (or $(q, 2)$)-planes of order q^2 and arbitrary dimension. Determine the subclass where there are many Baer axes (or many elation axes).

(5.2) Determine the *derivable* Baer-Elation planes of type $(2, q)$ or type $(q, 2)$ and order q^2 and of dimension 2.

(5.3) If π is a Baer-Elation plane of order q^2, $q = 2^r$ and type $(2^k, 2^{r+1-k})$ show the type is $(2, q)$ or $(q, 2)$.

(5.4) It is possible to have a $(\geqslant 4, \geqslant 4)$-Baer-Elation plane of any dimension?

(5.5) If π is a Baer-Elation plane of order q^2 and type $(2^b, 2^e)$, is it possible that $2^b \cdot 2^e > 2q$?

(5.6) If π is a Baer-Elation plane of dimension 2, derive the dual of π to obtain various semi-translation planes. Try to recover a Baer-elation plane by properties of an associated semi-translation plane.